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Volume 1

Semi-Markov Migration Models for Credit Risk

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coordinated by
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Contents

Introduction

This book is a summary of several papers that the authors wrote on credit risk starting from 2003 to 2016.

Credit risk problem is one of the most important contemporary problems that has been developed in the financial literature. The basic idea of our approach is to consider the credit risk of a company like a reliability evaluation of the company that issues a bond to reimburse its debt.

Considering that semi-Markov processes (SMPs) were applied in the engineering field for the study of reliability of complex mechanical systems, we decided to apply this process and develop it for the study of credit risk evaluation.

Our first paper [D'AM 05] was presented at the 27th Congress AMASES held in Cagliari, 2003. The second paper [D'AM 06] was presented at IWAP 2004 Athens. The third paper [D'AM 11] was presented at QMF 2004 Sidney. Our remaining research articles are as follows: [D'AM 07, D'AM 08a, D'AM 08b, SIL08, D'AM 09, D'AM 10, D'AM 11a, D'AM 11b, D'AM 12, D'AM 14a, D'AM 14b, D'AM 15, D'AM 16a] and [D'AM 16b].

Other credit risk studies in a semi-Markov setting were from [VAS 06, VAS 13] and [VAS 13]. We should also outline that up to now, at author's knowledge, no papers were written for outline problems or criticisms to the applications of SMPs to the migration credit risk.

The study of credit risk began with so-called structural form models (SFM). Merton [MER 74] proposed the first paper regarding this approach. This paper was an application of the seminal papers by Black and Scholes [BLA 73]. According to Merton's paper, default can only happen at the maturity date of the debt. Many criticisms were made on this approach. Indeed, it was supposed that there are no transaction costs, no taxes and that the assets are perfectly divisible. Furthermore, the short sales of assets are allowed. Finally, it is supposed that the time evolution of the firm's value follows a diffusion process (see [BEN 05]).

In Merton's paper [MER 74], the stochastic differential equation was the same that could be used for the pricing of a European option. This problem was solved by Black and Cox [BLA 76] by extending Merton's model, which allowed the default to occur at any time and not only at the maturity of the bond. In this book, techniques useful for the pricing of American type options are discussed.

Many other papers generalized the Merton and Black and Cox results. We recall the following papers: [DUA 94, LON 95, LEL 94, LEL 06, JON 84, OGD 87, LYD 00, EOM 03] and [GES 77].

The second approach to the study of credit risk involves reduced form models (RFMs). In this case, pricing and hedging are evaluated by public data, which are fully observable by everybody. In SFM, the data used for the evaluation of risk are known only within the company. More precisely, [JAR 04] explains that in the case of RFM, the information set is observed by the market, and in the case of SFM, the information set is known only inside the company.

The first RFM was given in [JAR 92]. In the late 1990s, these models developed. The seminal paper [JAR 97] introduced Markov models for following the evolution of rating. Starting from this paper, although many models make use of Markov chains, the problem of the poorly fitting Markov processes in the credit risk environment has been outlined.

Ratings change with time and a way of following their evolution their by means of Markov processes (see, for example, [JAR 97, ISR 01, HU 02]. In this environment, Markov models are called migration models. The problem of poorly fitting Markov processes in the credit risk environment has been outlined in some papers, including [ALT 98, CAR 94] and [LAN 02].

These problems include the following:

– *the duration inside a state*: actually, the probability of changing rating depends on the time that a firm remains in the same rating. Under the Markov assumption, this probability depends only on the rank at the previous transition;

– *the dependence of the rating evaluation from the epoch of the assessment*: this means that, in general, the rating evaluation depends on when it is done and, in particular, on the business cycle;

– *the dependence of the new rating from all history of the firm's rank evolution, not only from the last evaluation*: actually, the effect exists only in the downward cases but not in the case of upward ratings in the sense that if a firm gets a lower rating (for almost all rating classes), then there is a higher probability that the next rating will be lower than the preceding one.

