Maria Colombo

Flows of Non-Smooth Vector Fields and Degenerate Elliptic Equations

with Applications to the Vlasov-Poisson and Semigeostrophic Systems

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To my father

Contents

Introduction

In the last centuries, partial differential equations have been used to model many physical problems: the Navier-Stokes and Euler equations in fluid dynamics, the Boltzmann and Vlasov equations in statistical mechanics, the Schrodinger equation in quantum physics, and many other PDEs concerning, for instance, material science or meteorology. The richness of mathematical structure in these equations is always reason of surprise.

As a motivating example, we introduce the Vlasov-Poisson system. It describes the evolution of particles under their self-consistent electric or gravitational field. It is the continuous counterpart of the N-body problem, which describes the motion of N mass points under the influence of their mutual attraction governed by Newton's law of gravity. The N-body problem has applications in astronomy and plasma physics; for instance, it describes the solar system or the motion of galaxies. In the gravitational models, each element of unit mass with position x and velocity v obeys the equation

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\partial_x V_t(x), \end{cases}$$

where $V_t(x)$ is the gravitational potential depending on time t and position x. Collisions between different masses are considered as an extremely unlikely event and are therefore neglected. Since the number of involved elements in a galaxy can be of order 10^{10}–10^{12}, the galaxy is described in the Vlasov-Poisson system in a statistical way rather than keeping track of each mass point. For this reason, we introduce the quantity $f_t(x, v)$, which describes the distribution of particles with given position x and velocity v at time t. The density f_t solves a first order conservation law on phase space

$$\partial_t f_t + v \cdot \nabla_x f_t - \nabla_x V_t \cdot \nabla_v f_t = 0 \qquad \text{in } (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \qquad (1)$$

whose characteristics are the equations of motion of a single test particle.

In turn, the gravitational potential V_t is obtained from the physical density

$$\rho_t(x) = \int_{\mathbb{R}^d} f_t(x, v)\, dv \qquad \text{in } (0, \infty) \times \mathbb{R}^d \tag{2}$$

by solving the Poisson equation

$$-\Delta V_t = \sigma \rho_t \qquad \text{in } \mathbb{R}^d, \qquad \lim_{|x|\to\infty} V_t(x) = 0. \tag{3}$$

Here, $\sigma \in \{\pm 1\}$ distinguishes the gravitational (attractive) and the electrostatic (repulsive) problem.

The nonlinear system of partial differential equations (1), (2), and (3) has a transport structure: indeed it can be rewritten as

$$\partial_t f_t + \boldsymbol{b}_t \cdot \nabla_{x,v} f_t = 0, \tag{4}$$

where the vector field $\boldsymbol{b}_t(x, v) = (v, E_t(x)) : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is coupled to f_t via the relation $E_t = \sigma c_d \rho_t * (x/|x|^d)$ and c_d is a dimensional constant. Indeed, the force field E_t is obtained as $-\nabla_x V_t$ and V_t can be written as the convolution of ρ_t with a singular kernel by solving (3). Since the vector field is divergence free, it can be also rewritten as a continuity equation

$$\partial_t f_t + \nabla_{x,v} \cdot \big(\boldsymbol{b}_t f_t\big) = 0. \tag{5}$$

Solutions of (5), when considering a fixed vector field $\boldsymbol{b}$, turn out to be obtained by flowing the initial datum f_0 along the characteristics of the vector field $\boldsymbol{b}$. The deep connection between the transport/continuity equation (Eulerian point of view) and the notion of flow (Lagrangian point of view) is one of the most fascinating aspects of this theory. It is the basis of many results regarding the continuity equation and the flows even in a non-smooth setting, starting from the fundamental papers of DiPerna and Lions [78] and Ambrosio [5].

Many questions regarding the Vlasov-Poisson equation are nowadays little understood and some of them are deeply related to the dual, Lagrangian and Eulerian, nature of the equation. One of the main open problems in statistical mechanics is, for instance, the rigorous derivation of the equation. It amounts in proving that, when a sequence of configurations with finitely many particles approximates a continuous initial distribution of particles, the solutions of the approximate systems converge to the solution of the Vlasov-Poisson equation. As well as the Boltzmann equation, the Vlasov equation has been rigorously derived only under restrictive smallness assumptions on the time of observation, the total mass of matter, or the distance of the distribution function to equilibrium. Moreover, all derivations of the Vlasov equation assume that the interaction at small scales is either smooth or not too singular.

As we saw above, the Vlasov-Poisson equation can be seen as a transport equation in the phase space, coupled with a PDE which determines the gravitational field in terms of the distribution of particles. The main scope of our thesis is a further step in understanding some aspects of the interaction between transport equations and PDEs. More precisely, we consider the following problems, which regard the DiPerna-Lions theory and the regularity of degenerate elliptic equations, together with the analysis of the interaction between these points of view in models coming from mathematical physics.

- The Di Perna-Lions and Ambrosio theory for flows of non-smooth vector fields: We develop a local version of the DiPerna-Lions theories for ODE's, providing a complete analogy with the Cauchy-Lipschitz theory. More precisely, we prove existence and uniqueness of a maximal regular flow for non-smooth vector fields using only local regularity and summability assumptions on the vector field, in analogy with the classical theory, which uses only local regularity assumptions.
- The quantitative estimates for the ODE: They constitute a different approach to the DiPerna-Lions theory, this time relying on a priori estimates on solutions of the ODE rather than on the connection between Lagrangian and Eulerian structure. We apply these estimates in the Eulerian setting to obtain renormalized solutions of the continuity equation with a linear source term; this equation is not easily covered by the methods of DiPerna and Lions.
- The regularity of very degenerate elliptic equations: This problem comes from a model in traffic dynamic and it is a variant of the optimal transport problem, which takes into account congestion effects in the transportation. It leads to different equivalent formulations; they employ in one case some concepts related to flows of vector fields, in another case the minimization of a variational integral, where the convexity of the integrand degenerates on a full convex set. We are interested in the regularity of solutions.
- The Vlasov-Poisson system: This equation, introduced above, couples the transport structure in the phase space (namely, the space of positions and velocities of particles) with the Laplace equation, which describes the force field. The existence of classical solutions is limited to dimensions $d \leq 3$ under strong assumptions on the initial data, while weak solutions are known to exist under milder conditions. However, in the setting of weak solutions it is unclear whether the Eulerian description provided by the equation physically corresponds to a Lagrangian evolution of the particles. Through general tools concerning the Lagrangian structure of transport equations with non-smooth vector

fields, we show that weak solutions of Vlasov-Poisson are Lagrangian and we obtain global existence of weak solutions under minimal assumptions on the initial data.
- The semigeostrophic system: It was introduced in meteorology to describe atmospheric/ocean flows. After a suitable change of variable, it has a dual version which couples a transport equation with a nonlinear elliptic PDE, namely the Monge-Ampère equation. We study the problem of existence of distributional solutions to the original system.

