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V.I. Bogachev
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Topological Vector Spaces and Their Applications

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Preface

This book gives a concise exposition of the fundamentals of the theory of topological vector spaces, complemented by a survey of the most important results of a more subtle nature, which cannot be qualified as basic, but knowledge of which is useful for applications, and, finally, some of such applications connected with differential calculus in infinite-dimensional spaces and measure theory. Almost half of the book is devoted to these applications, which makes it very different from the whole series of known texts on topological vector spaces. Another notable difference between this book and known treatises like Bourbaki [**87**], Edwards [**150**], Grothendieck [**207**], Jarchow [**237**], Kelley, Namioka [**270**], Köthe [**292**], Narici, Beckenstein [**365**], Pérez Carreras, Bonet [**385**], Robertson, Robertson [**420**], Schaefer [**436**], Trèves [**530**], and Wilansky [**567**] is that we decided to include also some results without proofs (this does not concern the fundamentals, of course) with references instead, which enables us to inform the reader about many relatively recent achievements; some of them are disguised as exercises (with references to the literature), such exercises should not be confused with usual exercises marked by the symbol °. Thus, with respect to the presented information, our book is not covered by any other book on this subject (though, we cannot claim that it covers any such book).

Chapter 1 contains the fundamentals of the theory, including a large list of concrete examples, some general concepts (convex sets, seminorms, linear mappings) and a number of facts, the most important of which is the Hahn–Banach theorem on extensions of functionals in its diverse versions.

The main material of Chapter 2 is connected with projective and inductive limits (including strict inductive limits and inductive limits with compact embeddings, which is not sufficiently discussed in the existing literature), and also Grothendieck's method of constructing Banach spaces embedded into locally convex spaces.

Chapter 3 contains the classical material related to the so-called duality theory, i.e., introduction of different locally convex topologies on a given space giving the same set of continuous linear functionals. The central topics here are the Mackey–Arens theorem on topologies compatible with duality, the results on weak compactness, including the Eberlein–Šmulian and Krein–Šmulian theorems, and also some concepts and facts connected with completeness of locally convex spaces.

Chapter 4 is devoted to the fundamentals of the differentiation theory in locally convex spaces. It presents a general scheme of differentiability with respect to a system of sets (partial cases of which are Gâteaux, Hadamard and Fréchet differentiabilities) and a thorough discussion of important for applications differentiabilities with respect to systems of bounded and compact sets.

Chapter 5 gives a concise introduction to measure theory on locally convex spaces. Here we discuss extensions of cylindrical measures, the Fourier transform and conditions for the countable additivity in its terms (in particular, the Minlos and Sazonov theorems and their generalizations), covariance operators, measurable linear functionals and operators, measurable polynomials, and some important classes of measures (such as Gaussian, stable, and convex).

Each chapter opens with a brief synopsis of its content. All chapters contain many additional subsections with some more specialized information related to the main themes of the chapter, and also many exercises are given (more difficult ones are provided with hints or references). The book ends with the historic-bibliographic comments, the list of references (with indication of page numbers of citing the included works), and the author and subject indices.

The prerequisites for the first chapter of this book are just a grasp knowledge of calculus and linear algebra and some experience with basic concepts of topology, but for a thorough study it is advisable to be acquainted with a university course of functional analysis (following any text, e.g., Kolmogorov, Fomin [**284**] or Rudin [**425**]).

We are very grateful to T.O. Banakh, E.D. Kosov, I. Marshall, S.N. Popova, A.V. Shaposhnikov, A.S. Tregubov and E.V. Yurova for useful remarks and corrections.

Our work on this book began 25 years ago by the initiative of Vladimir Ivanovich Sobolev (1913–1995), the author of a series of widely known texts on functional analysis (including one of the first Russian texts, published as early as in 1951), and its completion is a tribute to the memory of this remarkable scientist and teacher.

CHAPTER 1

Introduction to the theory of topological vector spaces

In this chapter we present basic concepts and examples connected with topological vector spaces.

1.1. Linear spaces and topology

A topological vector space is a linear (or vector) space equipped with a topology which agrees with the linear structure. So we first recall separately basic concepts related to linear spaces and topological spaces. Let $\mathbb{K}$ be an algebraic field (throughout we deal with the field $\mathbb{R}$ of real numbers or with the field $\mathbb{C}$ of complex numbers; so a reader not acquainted with the general notion of an algebraic field can safely ignore it). A set E is called a *linear* (or *vector*) space over the field $\mathbb{K}$ if the elements of E (called vectors) can be added and multiplied by the elements of $\mathbb{K}$, i.e., we are given mappings

$$E\times E \to E,\ (u,v)\mapsto u+v, \quad \mathbb{K}\times E \to E,\ (\lambda,v)\mapsto \lambda v$$

satisfying the following conditions:

(i) $u+v=v+u$ for all $u,v\in E$,

(ii) there is a unique element $0\in E$ (the zero element) for which $v+0=v$ for all $v\in E$,

(iii) for every $v\in E$ there is a unique element $-v$ for which $v+(-v)=0$,

(iv) $\lambda(u+v)=\lambda u+\lambda v$, $\lambda(\mu v)=(\lambda\mu)v$, $(\lambda+\mu)u=\lambda u+\mu u$ and $0v=\lambda 0=0$ for all $u,v\in E$ and $\lambda,\mu\in\mathbb{K}$.