All these problems were solved by means of models that applied the SMPs, generalizing the Markov migration models.

This book is self-contained and is divided into nine chapters.

The first part of the Chapter 1 briefly describes the rating evolution and introduces to the meaning of migration and the importance of the evaluation of the probability of default for a company that issues bonds. In the second part, Markov chains are described as a mathematical tool useful for rating migration modeling. The subsequent step shows how rating migration models can be constructed by means of Markov processes.

Once the Markov limits in the management of migration models are defined, the chapter introduces the homogeneous semi-Markov environment. The last tool that is presented is the non-homogeneous semi-Markov model. Real-life examples are also presented.

In Chapter 2, it is shown how it is possible to take into account simultaneously recurrence times, i.e. backward and forward processes at the beginning and at the end of the time in which the credit risk model is observed. With such a generalization, it is possible to consider what happens inside the time before and after each transition to provide a full understanding of durations inside states of the studied system. The model is presented in a discrete time environment.

Chapter 3 presents the application of recurrence times in credit risk problems. Indeed, the first criticisms of Markov migration models were on the independence of the transition probabilities with respect to the duration of waiting time inside states (see [CAR 94, DUF 03]). SMP overcomes this problem but the introduction of initial and final backward and forward times allows for a complete study of the duration inside states. Furthermore, the duration of waiting time in credit risk problems is a fundamental issue in the construction of credit risk models.

In this chapter, real data examples are presented that show how the results of our semi-Markov models are sensitive to recurrence times.

Some papers have outlined the problem of unsuitable fitting of Markov processes in a credit risk environment. Chapter 4 presents a model that overcomes all the inadequacies of the Markov models. As previously mentioned, the full introduction of recurrence times solves the duration problem. The time dependence of the rating evaluation can be solved by means of the introduction of non-homogeneity. The downward problem is solved by means of the introduction of six states. The randomness of waiting time in the transitions of states is considered, thus making it possible to take into account the duration completely inside a state. Furthermore, in this chapter, both transient and asymptotic analyses are presented. The asymptotic analysis is performed by using a mono-unireducible topological structure. At the end of the chapter, a real data application is performed using the historical database of Standard & Poor's as the source.

Chapter 5 presents a model to describe the evolution of the yield spread by considering the rating evaluation as the determinant of credit spreads. The underlying rating migration process is assumed to be a non-homogeneous discrete time semi-Markov non-discounted reward process. The rewards are given by the values of the spreads.

The calculation of the total sum of mean basis points paid within any given time interval is also performed.

From this information, we show how it is possible to extract the time evolution of expected interest rates and discount factors.

In Chapter 6, a discrete time non-homogeneous semi-Markov model for the rating evolution of the credit quality of a firm C is considered (see [D'AM 04]). The credit default swap spread for a contract between two parties, A and B, that sell and buy a protection about the failure of the firm C is determined. The work, both in the case of deterministic and stochastic recovery rate, is calculated. The link between credit risk and reliability theory is also highlighted.

Chapter 7 details two connected problems, as follows:

– the construction of an appropriate multivariate model for the study of counterparty credit risk in the credit rating migration problem is presented. For this financial problem, different multivariate Markov chain models were proposed. However, the Markovian assumption may be inappropriate for the study of the dynamics of credit ratings, which typically shows non-Markovian-like behavior. In this first part of the chapter, we develop a semi-Markov approach to study the counterparty credit risk by defining a new multivariate semi-Markov chain model. Methods are proposed for computing the transition probabilities, reliability functions and the price of a risky credit default swap;

– the construction of a bivariate semi-Markov reward chain model is presented. Equations for the higher order moments of the reward process are presented for the first time and applied to the problem for modeling the credit spread evolution of an obligor by considering the dynamic of its own credit rating and that of a dependent obligor called the counterpart. How to compute the expected value of the accumulated credit spread (expressed in basis points) that the obligor should expect to pay in addition to the risk free interest rate is detailed. Higher order moments of the accumulated credit spread process convey important financial information in terms of variance, skewness and kurtosis of the total basis points the obligor should pay in a given time horizon. This chapter contributes to the literature by extending on previous results of semi-Markov reward chains. The models and the validity of the results are illustrated through two numerical examples.