In the following, we give a quick overview on all these problems and an outline of the thesis' content, postponing a more detailed mathematical and bibliographical description of the single problems to the beginning of each chapter. The results in this thesis are the final outcome of several collaborations developed during the PhD studies and have been presented in a series of papers, already published or submitted.

Flows of non-smooth vector fields. Given a vector field $\boldsymbol{b} : (0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ we consider the ordinary differential equation

$$\begin{cases} \partial_t \boldsymbol{X}(t,x) = \boldsymbol{b}(t,\boldsymbol{X}(t,x)) & \forall t\in(0,T) \\ \boldsymbol{X}(0,x) = x, \end{cases} \tag{6}$$

which is strictly related (via the method of characteristics) to the continuity equation

$$\begin{cases} \partial_t u + \nabla\cdot(\boldsymbol{b}u) = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ u_0 = \bar{u} \quad \text{given}, \end{cases} \tag{7}$$

where $u : (0,T)\times\mathbb{R}^d \to \mathbb{R}$. If the vector field $\boldsymbol{b}$ is Lipschitz with respect to space uniformly in time, the Cauchy-Lipschitz theory and classical PDE arguments provide existence and uniqueness of a solution to (6) and (7). In their fundamental papers, exploiting the connection between (6) and (7), Di Perna and Lions [81] and Ambrosio [5] proved existence and uniqueness of a so called *regular lagrangian flow*, namely a certain solution to (6), even in the case of Sobolev and BV vector fields. However, the Cauchy-Lipschitz theory is not only pointwise but also purely local, meaning that existence and uniqueness for small intervals of time depend *only* on local regularity properties of the vector fields $\boldsymbol{b}_t(x)$. On the other hand, not only the DiPerna-Lions theory is an almost everywhere theory (and this really seems to be unavoidable) but also the existence results for the flow depend on *global* in space growth estimates on $|\boldsymbol{b}|$, the most typical one being

$$\frac{|\boldsymbol{b}_t(x)|}{1+|x|} \in L^1\big((0,T);L^1(\mathbb{R}^d)\big) + L^1\big((0,T);L^\infty(\mathbb{R}^d)\big),$$

which prevent the trajectories of the flow from blowing up in finite time. In Chapter 2, based on a joint work [10] with Ambrosio and Figalli, under purely local and natural assumptions on the vector field, we prove existence of a unique *maximal regular flow* $\boldsymbol{X}(t,x)$, defined up to a maximal time $T_{\boldsymbol{X}}(x)$ which is positive $\mathscr{L}^d$-a.e. in $\mathbb{R}^d$, with

$$\limsup_{t\to T_{\boldsymbol{X}}(x)} |\boldsymbol{X}(t,x)| = \infty \qquad \text{for } \mathscr{L}^d\text{-a.e. } x\in\mathbb{R}^d \text{ such that } T_{\boldsymbol{X}}(x) < T.$$

We then study, in Chapter 3, the natural semigroup and stability properties of this object; finally we analyze the blow-up of the maximal regular flow $\boldsymbol{X}(\cdot,x)$ at the maximal time $T_{\boldsymbol{X}}(x)$. Surprisingly enough, indeed, the proper blow up of trajectories, namely

$$\lim_{t\to T_{\boldsymbol{X}}(x)} |\boldsymbol{X}(t,x)| = \infty \qquad \text{for } \mathscr{L}^d\text{-a.e. } x\in\mathbb{R}^d \text{ such that } T_{\boldsymbol{X}}(x) < T$$

happens only under a global bound on the divergence of $\boldsymbol{b}$, whereas there are counterexamples if only local bounds are assumed.

Quantitative estimates for the continuity equation. Another aspect of the theory of regular lagrangian flows are the so called "quantitative estimates", developed in the Lagrangian case (namely, for solutions of (6)) by Ambrosio, Lecumberry, and Maniglia [22], Crippa and De Lellis [67]. This theory allows to prove uniqueness and stability of flows, in an independent way with respect to the analysis of the solutions to the continuity equation. More precisely, the fundamental a-priori estimate is the following: given a small parameter $\delta > 0$, if $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are the flows of two vector fields $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ we consider the functional

$$\Phi_\delta(t) := \int_{\mathbb{R}^d} \log\left(1 + \frac{|\boldsymbol{X}_1(t,x) - \boldsymbol{X}_2(t,x)|^2}{\delta}\right) dx,$$

whose time derivative is bounded independently on δ under suitable assumptions on the vector fields. A similar functional can be employed also in the Eulerian setting to estimate the distance of two solutions of the continuity equation (7). This approach is followed in joint works with Crippa and Spirito [56, 57], presented in Chapter 5, where we consider (7) with a non-smooth vector field and a linear source term, called *damping term* (although its sign may be either positive or negative), namely a right-hand side of the form cu with $c : (0,T)\times\mathbb{R}^d \to \mathbb{R}$. In their fundamental paper [81], DiPerna and Lions proved that, when c is bounded in space and time, the equation is well posed in the class of distributional solutions and the solution is transported by suitable characteristics of the vector field. Thanks to the quantitative estimates for the solution of the continuity equation, existence and uniqueness of solutions holds under more general assumptions on the data, for instance, assuming only integrability of the damping term.

Regularity of degenerate elliptic PDEs. In Chapter 6 and 7 we study the gradient regularity of local minimizers of the functional

$$\int_\Omega \mathcal{F}(\nabla u) + fu, \tag{8}$$

where we are given a bounded open subset Ω of $\mathbb{R}^d$, a convex function $\mathcal{F} : \mathbb{R}^d \to \mathbb{R}$ which exhibits a large degeneracy set, and an integrable function $f : \Omega \to \mathbb{R}$. Our model function is

$$\mathcal{F}(v) = \frac{1}{p}(|v| - 1)_+^p \qquad \forall\, v \in \mathbb{R}^d, \tag{9}$$

so that the degeneracy set is the entire unit ball. This problem comes from a model by Beckmann [30], where, given an urban area where people move from home to work, the optimal traffic flow σ solves the minimum problem

$$\min\left\{\int_\Omega \mathcal{F}^*(\sigma) : \sigma \in L^{p'}(\Omega),\ \nabla\cdot\sigma = f,\ \sigma\cdot\nu_{\partial\Omega} = 0\right\}. \tag{10}$$

Here, $\mathcal{F}^*$ denotes the convex conjugate of the function $\mathcal{F}$; by the choice of $\mathcal{F}$ in (9), we have that

$$\mathcal{F}^*(\sigma) = |\sigma| + \frac{1}{p'}|\sigma|^{p'} \qquad \forall \sigma \in \mathbb{R}^d$$

where p' satisfies $1/p + 1/p' = 1$. The function $\mathcal{F}$ is chosen so that its convex conjugate $\mathcal{F}^*$ has more than linear growth at infinity (so to avoid "congestion") and satisfies $\liminf_{w\to 0} |\nabla\mathcal{F}^*(w)| > 0$ (which means that moving in an empty street has a nonzero cost).