In Chapter 8, as in the previous chapters, the credit risk problem is placed in a reliability environment. One of the main applications of SMPs is, as it is well known, in the field of reliability. For this reason, it is quite natural to construct semi-Markov credit risk migration models.

This chapter details the first results that were obtained by the research group by the application of Monte Carlo simulation methods. How to reconstruct the semi-Markov trajectories using Monte Carlo methods and how to obtain the distribution of the random variable of the losses that the bank should support in the given horizon time are also explained in this chapter. Once this random variable is reconstructed, it will be possible to have all the moments of it and all the variability indices including the VaR. As it is well known, the VaR construction represents the main risk indicator in the Basel I–III committee agreements.

1

Semi-Markov Processes Migration Credit Risk Models

This chapter presents a very concise presentation of the credit risk problem and basic stochastic models used to solve it, mainly homogeneous and non-homogeneous semi-Markov models illustrated with some numerical examples.

These models will be discussed in the following chapters.

1.1. Rating and migration problems

1.1.1. *Ratings*

As mentioned by Solvency II and Basel III Committees, the credit risk problem is one of the most important contemporary problems for banks and insurance companies. Indeed, for banks, for example, more than 40% of their equities are necessary to cover this risk.

When a bank has a loan or when a financial institution issues bonds bought by a firm, this bank or this firm risk not being able to recover their money totally or partially. This risk is called *default risk*. A lot of work has been done to build stochastic models to evaluate the probability of default. One of the first models is the *Merton* model [MER 74], or the *firm model,* considering the case of a firm that borrows an amount M of money at time 0,

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for example in the form of a zero coupon bond with facial value F (interests included) representing the amount to reimburse at time T.

It is clear that a smaller probability of default is better for the issuing company as it makes buying their bonds more attractive.

As the default risk of a firm is difficult to evaluate and since its value can change with time up to the maturity time of the bond, this problem is studied by big *agencies of rating* such as Standard and Poor's, Fitch and Moody. The agencies play an important role in financial and economic worlds.

In the case of Standard and Poor's, there are the nine different classes of rating and so we have to consider the following set of states:

$$E = \{\text{AAA, AA, A, BBB, BB, B, CCC, D, NR}\}. \qquad [1.1]$$

The first seven states are working states (good states) and the last two are bad states giving the two following subsets:

$$U = \{\text{AAA, AA, A, BBB, BB, B, CCC}\}, \ \ D = \{\text{ D, NR}\}. \qquad [1.2]$$

The up states represent the long-term ratings given by Standard and Poor's (S&P) to the firm that have bonds on the market and that regularly reimburse their bonds. Clearly, the worse the rating, the higher the interest rate will be that the firm that issues the bonds must pay in term of basic points. The two down states represent, respectively, the Default state and the No Rating (NR) state. The former happens when the firm could not reimburse, partially or totally, the bonds. The second down state represents a firm to which the agency does not give the rating evaluation.

It is clear that the rate given by an agency at a time t_1 can be revised at a time t_2 and that so this rate has a time evolution modeled by stochastic models called *migration models*.

The main problem in the credit risk environment is the study of *default probability*. For this reason, many migration models do not consider the NR state and transform the default state D in an absorbing state.

The state set becomes the following:

$$E = \{\text{AAA, AA, A, BBB, BB, B, CCC, D}\}, \qquad [1.3]$$

and the subset of the down states will be formed only by the default state D.

In real economic life, credit rating agencies play a crucial role; they compile data on individual companies or countries to estimate their probability of default, represented by their scale of credit ratings at a given time and also by the probability of transitions for successive credit ratings.

1.1.2. *Migration problem*

A change in the rating is called a *migration*.

Clearly, a migration to a higher rating will increase the value of a company's bond and decrease its yield, giving what we call a negative *spread*, as it has a lower probability of default, and the inverse is true with a migration toward a lower grade with a consequently positive spread.

In the following, we give an example of a possible *transition matrix* for migration from 1 year to the next.

	AAA	AA	A	BBB	BB	B	CCC	D	Total
AAA	0.90829	0.08272	0.00736	0.00065	0.00066	0.00014	0.00006	0.00012	1
AA	0.00665	0.9089	0.07692	0.00583	0.00064	0.00066	0.00029	0.00011	1
A	0.00092	0.0242	0.91305	0.05228	0.00678	0.00227	0.00009	0.00041	1
BBB	0.00042	0.0032	0.05878	0.87459	0.04964	0.01078	0.0011	0.00149	1
BB	0.00039	0.00126	0.00644	0.0771	0.81159	0.08397	0.0097	0.00955	1
B	0.00044	0.00211	0.00361	0.00718	0.07961	0.80767	0.04992	0.04946	1
CCC	0.00127	0.00122	0.00423	0.01195	0.0269	0.11711	0.64479	0.19253	1
D	0	0	0	0	0	0	0	1	1

Table 1.1. *Example of transition matrix of credit ratings (source: [JAN 07])*

The elements of the first diagonal row give the probabilities of no migration and are the highest elements of the matrix, but they decrease with poorer quality ratings.

Here, we see, for example, that a company with rank A has more or less nine chances out of 10 to maintain its rating for the following year, but its chances of going up to rank AA is only two out of 100.

On the other hand, the chances of a company with a CCC rating defaulting in next year is 20 out of 100.

Table 1.2 gives the transition probability matrix of credit ratings of *Standard and Poor's* for 1998 (see ratings performance, Standard and Poor's) for a sample of 4,014 companies.

As mentioned previously, let us point out the presence of an NR state (*rating withdrawn*), meaning that for a company in such a state, the rating has been withdrawn and that this event does not necessarily lead to default the following year, thus explaining the last row of Table 1.1.

Effec.		AAA	AA	A	BBB	BB	B	CCC	D	N.R.	Total
165	AAA	90.3	6.1	0	0.61	0	0	0	0	3.03	100
560	AA	0.18	90	5.71	0.18	0	0	0	0	4.29	100
1,095	A	0.09	1.5	87.22	5.11	0.18	0	0	0	5.94	100
896	BBB	0	0	2.79	84.93	4.46	0.67	0.22	0.34	6.59	100
619	BB	0.32	0.2	0.16	5.33	75.44	5.98	2.75	0.65	9.21	100
649	B	0	0	0.15	0.62	6.16	76.3	5.09	4.47	7.24	100
30	CCC	0	0	3.33	0	0	20	33.3	36.67	6.67	100
	NR	0	0	0	0	0	0	0	0	100	100
4,014											

Table 1.2. *Example of withdrawn rating (source: [JAN 07])*

Here, we see, for example, that companies in state A will not be in default the next year but that 5.1% of them will degrade to BBB.

1.1.3. *Impact of rating on spreads for zero bonds*

To understand the importance of ratings and migration, let us recall their impact on the spread, that is the difference between the interest paid by the issuer and the non-risky rate represented by δ is the constant instantaneous intensity of interest rate. Let us recall that a *zero-coupon bond* is a contract paying a known fixed amount called the *principal*, at some given future date, called the *maturity date*. If the principal is one monetary unit and T is the maturity date, the value of this zero-coupon at time 0 is given by:

$$B(0,T) = e^{-\delta T} . \qquad [1.4]$$

of course, the investor in zero-coupons must take into account the risk of default of the issuer. To do so, Janssen and Manca [JAN 07] consider that, in a risk neutral framework, the investor has no preference between the two following investments:

i) to receive almost certainly at time 1 the amount e^{δ} as counterpart of the investment at time 0 of one monetary unit;

ii) to receive at time 1 the amount $e^{(\delta+s)}(s>0)$ with probability $(1-p)$ or 0 with probability *p*, as counterpart of the investment at time 0 of one monetary unit, *p* being the default probability of the issuer.