Problem 10 is equivalent to the problem of minimizing the energy (8) with the particular choice of $\mathcal{F}$ given by (9). The unique optimal minimizer $\bar\sigma$ in problem (10) turns out to be exactly $\nabla\mathcal{F}(\nabla u)$, where $\mathcal{F}$ is defined by (9). The continuity of $\bar\sigma$ is meaningful in terms of traffic models, as shown in [49]. Indeed, one can consider measures on the space of possible paths and select an optimal measure which satisfies a Wardrop equilibrium principle: no traveler wants to change his path, provided all the other ones keep the same strategy. According to this optimal measure, every path is a geodesic with respect to a metric on Ω of the form

$g(|\bar{\sigma}(x)|)Id$ (where $g(t) = 1 + t^{p-1}$ is the so-called "congestion function"), which is defined in terms of the optimal traffic distribution itself. The continuity of $\bar{\sigma}$ and, therefore, of the metric allows to set and study the geodesic problem in the usual sense.

In order to understand the regularity of minimizers of functionals as in (8), we first recall that, when $\nabla^2\mathcal{F}$ is uniformly elliptic, namely there exist $\lambda, \Lambda > 0$ such that $\lambda Id \leq \nabla^2\mathcal{F} \leq \Lambda Id$, the regularity results of u rely on De Giorgi theorem and Schauder estimates. If the ellipticity of $\mathcal{F}$ degenerates at only one point, then several results are still available. For instance, in the model case of the p-Laplace equation, that is when $\mathcal{F}(v) = |v|^p$ and $f = 0$, the $C^{1,\alpha}$ regularity of u has been proved by Uraltseva for $p \geq 2$, initiating a wide literature.

With the choice of $\mathcal{F}$ in (9), the Lipschitz regularity of a local minimizer u follows by standard techniques [87], since the equation is the classical p-Laplace equation when the gradient is large. In general no more regularity than L^∞ can be expected on ∇u. Indeed, when $\mathcal{F}$ is given by (6.5) and f is identically 0, every 1-Lipschitz function is a global minimizer of (6.4). However, in Chapters 6 and 7, based on joint works with Figalli [55, 59] we prove the continuity of $\nabla\mathcal{F}(\nabla u)$, extending a previous result of Santambrogio and Vespri [114] which holds only in dimension 2.

The Vlasov-Poisson system. The structure of transport equation hidden in the nonlinear Vlasov-Poisson system, presented at the beginning of this Introduction, has been exploited in a huge literature, in order to obtain existence and uniqueness of classical solutions, namely, solutions where all the relevant derivatives exist. The first existence results were obtained in dimension 1 by Iordanskii [99], in dimension 2 by Ukai and Okabe [120], in dimension 3 for small data by Bardos and Degond [26], and for symmetric initial data in [29, 123, 95, 116]. Finally, in 1989 Pfaffelmöser [111] and Lions and Perthame [105] were able to prove global existence of classical solutions starting from general data. Moreover, the uniqueness problem has been addressed under more restrictive assumptions on the initial datum in [105] and [108], and both proofs employ the Lagrangian flow associated to the solution, which is regular enough under a global bound on the space density.

In recent years, an interesting direction of research in the context of the Vlasov-Poisson system is given by the analysis of existence, uniqueness and properties of weak solutions. In particular, when one drops the assumption of boundedness of the initial density (this assumption is preserved along solutions thanks to the transport structure of the equation) and assumes only that $f_t \in L^1(\mathbb{R}^{2d})$, the term $E_t f_t$ appearing in the equation is not even locally integrable. For this reason, Di Perna and Lions [78] introduced the concept of *renormalized solution*, which is equivalent to the notion of weak (distributional) solution under suitable integrability assumptions on f_t. In this context, DiPerna and Lions announced global existence of solutions when the total energy is finite and $f_0 \log(1 + f_0) \in L^1(\mathbb{R}^{2d})$.

In the setting of weak solutions, due to the low regularity of the density and of the vector field, it is unclear whether the *Eulerian description* provided by the equation physically corresponds to a *Lagrangian evolution* of the particles. In Chapter 8 (based on a joint work with Ambrosio and Figalli [11]), we investigate this problem and we apply the general tools developed in Chapter 4 to prove that the Lagrangian structure holds even in the context of weak/renormalized solutions. We obtain also global existence of weak solutions under minimal assumptions on the initial data and improve the result in [78], dropping the hypothesis $f_0 \log(1 + f_0) \in L^1(\mathbb{R}^{2d})$ and assuming only the finiteness of energy.

The semigeostrophic system. The semigeostrophic system models athmosperic/ocean flows on large scales. The problem can be described in the case of periodic solutions in $\mathbb{R}^2$, namely on the 2-dimensional torus $\mathbb{T}^2$

$$\begin{cases} \partial_t \nabla P_t(x) + \big(u_t(x)\cdot\nabla\big)\nabla P_t(x) = J(\nabla P_t(x) - x) & (x,t)\in\mathbb{T}^2\times(0,\infty) \\ \nabla\cdot u_t(x) = 0 & (x,t)\in\mathbb{T}^2\times[0,\infty) \\ P_0(x) = P^0(x) & x\in\mathbb{T}^2. \end{cases} \tag{11}$$

where P^0 is the initial datum, $J \in \mathbb{R}^{2\times 2}$ is a rotation matrix, u_t represents the velocity, and ∇P_t is related to the pressure of the fluid.