The positive quantity *s* is called the *spread* with respect to the non-risky instantaneous interest rate δ as counterpart of this risky investment in zero-coupon bonds.

From the indifference mentioned above, we obtain the following relation:

$$e^{\delta} = (1-p)e^{(\delta+s)} \qquad [1.5]$$

or

$$1 = (1-p)e^{s}, \qquad [1.6]$$

from which it follows that

$$s = -\ln(1-p). \tag{1.7}$$

And so at a first-order approximation, we see that *the spread is more or less equal to the probability of default*:

$$s \approx p. \tag{1.8}$$

The more precise second-order approximation gives:

$$s \cong p + \frac{1}{2}p^2. \tag{1.9}$$

Let us now consider a more positive and realistic situation in which the investor can have a *recovery percentage*, i.e. he can recover an amount $\alpha, (0 < \alpha < 1)$ if the issuer defaults at maturity or before.

In this case, the expectation equivalence principle relation [1.5] becomes:

$$e^{\delta} = (1-p)e^{\delta+s} + p\alpha e^{\delta}, \tag{1.10}$$

or

$$1 = (1-p)e^{s} + p\alpha. \tag{1.11}$$

It follows that in this case the value of the spread satisfies the equation

$$e^{s} = \frac{1-p\alpha}{1-p} \tag{1.12}$$

and so the spread value is

$$s = \ln\frac{1-p\alpha}{1-p}. \tag{1.13}$$

As above, using the Mac Laurin formula, respectively, of order 1 and 2, we obtain the two following approximations for the spread:

$$\begin{aligned} &s \approx p(1-\alpha), \\ &s \approx p(1-\alpha) + \frac{1}{2}(p^2(1-\alpha^2)) \end{aligned} \tag{1.14}$$

Now, we see that, at the first order approximation, the spread decreases by an amount $p\alpha$.

1.1.4. *Homogeneous Markov chain model*

In the 1990s, Markov models were introduced to study credit risk problems. Many important papers on these kinds of models were published (see [JAR 95, JAR 97, NIC 00, ISR 01, HU 02]), mainly for solving the problem of the evaluation of the transition matrices.

Under the assumption of a homogeneous Markov chain for the migration process, we can follow the rate under a time dynamic point of view and as such evaluate the probability distribution of the rate after t years. We can also compute mean rates, variances and also VaR values (see Chapter 7 of [DEV 15]).

For example, using Table 1.2, we obtain the following results:

i) the probability that an AA company defaults after 2 years:

$$P^{(2)}(D/AA) = 0.0018 \cdot 0.0034 = 0.0006\%,$$

which is still very low.

ii) the probability that a BBB company defaults in one of the next 2 years:

$$\begin{aligned} P(D/BBB;2) &= P(D/BBB) + P(BBB/BBB)P(D/BBB) \\ &\quad + P(BB/BBB)P(D/BB) + P(B/BBB)P(D/B) \\ &\quad + P(CCC/BBB)P(D/CCC) \\ &= 0.34\% + (84.93\% \cdot 0.34\%) \\ &\quad + (4.46\% \cdot 0.65\%) + (0.67\% \cdot 4.47\%) + (0.22\% \cdot 36.67\%) \\ &= 0.77\%. \end{aligned}$$

iii) the probability for a company BBB to default between years 1 and 2:

Using the standard definition of conditional probability, we get

$$\begin{aligned} \text{P(D at 2/non-def. at 1)} &= \text{P(D at 2 and non-def. at 1)/ P(non-def. at 1)} \\ &= (0.77\% - 0.34\%)/(1 - 0.34\%) \\ &= 0.43\%. \end{aligned}$$

1.1.5. *Migration models*

1) *Credit risk and reliability problems*

Homogeneous semi-Markov processes (HSMPs) were defined by Levy [LEV 54] and Smith [SMI 55], independently.