Energetic considerations show that it is natural to assume the convexity of the function $P_t(x)$. The system (11) has a *dual formulation* obtained with a *change of variable*

$$\begin{cases} \partial_t \rho_t + \nabla \cdot (U_t \rho_t) = 0 \\ U_t(x) = J(x - \nabla P_t^*(x)) \\ \rho_t = (\nabla P_t)_\sharp \mathscr{L}_{\mathbb{T}^2} \\ P_0(x) = p^0(x) + |x|^2/2, \end{cases}$$

where P_t^* is the convex conjugate of P_t. The existence of dual solutions was proved in 1998 by Benamou and Brenier [31], and, starting from the lagrangian solutions of the dual equation, in [69] the authors managed to build a very weak solution of (11) of lagrangian type, by reversing the change of variables. The formal expression for the velocity u_t of the original system, given a solution (P_t, ρ_t) of the dual system, is given by

$$u_t(x) := [\partial_t \nabla P_t^*](\nabla P_t(x)) + [\nabla^2 P_t^*](\nabla P_t(x)) J(\nabla P_t(x) - x). \quad (12)$$

However, the existence of distributional solutions to (11) stayed as an open problem due to the low regularity of the change of variable, since a priori $\nabla^2 P_t^*$ is only a matrix-valued measure and one needs also differentiability in time of ∇P_t^* to give a meaning to (12). The existence of Eulerian solutions is shown in joint works with Ambrosio, De Philippis, and Figalli [7, 8], thanks to the recent regularity results on solutions of the Monge-Ampère equation [73], and it is the content of Chapter 9.

In the final part of this introduction, we outline other works developed during the PhD that present some common underlying ideas and techniques with the ones outlined above in this introduction.

Regularity of double phase variational problems. Degenerate elliptic problems arise also to model strongly anisotropic materials. Given $\Omega \subset \mathbb{R}^d$, $d \geq 2$, we are here interested in the regularity of local minimizers $u\colon \Omega \to \mathbb{R}$ of a class of variational integrals whose model is given by the functional

$$\mathcal{P}(w) := \int_\Omega (|Dw|^p + a(x)|Dw|^q)\, dx \ , \quad (13)$$

which is naturally defined on $W^{1,1}(\Omega)$, where

$$1 < p < q \,, \qquad 0 \leq a \in C^{0,\alpha}(\Omega)\,, \qquad \alpha \in (0, 1]\,.$$

The functional $\mathcal{P}$ belongs to the class of functionals with non-standard growth conditions, which have been widely studied in recent years. These are integral functionals of the type

$$w \mapsto \int_\Omega f(x, Dw)\, dx \ ,$$

where the integrand $f\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfies unbalanced polynomial growth conditions of the type

$$|z|^p \lesssim f(x, z) \lesssim |z|^q + 1 \qquad \text{for every } z \in \mathbb{R}^d.$$

In (13), the coefficient $a(x)$ describes the geometry of a composite, made of two different materials, with power hardening of rate p and q, respectively. From the mathematical viewpoint, the integrand of (13) switches between two different types (phases) of elliptic behaviors according to the coefficient $a(\cdot)$. Since a interacts directly with the ellipticity of the problem, the presence of x is not any longer a perturbation, and this has direct consequences on the regularity of minimizers. More precisely, the regularity of the minimizer holds if the gap between the exponents p and q is controlled in terms of the regularity of a by

$$q \leq p + \alpha. \tag{14}$$

This condition is sharp, as shown in the counterexample in [83]. In [62], Mingione and I proved that bounded local minimizers of (13) under the assumption (14) have Hölder continuous gradients, namely $\nabla u \in C^{0,\beta}$ for some $\beta > 0$. Boundedness is a rather common feature since it for instance follows by maximum principle when considering solutions of Dirichlet problems involving a bounded boundary datum $u_0 \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$. In a companion paper [61] we prove that the same regularity holds also in the case of unbounded local minimizers, but this time we assume a different relation between the exponents p, q and the regularity of a:

$$q < p + \frac{\alpha p}{n}.$$

The proofs in [61, 62] rely on many different technical tools, going from the p-harmonic approximation lemma to a fractional Caccioppoli inequality. A common underlying idea is to consider, *at each scale*, namely on every ball $B_R \subset \Omega$, an alternative according to the fact that

$$\sup_{x \in B_R} \frac{a(x)}{R^\alpha} \leq M$$

holds or not, for a threshold M to be chosen. If it holds, then *at this fixed scale* we are in the *p-phase* and we compare our minimizer to a solution of the p-Laplace equation in the same ball. Otherwise, we are in the *(p, q)-phase* and the solution is compared to the solution of a functional like (13) with frozen coefficient $a(\cdot) = a_0$. The regularity for the frozen problem has been studied in [104].

Many questions arise from the results presented above. For instance, in collaboration with Baroni and Mingione [27, 28], we see that Harnack inequalities, in analogy with the results of [76], hold also for minimizers of double phase integrals and that the regularity theory developed in [61] can be generalized to different ellipticity types. In particular, we consider a functional of the type

$$\mathcal{P}_{ln}(w) := \int_\Omega \left[|Dw|^p + a(x)|Dw|^p \ln(1 + |Dw|)\right] dx$$

and correspondingly, the coefficient a is allowed to have a logarithmic modulus of continuity in order to obtain the Hölder continuity of the minimizer.

Optimal transport with Coulomb cost. In some recent papers, Buttazzo, De Pascale and Gori-Giorgi [40] and Cotar, Friesecke and Klüppelberg [66] consider a mathematical model for the strong interaction limit of the density functional theory (DFT). In particular, the model for the minimal interaction of N electrons is formulated in terms of a multimarginal Monge transport problem. Let $c : (\mathbb{R}^d)^N \to \mathbb{R}$ be the Coulomb cost function

$$c(x_1, \ldots, x_N) = \sum_{1 \le i < j \le N} \frac{1}{|x_i - x_j|} \qquad \forall (x_1, \ldots, x_N) \in (\mathbb{R}^d)^N, \quad (15)$$

$\rho \in \mathcal{P}(\mathbb{R}^d)$ be a given probability measure on $\mathbb{R}^d$, and $\mathcal{T}(\rho)$ be the set of transport maps $\mathcal{T}(\rho) = \{T : \mathbb{R}^d \to \mathbb{R}^d \text{ Borel} : T_\sharp \rho = \rho\}$, where $T_\sharp \rho$ represents the pushforward measure of the measure ρ through the Borel map T. We consider the Monge multimarginal problem

$$(M) = \inf \left\{ \int_{\mathbb{R}^d} c(x, T_2(x), \ldots, T_N(x))\, d\mu(x) : T_2, \ldots, T_N \in \mathcal{T}(\rho) \right\}$$

and its cyclical version

$$(M_{cycl}) = \inf \left\{ \int_{\mathbb{R}^d} c(x, T(x), \ldots, T^{(N-1)}(x))\, d\mu(x) : T \in \mathcal{T}(\rho), T^{(N)} = Id \right\},$$

which is meaningful since the cost function is symmetric. Following the standard theory of optimal transport, we introduce the set of transport plans

$$\Pi(\rho) = \left\{ \gamma \in \mathcal{P}(\mathbb{R}^{dN}) \;:\; \pi^i_\sharp \gamma = \rho,\; i = 1, \ldots, N \right\},$$

where $\pi^i : (\mathbb{R}^d)^N \to \mathbb{R}^d$ are the projections on the i-th component for $i = 1, \ldots, N$, and the Kantorovich multimarginal problem

$$(K) = \min \left\{ \int_{(\mathbb{R}^d)^N} c(x_1, \ldots, x_N) d\gamma(x_1, \ldots, x_N) : \gamma \in \Pi(\rho) \right\},$$

where, in contrast with (M), we allow the splitting of mass. To every $(N-1)$-uple of transport maps $T_2, \ldots, T_N \in \mathcal{T}(\rho)$ we associate the transport plan

$$\gamma = (Id, T_2, \ldots, T_N)_\sharp \rho \in \Pi(\rho).$$

We remark that the existence of an optimal transport plan, namely a minimizer of (K), follows from the lower semicontinuity of the cost, from the linearity of the cost of a plan γ with respect to γ and from the fact that the admissible plans form a tight subset of the set of measures on $(\mathbb{R}^d)^N$. In a joint paper with Di Marino [53], under the sharp assumption that ρ is non-atomic, we prove that $(K) = (M) = (M_{cycl})$. In particular, if an optimal transport map exists, it has the cyclical structure that appears in (M_{cycl}). This result reduces the optimization problem (K) over measures on $\mathbb{R}^{Nd}$ to the problem (M_{cycl}) over functions on $\mathbb{R}^N$ and is useful in deriving numerical methods to compute the value of (K). In a companion paper [54], joint work with Di Marino and De Pascale, we address the problem of existence of optimal transport maps in dimension $d = 1$, providing an explicit construction of the optimal map. For $N = 2$, in any dimension, existence follows from the standard optimal transport theory (see [124]) since the so called "twist condition" is formally satisfied by the Coulomb cost (15). In the multimarginal case $N \geq 3$, there is no general theory for the existence of optimal maps and the construction in [54] heavily relies on the assumption $d = 1$. The generalization of this result to higher dimensions is open. Finally, in a paper with Stra [64] we begin the analysis of the case of spherically symmetric data, which model for instance Litium and Berillium atoms. We disprove a conjecture on the structure of the optimal transport, showing that some special maps, introduced by Seidl, Gori Giorgi and Savin, are not always optimal in the corresponding transport problem. We also provide examples of maps satisfying optimality conditions for special classes of data.

Geometric characterizations of rigidity in symmetrization inequalities and nonlocal perimeters. Symmetrization inequalities are among the most basic tools of the Calculus of Variations. They include the Polya-Szego inequality for the Dirichlet energy, the Steiner symmetrization and its analogous in the Gaussian setting, named Ehrhard symmetrization, which is a well-known tool in Probability Theory, arising in the study of geometric variational problems in Gauss space.

The study of their equality cases plays a fundamental role in the explicit characterization of minimizers, thus in the computation of optimal constants in geometric and functional inequalities. Although it is usually easy to derive useful necessary conditions for equality cases, the analysis of *rigidity of equality cases* (that is, the situation when every set realizing equality in the given symmetrization inequality turns out to be symmetric) is a much subtler issue. Sufficient conditions for rigidity have been known, and largely used, in the case of the Polya-Szego inequality for the Dirichlet energy [39], and of Steiner inequality for perimeter [50]. However, these sufficient conditions fail to be also necessary: for example, the one proposed in [50] fails to characterize rigidity even in the class of polyhedra in $\mathbb{R}^3$. A preliminary analysis of some examples indicates that, in order to formulate geometric conditions which could possibly be suitable for characterizing rigidity, one needs a measure-theoretic notion which describes when a Borel set "disconnects" another Borel set. This notion, called *essential connectedness*, was first introduced in a joint paper with Cagnetti, De Philippis, and Maggi [47] and it is inspired by the notion of indecomposable current adopted in Geometric Measure Theory (see [86, 4.2.25]). It allows to formulate in its terms a simple geometric condition that characterizes rigidity in Ehrhard inequality for Gaussian perimeter. The same notion can be employed, together with a fine analysis of the differentiability properties of the barycenter function of a set of finite perimeter whose sections are segments, to provide various characterizations of rigidity in Steiner inequality for Euclidean perimeter. This was done in collaboration with Cagnetti, De Philippis, and Maggi [48].

Chapter 1
An overview on flows of vector fields and on optimal transport

The aim of this Chapter is twofold. On one side, we give an overview on the classical results regarding flows of vector fields, the regularity of degenerate elliptic PDEs and, in particular, the Monge-Ampère equation. These results and ideas will be fundamental for the development of all the subsequent chapters. On the other side, we present the classical theory according to a point of view that will be useful in the rest of this thesis, showing refinements of the known theorems that suit the subsequent discussions.

1.1. Classical and nonsmooth theory

Given a vector field $\boldsymbol{b} : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$ we consider the ordinary differential equation

$$\begin{cases} \partial_t \boldsymbol{X}(t,x) = \boldsymbol{b}_t(\boldsymbol{X}(t,x)) & \forall t \in (0,T) \\ \boldsymbol{X}(0,x) = x, \end{cases} \tag{1.1}$$

In the smooth setting, namely when $\boldsymbol{b}$ is locally Lipschitz with respect to the space variable, existence and uniqueness of a solution to (1.1) is guaranteed by the Cauchy-Lipschitz theorem.

Theorem 1.1 (Cauchy-Lipschitz). *Let $T > 0$, $\boldsymbol{b} \in L^1((0,T);\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d))$. Then for every $x \in \mathbb{R}^d$ there exists a unique maximal solution $\boldsymbol{X}(\cdot,x)$ of (1.1) defined in a nonempty maximal existence time $[0,T_{\boldsymbol{X}}(x))$. Moreover, the map $T_{\boldsymbol{X}}$ is lower semicontinuous, for every $x \in \mathbb{R}^d$ such that $T_{\boldsymbol{X}}(x) < T$ the trajectory $\boldsymbol{X}(\cdot,x)$ blows up properly, namely*

$$\lim_{t \to T_{\boldsymbol{X}}(x)} |\boldsymbol{X}(t,x)| = \infty,$$

and the map $\boldsymbol{X}(t,\cdot)$ is locally Lipschitz in space on its finiteness domain.