A detailed theoretical analysis of semi-Markov processes (SMPs) is given in [HOW 71, JAN 06, JAN 07].

As specified in [HOW 71, LIM 01] and more recently in [JAN 07, DEV 15], one of the most important applications of SMPs in engineering is in the field of reliability.

In a reliability problem, we consider a system S that could be a mechanical or an electronic material, for example, and which can be in m different states represented by the set

$$I = \{1, ..., m\}.$$

This state set can be partitioned into two subsets. The first is formed by the states in which the system can function and the second by the states in which the system is partially functioning or totally malfunctioning in case of a fatal failure.

We can compare the ratings given to an issuer of bonds to the successive state of a virtual reliability system S so that the state m of total failure corresponds to the default rate D.

The credit risk problem can be positioned in the reliability environment as shown in section 1.3. The rating process, done by the rating agency, gives the reliability degree of a bond. For example, in the case of Standard and Poor's, we have set of eight different classes of rating and so the set of states is

$$E = \{\text{AAA, AA, A, BBB, BB, B, CCC, D}\}.$$

The first seven states are working states (good states) and the last is the only bad state. The two subsets are:

$$U = \{\text{AAA, AA, A, BBB, BB, B, CCC}\}, \quad D = \{\text{ D}\}. \qquad [1.15]$$

Reliability in real problems can also be dealt with successfully by means of SMPs (see e.g. [BLA 04]).

The rating level changes over time and one way to follow the time evolution of ratings is by means of Markov processes (see [JAR 97]). In this environment, Markov models are called "migration models". Other papers (see, e.g. [NIC 00, ISR 01, HU 02]) followed this approach working mainly on the generation of a transition matrix.

The default problem can be included in the more general problem of the reliability of a stochastic system. In the credit risk migration model, the rating agencies giving the rating estimate the reliability of the firm that issued the bonds. The default state can be seen as a non-working state that, in this special case, is also an absorbing state.

In this chapter, the semi-Markov reliability model, presented in [BLA 04] is applied in order to solve the credit risk problem.

2) *Main questions in migration*

The problem of the suitability of Markov processes in the credit risk environment has been addressed (see [ALT 98, NIC 00, KAV 01, LAN 02]).

Nevertheless, Markov processes only constitute a first approach but are not entirely satisfactory to describe migrations problems in a more realistic way as they do not consider some important facts such as:

i) *the duration inside a state*: the probability of changing rating depends on the time a company maintains the same rating (see, e.g. [CAR 94]). To be more precise, quoting [DUF 03, p. 87]: "there is dependence of transition probabilities on duration in a rating category or age";

ii) *the time dependence of the rating evaluation*: this means that in general the rating evaluation depends on time t and, in particularly, on the business cycle (see [NIC 00]). A rating evaluation carried out at time t is generally different from one carried out at time s, if $s \neq t$;

iii) *the dependence of the new rating*: it can in general depend on all the previous ones and not only on the last one (see [CAR 94, NIC 00]).

As mentioned in [D'AM 05]), the *first problem* can be satisfactorily solved by means of SMPs. In fact, in SMP the transition probabilities are a function of the waiting time spent in a state of the system. In [CAR 94], in particular, a Weibull distribution is used in order to investigate the duration effect for time spent continuously at a given credit rating. The *second problem* can be dealt with in a more general way by means of a non-homogeneous environment.

The *third problem* exists in the case of downward moving ratings but not in the case of upward moving ratings (see [KAV 01]). More precisely, if a company gets a lower rating then there is a higher probability that its subsequent rating will also be lower than the preceding one. In the case of upward movement, this phenomenon does not hold.

In this chapter, we present models that can completely solve the first and second problems based on HSMP and non-homogeneous semi-Markov process (NHSMP).

Semi-Markov models were introduced by Janssen *et al*. [JAN 05] and Janssen and Manca [JAN 07] first in the homogeneous case. The non-homogeneous case was developed in [JAN 04] and [JAN 07]. With these new models, it is possible to generalize the Markov models introducing the randomness of time for transitions between the states.