The ODE (1.1) is strictly related (via the method of characteristics) to the transport equation

$$\begin{cases} \partial_t u + \boldsymbol{b} \cdot \nabla u = 0 & \text{in } (0,T) \times \mathbb{R}^d \\ u_0 = \bar{u} & \text{given.} \end{cases} \tag{1.2}$$

Indeed, if u is a smooth solution of (1.2) and $x \in \mathbb{R}^d$, we have

$$\begin{aligned} \frac{d}{dt} u_t(\boldsymbol{X}(t,x)) &= \partial_t u_t(\boldsymbol{X}(t,x)) + \partial_t \boldsymbol{X}(t,x) \cdot \nabla u_t(\boldsymbol{X}(t,x)) \\ &= \partial_t u_t(\boldsymbol{X}(t,x)) + \boldsymbol{b}_t(\boldsymbol{X}(t,x)) \cdot \nabla u_t(\boldsymbol{X}(t,x)) = 0, \end{aligned}$$

so that u is constant along the characteristics of $\boldsymbol{b}$. Hence, given an initial datum $u_0 = \bar{u}$, we expect

$$u_t(x) = \bar{u}(\boldsymbol{X}(t,\cdot)^{-1}(x))$$

to be a solution of the transport equation, and this can be easily checked by direct computation. In the last thirty years, a huge effort has been made in order to develop a theory of flows of vector fields in the non-smooth setting, in view of applications to physical systems. In the following, we precise the meaning of the ODE (1.1) and of the continuity and transport equation in a non-smooth setting. The continuity equation is

$$\begin{cases} \partial_t u + \nabla \cdot (\boldsymbol{b} u) = 0 & \text{in } (0,T) \times \mathbb{R}^d \\ u_0 = \bar{u} \text{ given,} \end{cases} \tag{1.3}$$

where $u : (0,T) \times \mathbb{R}^d \to \mathbb{R}$; in the case of a divergence-free vector field, it is equivalent to the transport equation (1.2). We mostly use standard notation, denoting by $\mathscr{L}^d$ the Lebesgue measure in $\mathbb{R}^d$, and by $f_\# \mu$ the push-forward of a Borel nonnegative measure μ under the action of a Borel map f, namely $f_\# \mu(B) = \mu(f^{-1}(B))$ for any Borel set B in the target space. We denote by $\mathcal{B}(\mathbb{R}^d)$ the family of all Borel sets in $\mathbb{R}^d$. In the family of positive finite measures in an open set Ω, we will consider both the weak topology induced by the duality with $C_b(\Omega)$ that we will call *narrow* topology, and the *weak* topology induced by $C_c(\Omega)$. Also, $\mathscr{M}_+(\mathbb{R}^d)$ will denote the space of finite Borel measures on $\mathbb{R}^d$, while $\mathscr{P}(\mathbb{R}^d)$ denotes the space of probability measures.

In the non-smooth setting, given a Borel vector field $\boldsymbol{b} : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$, an integral curve $\gamma : [0,T] \to \mathbb{R}^d$ of the equation $\partial_t \gamma = \boldsymbol{b}_t(\gamma)$ (see (1.1)) is an absolutely continuous curve in $AC([0,T];\mathbb{R}^d)$ which satisfies the previous ODE for almost every $t \in [0,T]$. The continuity equation is intended in distributional sense, according to the following definition.

Definition 1.2 (Distributional solutions). A family $\{\mu_t\}_{t\in[0,T]}$ of locally finite signed measures on $\mathbb{R}^d$ such that $\boldsymbol{b}_t\mu_t$ is a locally finite measure is a solution of the continuity equation if it solves

$$\partial_t\mu_t + \nabla\cdot(\boldsymbol{b}_t\mu_t) = 0$$

in the sense of distributions, namely for every $\phi \in C_c^\infty((0,T)\times\mathbb{R}^d)$

$$\int_0^T\int_{\mathbb{R}^d}\big[\partial_t\phi_t(x) + \nabla_x\phi_t(x)\cdot\boldsymbol{b}_t(x)\big]\,d\mu_t(x)\,dt = 0.$$

The family $\{\mu_t\}_{t\in[0,T]}$ is a solution of the continuity equation with initial datum μ_0 if for every $\phi \in C_c^\infty([0,T)\times\mathbb{R}^d)$

$$\int_{\mathbb{R}^d}\phi_0(x)\mu_0(x) + \int_0^T\int_{\mathbb{R}^d}\big[\partial_t\phi_t(x) + \nabla_x\phi_t(x)\cdot\boldsymbol{b}_t(x)\big]\,d\mu_t(x)\,dt = 0.$$

When we consider possibly singular measures μ_t, the vector field $\boldsymbol{b}_t$ has to be defined pointwise and not only $\mathscr{L}^d$-a.e., since the product $\boldsymbol{b}_t\mu_t$ is sensitive to modifications of $\boldsymbol{b}_t$ in $\mathscr{L}^d$-negligible sets. In the following, in particular with Sobolev or BV vector fields, we will often consider only measures μ_t which are absolutely continuous with respect to $\mathscr{L}^d$, so everything is well posed and does depend only on the equivalence class of $\boldsymbol{b}$ in $L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^d)$.

If we consider a function $\beta \in C^1(\mathbb{R})$ and we multiply the transport equation (1.2) by $\beta'(u)$, we see that, if u is a smooth solution of the transport equation, so is $\beta(u)$. The previous observation is encoded in the following definition.

Definition 1.3 (Renormalized solutions). Let $\boldsymbol{b}\in L^1_{\mathrm{loc}}((0,T);L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d))$ be a vector field with $\operatorname{div}\boldsymbol{b}\in L^1_{\mathrm{loc}}((0,T);L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d))$. Let $u\in L^\infty_{\mathrm{loc}}((0,T);L^\infty_{\mathrm{loc}}(\mathbb{R}^d))$ and assume that, in the sense of distributions, there holds

$$c := \partial_t u + \boldsymbol{b}\cdot\nabla u \in L^1_{\mathrm{loc}}((0,T);L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)). \tag{1.4}$$

Then, u is a renormalized solution of (1.4) if for every $\beta \in C^1(\mathbb{R})\cap L^\infty(\mathbb{R})$

$$\partial_t\beta(u) + \boldsymbol{b}\cdot\nabla\beta(u) = c\beta'(u).$$

in the sense of distributions. Analogously, we say that u is a renormalized solution starting from a Borel function $u_0 : \mathbb{R}^d \to \mathbb{R}$ if

$$\int_{\mathbb{R}^d}\phi_0(x)\beta(u_0(x))dx+\int_0^T\int_{\mathbb{R}^d}[\partial_t\phi_t(x)+\nabla\phi_t(x)\cdot\boldsymbol{b}_t(x)]\beta(u_t(x))dxdt=0$$

for all $\phi \in C_c^\infty([0,T)\times\mathbb{R}^d)$ and all $\beta \in C^1\cap L^\infty(\mathbb{R})$.

The renormalization property describes a property of solutions of a wide class of PDEs related to the transport equation (1.2); for this reason, we will introduce in the following Chapters a few definitions of renormalized solutions that capture better the features of each single problem. The renormalization property can be also used to give a meaning to equation (1.3) when the boundedness (or even the integrability) of u is not any more assumed as an assumption. Indeed, although the product $\boldsymbol{b}_t u_t$ may not even be locally integrable if $\boldsymbol{b}_t \in L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^d)$ and $u_t \in L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^d)$, the term $\boldsymbol{b}_t\beta(u_t)$ appearing in (5.11) is always locally integrable. This will be used in Chapter 8 to give a general notion of solution to the Vlasov-Poisson equation and in Chapter 5 for the continuity equation with an integrable damping term (see Definition 8.1 and 5.3 respectively).

If the vector field $\boldsymbol{b}$ is not assumed to be smooth, namely locally Lipschitz in space, but only Sobolev or BV, easy one dimensional examples show that the uniqueness of trajectories of the ODE 1.1 fails. For instance, if we consider the autonomous vector field $\boldsymbol{b}(x) = \sqrt{|x|}$, $x \in \mathbb{R}$, then we have many solutions of the ODE, which start from $x_0 = -c^2 < 0$, reach the origin in time $2c$, stay at the origin for any time $T \geq 0$, and continue as $(t - T - 2c)^2$.

However, one can still associate to the vector field $\boldsymbol{b}$ a notion of flow, made of a selection of trajectories of the ODE. Among all possible selections, we prefer the ones that do not allow for concentration, as presented in the following definition.

Definition 1.4. Let $T > 0$ and $\boldsymbol{b} : (0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ a Borel, locally integrable vector field. We say that the Borel map $\boldsymbol{X} : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ is a regular Lagrangian flow of $\boldsymbol{b}$ if the following two properties hold:

(i) for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$, $\boldsymbol{X}(\cdot, x) \in AC([0,T];\mathbb{R}^d)$ and solves the ODE $\dot{x}(t) = \boldsymbol{b}_t(x(t))$ $\mathscr{L}^1$-a.e. in $(0,T)$, with the initial condition $\boldsymbol{X}(0,x) = x$;
(ii) there exists a constant $C = C(\boldsymbol{X})$ satisfying $\boldsymbol{X}(t,\cdot)_\# \mathscr{L}^d \leq C\mathscr{L}^d$ for every $t \in [0,T]$.

It can be easily checked that the definition of regular Lagrangian flow depends on the equivalence class of $\boldsymbol{b}$ in $L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^d)$ rather then on the pointwise values of $\boldsymbol{b}$.

The well-celebrated papers of DiPerna and Lions [81] and Ambrosio [5] provide existence and uniqueness of the regular Lagrangian flow assuming local Sobolev or BV regularity of $\boldsymbol{b}$, boundedness of the distributional divergence $\operatorname{div}\boldsymbol{b}$, and some growth conditions on $\boldsymbol{b}$.

Theorem 1.5. *Let* $\boldsymbol{b} \in L^1((0,T); BV_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d))$ *be a vector field that satisfies the bound on the divergence* $(\operatorname{div}\boldsymbol{b})_- \in L^1((0,T); L^\infty(\mathbb{R}^d))$ *and the growth condition*

$$\frac{|\boldsymbol{b}_t(x)|}{1+|x|} \in L^1((0,T); L^1(\mathbb{R}^d)) + L^1((0,T); L^\infty(\mathbb{R}^d)).$$

Then there exists a unique regular Lagrangian flow $\boldsymbol{X}$ *of* $\boldsymbol{b}$.

The previous theorem has been extended to different classes of vector fields; some of them are listed in Remark 1.9 below. Thanks to the existence and uniqueness of a regular Lagrangian flow, it is possible to define the notion of Lagrangian solution for the continuity and transport equation. These are solutions obtained by flowing the initial datum according to the regular Lagrangian flow of $\boldsymbol{b}$.

The proof of the previous theorem is based on the interaction between the PDE point of view on the continuity equation and the Lagrangian techniques. In the following two sections, we present two key ideas behind Theorem 1.5, which in turn will be fundamental in order to develop a local version of Theorem 1.5.

1.2. A bridge between Lagrangian and Eulerian solutions: the superposition principle

This section is devoted to the so called "superposition principle", which encodes the connection between the Eulerian and the Lagrangian formulation of the continuity equation, namely between nonnegative distributional solutions of the PDE and solutions transported by a set of (possibly branching) curves. The aim of Section 1.3 is, then, to show that, under more restrictive assumptions on the vector field, this set of curves is given exactly by the flow of $\boldsymbol{b}$.

Let us fix $T \in (0,\infty)$ and consider a weakly continuous family $\mu_t \in \mathscr{M}_+\big(\mathbb{R}^d\big)$, $t \in [0,T]$, solving in the sense of distributions the continuity equation

$$\frac{d}{dt}\mu_t + \nabla\cdot(\boldsymbol{b}_t\mu_t) = 0 \qquad \text{in } (0,T)\times\mathbb{R}^d$$

for a Borel vector field $\boldsymbol{b} : (0,T)\times\mathbb{R}^d \to \mathbb{R}^d$, locally integrable with respect to the space-time measure $\mu_t dt$. When we restrict ourselves to probability measures μ_t, then weak and narrow continuity with respect to. t are equivalent; analogously, we may equivalently consider compactly supported test functions $\varphi(t,x)$ in the weak formulation of the continuity equation, or functions with bounded C^1 norm whose support is contained in $I\times\mathbb{R}^d$ with $I \Subset (0,T)$. If $J \subset \mathbb{R}$ is an interval and $t \in J$, we denote by $e_t : C(J;\mathbb{R}^d) \to \mathbb{R}^d$ the evaluation map at time t, namely $e_t(\eta) := \eta(t)$ for any continuous curve $\eta : J \to \mathbb{R}^d$.