1.2. Homogeneous semi-Markov processes

1.2.1. *Basic definitions*

In this section, we follow the notation given in [DEV 15] to recall basic definitions and properties of discrete homogeneous semi-Markov process (DHSMP).

Let us consider a *physical* or *economic system* called S with m possible states forming the set $I = \{1, ..., m\}$.

At time 0, system S starts from an initial state represented by the r.v. J_0, stays a non-negative random length of time X_1 in this state, and then goes into another state J_1 for a non-negative length of time X_2 before going into J_2, etc.

So we have a two-dimensional stochastic process in discrete time called a *positive* (*J-X*) *process* or simply (*J-X*) *process*

$$(J - X) = ((J_n, X_n), n \geq 0) \tag{1.16}$$

assuming

$$X_0 = 0,\ a.s. \tag{1.17}$$

where the sequence $(J_n, n \geq 0)$ gives the successive *states* of S in time and the sequence $(X_n, n \geq 0)$ gives the successive *sojourns* in state J_{n-1} or the *interarrival times* between two successive transitions.

Times at which transitions occur are given by the sequence $(T_n, n \geq 0)$ where:

$$T_0 = 0,\ T_1 = X_1, \ldots, T_n = \sum_{r=1}^{n} X_r \tag{1.18}$$

such that

$$X_n = T_n - T_{n-1}, n \geq 1. \tag{1.19}$$

The process $((J_n, T_n), n \geq 0)$ is called a *Markov renewal process* (MRP).

On the complete probability space $(\Omega, \Im, P)$, the stochastic dynamic evolution of the considered (*J-X*) process is completely defined by the knowledge of the initial probability distribution

$$\mathbf{p} = (p_1, \ldots, p_m),$$
$$p_i \geq 0, i \in I; \sum_{i=1}^{m} p_i = 1$$

with

$$\begin{aligned} &p_i = P[J_0] = i, i \in I, \\ &X_0 = 0, a.s. \end{aligned} \tag{1.20}$$

and moreover, for all n > 0, j=1,…,m, by the so-called *homogeneous semi-Markov condition*

$$P(J_n = j, X_n \le x \big| (J_k, X_k), k = 0,...,n-1) = Q_{J_{n-1}j}(x), \ a.s., \quad [1.21]$$

where any function Q_{ij} (i,j=1,…,m) is a non-decreasing real function null on $\mathbb{R}^+$ such that if

$$p_{ij} = \lim_{x \to +\infty} Q_{ij}(x), \ i, j \in I, \quad [1.22]$$

then

$$\sum_{j=1}^{m} p_{ij} = 1, \ i \in I. \quad [1.23]$$

We also have

$$\begin{aligned} &P(T_n - T_{n-1} \le x, J_n = j \big| (J_k, T_k), k = 0,...,n-1, J_{n-1} = i) \\ &= P(T_n - T_{n-1} \le x, J_n = j \big| J_{n-1} = i)(= Q_{ij}(x)). \end{aligned} \quad [1.24]$$

With matrix notation, we will write:

$$\mathbf{Q}(x) = \left[Q_{ij}(x) \right], \ \mathbf{P} = \left[p_{ij} \right] (= \mathbf{Q}(\infty)) \quad [1.25]$$

and it follows (see [PYK 61]) that

$$p_{ij} = \lim_{t \to \infty} Q_{ij}(t); \ i, j \in E, \ t \in \mathbb{N}, \quad [1.26]$$

where P = [p_{ij}] is the transition matrix of the *embedded Markov chain* $(J_n, n \ge 0)$ in the process.

The matrix Q is called a semi-Markov kernel.

The (J-X) is called *discrete* or in *discrete time* if the random variables X_n are in discrete time. This means that all the possible values of these variables are in the set $\{0, \Delta, ..., n\Delta, ...\}$, where Δ is the time unit (in the sequel we will take Δ = 1). Otherwise, we speak of *continuous time* (J-X) process.