We now recall the so-called superposition principle. We prove it under the general assumption that μ_t may a priori vanish for some $t \in [0,T]$, but satisfies (1.5); we see in Remark 1.7 that this assumption implies that there is no mass loss, namely $\mu_t(\mathbb{R}^d) = \mu_0(\mathbb{R}^d)$ for every $t \in [0,T]$. Remark 1.7 allows the reduction of the superposition principle, as stated below, to [12, Theorem 12], which presents the same result assuming that the family μ_t is made of probability measures. We mention also [19, Theorem 8.2.1], where a proof is presented in the even more special case of L^p integrability on $\boldsymbol{b}$ for some $p > 1$

$$\int_0^T \int_{\mathbb{R}^d} |\boldsymbol{b}_t(x)|^p \, d\mu_t(x)\, dt < \infty.$$

The superposition principle will play a role in the proof of the comparison principle stated in Proposition 1.11, in the blow-up criterion of Theorem 3.13 and in Theorem 4.9, where a completely local version of the superposition principle is presented.

Theorem 1.6 (Superposition principle and approximation).
Let $\boldsymbol{b} : (0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ be a Borel vector field. Let $\mu_t \in \mathscr{M}_+\big(\mathbb{R}^d\big)$, $0 \le t \le T$, with μ_t weakly continuous in $[0,T]$ solution to the equation $\frac{d}{dt}\mu_t + \operatorname{div}(\boldsymbol{b}\mu_t) = 0$ in $(0,T)\times\mathbb{R}^d$, with

$$\int_0^T \int_{\mathbb{R}^d} \frac{|\boldsymbol{b}_t(x)|}{1+|x|}\, d\mu_t(x)\, dt < \infty. \tag{1.5}$$

Then there exists $\boldsymbol{\eta} \in \mathscr{M}_+\big(C([0,T];\mathbb{R}^d)\big)$ satisfying:

(i) *$\boldsymbol{\eta}$ is concentrated on absolutely continuous curves η in $[0,T]$, solving the ODE $\dot\eta = \boldsymbol{b}_t(\eta)$ $\mathscr{L}^1$-a.e. in $(0,T)$;*
(ii) *$\mu_t = (e_t)_\#\boldsymbol{\eta}$ (so, in particular, $\mu_t(\mathbb{R}^d) = \mu_0(\mathbb{R}^d)$) for all $t \in [0,T]$.*

Moreover, there exists a family of measures $\mu_t^R \in \mathscr{M}_+\big(\mathbb{R}^d\big)$, narrowly continuous in $[0,T]$, solving the continuity equation and supported on $\overline{B}_R$, such that $\mu_t^R \uparrow \mu_t$ as $R \to \infty$ for all $t \in [0,T]$.

Remark 1.7. We show that, if μ_t and $\boldsymbol{b}_t$ are taken as in Theorem 1.6 then μ_t does not loose or gain mass, namely

$$\mu_t(\mathbb{R}^d) = \mu_0(\mathbb{R}^d) \qquad \forall t \in [0,T]. \tag{1.6}$$

Indeed, let $R \ge 1$ and $\chi_R \in C_c^\infty(B_{3R})$ be a cut-off function with $0 \le \chi_R \le 1$, $\chi_R \equiv 1$ on a neighborhood of B_R and $|\nabla\chi_R| \le \chi_{B_{3R}\setminus B_R}$. Since

μ_t solves the continuity equation and since $1/R \le 4/(1+|x|)$ for $|x| \in B_{3R} \setminus B_R$, we have

$$\begin{aligned}
\Big| \int_{\mathbb{R}^d} \chi_R \, d\mu_0 - \int_{\mathbb{R}^d} \chi_R \, d\mu_t \Big| &\le \int_0^T \Big| \frac{d}{dt} \int_{\mathbb{R}^d} \chi_R \, d\mu_t \Big| \, dt \\
&= \int_0^T \Big| \int_{B_{3R}\setminus B_R} \boldsymbol{b}_t \cdot \nabla \chi_R \, d\mu_t \Big| \, dt \\
&\le \frac{1}{R} \int_0^T \int_{B_{3R}\setminus B_R} |\boldsymbol{b}_t| \, d\mu_t \, dt \\
&\le 4 \int_0^T \int_{B_{3R}\setminus B_R} \frac{|\boldsymbol{b}_t(x)|}{1+|x|} \, d\mu_t(x) \, dt.
\end{aligned}$$

Hence we deduce that

$$\begin{aligned}
\mu_0(B_R) - \mu_t(B_{3R}) &\le \int_{\mathbb{R}^d} \chi_R \, d\mu_0 - \int_{\mathbb{R}^d} \chi_R \, d\mu_t \\
&\le 4 \int_0^T \int_{B_{3R}\setminus B_R} \frac{|\boldsymbol{b}_t(x)|}{1+|x|} \, d\mu_t(x) \, dt
\end{aligned} \tag{1.7}$$

and

$$\begin{aligned}
\mu_t(B_R) - \mu_0(B_{3R}) &\le \int_{\mathbb{R}^d} \chi_R \, d\mu_t - \int_{\mathbb{R}^d} \chi_R \, d\mu_0 \\
&\le 4 \int_0^T \int_{B_{3R}\setminus B_R} \frac{|\boldsymbol{b}_t(x)|}{1+|x|} \, d\mu_t(x) \, dt.
\end{aligned} \tag{1.8}$$

Letting $R \to \infty$ in (1.7) and (1.8), the right-hand sides converge to 0 by (1.5) and we find (1.6).

The proof of the superposition principle, as stated in Theorem 1.6, can be found in [12, Theorem 12], once Remark 1.7 is taken into account. The proof is based on a clever regularization argument: we consider a family of convolution kernels $\{\rho_\varepsilon\}_{\varepsilon \in (0,1)}$, having integral 1 and supported on the whole $\mathbb{R}^d$, and we define

$$\mu_t^\varepsilon := \mu_t * \rho^\varepsilon, \qquad \boldsymbol{b}^\varepsilon := \frac{(\boldsymbol{b}\mu_t) * \rho^\varepsilon}{\mu_t * \rho^\varepsilon}.$$

We call $\boldsymbol{X}^\varepsilon$ the flow of the vector field $\boldsymbol{b}^\varepsilon$, so that μ^ε solves the continuity equation and it is transported by $\boldsymbol{X}^\varepsilon$, since $\boldsymbol{b}^\varepsilon$ satisfies some local Lipschitz bounds, uniformly in time. Then, we define $\boldsymbol{\eta}^\varepsilon \in \mathscr{M}(AC([0,T];\mathbb{R}^d))$ as the law under μ_0^ε of the map $x \mapsto \boldsymbol{X}^\varepsilon(\cdot, x)$, namely $\boldsymbol{\eta}^\varepsilon := \boldsymbol{X}^\varepsilon(\cdot, x)_\# \mu_0^\varepsilon$. Assumption (1.5) (which holds uniformly also for $\boldsymbol{b}^\varepsilon$ and μ^ε) allows to