Now it is possible to define the *distribution function of the waiting time* in each state i, given that the next state is known:

$$G_{ij}(t)=\mathrm{P}[\,T_{n+1}-T_n \le t \mid J_n = i, J_{n+1} = j\,]. \qquad [1.27]$$

The related probabilities can be obtained by means of the following formula:

$$G_{ij}(t) = \begin{cases} Q_{ij}(t)/p_{ij} & \text{if } p_{ij} \neq 0, \\ 1 & \text{if } p_{ij} = 0 \end{cases}. \qquad [1.28]$$

Furthermore, it is necessary to introduce the *distribution function of the waiting time* in each state *i,* regardless of the next state:

$$H_i(t) = \mathrm{P}[\,T_{n+1} - T_n \le t \mid J_n = i]. \qquad [1.29]$$

Obviously, it results that:

$$H_i(t) = \sum_{j=1}^{m} p_{ij} G_{ij}(t) \qquad [1.30]$$

or by relation [1.28]

$$H_i(t) = \sum_{j=1}^{m} Q_{ij}(t).$$

In the semi-Markov theory, the functions $G_{ij}, i, j \in I$ and $H_i, i \in I$ are, respectively, called *conditional* and *unconditional waiting time distributions*.

In a semi-Markov model for credit risk developed in the following, the functions G_{ij} will be the distribution functions of the time between two consecutive ratings by the agency. Of course, the “transition” from i to i is possible meaning that the rate has remained unchanged.

1.2.2. *The Z SMP and the evolution equation system*

Finally, we have to introduce the *SMP* where $Z = (Z(t))$, representing, for each time t, the state occupied by the process i.e.:

$$Z(t) = J_{N(t)}, \text{ where } N(t) = \max\{n : T_n \le t\}. \quad [1.31]$$

For a discrete (J-X) process, the Z variables take their values in $\mathbb{N}$ and in this case we speak on DHSMP. Without specifying discrete or continuous time, we speak of HSMP.

Figure 1.1 gives a typical sample path of an SMP.

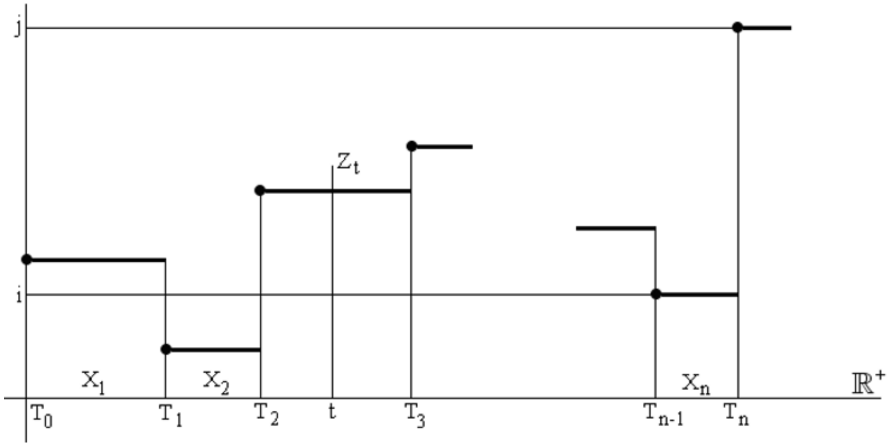

Figure 1.1. *A sample path of an SMP (source: [JAN 07])*

The *transition probabilities* of the Z process are defined by

$$\phi_{ij}(t) = \mathrm{P}[\, Z(t) = j \mid Z(0) = i \,]. \quad [1.32]$$

For DHSMP, they are obtained solving the following evolution equations:

$$\phi_{ij}(t) = \delta_{ij}(1 - H_i(t)) + \sum_{\beta=1}^{m} \sum_{\vartheta=1}^{t} q_{i\beta}(\vartheta)\phi_{\beta j}(t - \vartheta), \quad [1.33]$$