Below we often omit the explicit mentioning the field $\mathbb{K}$ and its elements will be called scalars, and in case of $\mathbb{K}\subset\mathbb{C}$ they will be called numbers. About general fields, see Kurosh [**306**].

1.1.1. Example. Let $\mathbb{K}=\mathbb{R}$ and let T be a nonempty set. Let $\mathbb{R}^T$ be the set of all real functions on T, where the linear operations are defined pointwise:

$$(f+g)(t) := f(t)+g(t), \quad (\lambda f)(t) := \lambda f(t).$$

Then $\mathbb{R}^T$ is a linear space; it is called the product of T copies of the real line or a *power of the real line*.

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Throughout, if it is not explicitly stated otherwise, we assume that $\mathbb{K}$ is a nondiscrete normed field. A norm on the field $\mathbb{K}$ is a mapping $\mathbb{K} \to [0, +\infty)$ (its value on the element $x \in \mathbb{K}$ is denoted by $|x|$) such that the following conditions are fulfilled: $|x| > 0$ for $x \in \mathbb{K} \setminus \{0\}$ (nondegeneracy), $|0| = 0$, $|xy| = |x|\,|y|$ (multiplicativity) and $|x + y| \leqslant |x| + |y|$ (the triangle inequality) for all $x, y \in \mathbb{K}$. A field equipped with a norm is called a *normed field*. For example, the field $\mathbb{C}$ of complex numbers is normed if $|a|$ is the usual absolute value of the number $a \in \mathbb{C}$. A field is *nondiscrete* if it has an element $k \neq 0$ with $|k| \neq 1$.

A *seminorm* on a vector space E is a function $p\colon\ E \to [0, \infty)$ such that

(1) $p(kx) = |k|p(x) \ \ \forall\, k \in \mathbb{K},\ x \in E$;
(2) $p(x_1 + x_2) \leqslant p(x_1) + p(x_2) \ \ \forall\, x_1 \in E,\ x_2 \in E$.

A seminorm p is called a *norm* if $p(x) > 0$ whenever $x \neq 0$. For example, $\|x\| = (x, x)^{1/2}$ is a norm on a *Euclidean* space E with an inner product $(\,\cdot\,,\,\cdot\,)$, i.e., $x \mapsto (x, y)$ is linear, $(x, y) \equiv (y, x)$ for real E and $(x, y) \equiv \overline{(y, x)}$ for complex E, $(x, x) \geqslant 0$ and $(x, x) = 0$ only for $x = 0$.

Two norms p and q are called *equivalent* if for some numbers $c_1, c_2 > 0$ and all x the inequality $c_1 p(x) \leqslant q(x) \leqslant c_2 p(x)$ holds.

A collection of vectors in a linear space is called *linearly independent* if the equality $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$, where v_i are distinct vectors in this collection and λ_i are scalars, implies that $\lambda_i = 0$ for all $i = 1, \ldots, n$; otherwise this collection is *linearly dependent*.

A linearly independent collection of vectors v_α is called an *algebraic basis* (*Hamel's basis*) in the space X if every vector in X is a finite linear combination of the vectors v_α. In the zero space the zero element is a basis by definition. Below we prove the existence of a Hamel basis in every linear space; moreover, different Hamel bases have the same cardinality. The cardinality of a Hamel basis is called the *dimension* of the space.

Let E and F be two vector spaces over the same field. A mapping $A\colon\ E \to F$ is called *linear* (or a *linear operator*) if

$$A(\lambda u + \mu v) = \lambda A(u) + \mu A(v)$$

for all vectors $u, v \in E$ and all scalars λ, μ. A linear mapping with values in the field of scalars is called a *linear functional*.

The set $\operatorname{Ker} A := A^{-1}(0)$ is called the *kernel* of the linear mapping A and the set $\operatorname{Ran} A := A(E)$ is called the *range* of A.

For every vector space E, the symbol E^* denotes the vector space of all linear functions on E; it is called the *algebraic dual* to E. The algebraic dual space should not be confused with the *topological dual* considered below and consisting of all continuous linear functions. For the general theory and applications, the topological dual spaces are most important, but the algebraic dual is useful for some examples and constructions.

The quotient (or the factorspace) E/E_1 of a vector space E by its its subspace E_1 is defined as follows: the elements of E/E_1 are equivalence classes of the set E, where $x \sim z \iff x - z \in E_1$. Thus, if $Z \in E/E_1$, then there exists (non unique) $z \in E$ such that $Z = z + E_1$. The linear operations in E/E_1 are defined as follows: suppose that $X = x + E_1$, $Z = z + E_1$, $\lambda \in \mathbb{K}$; then $X + Z = (x+z) + E_1$, $\lambda X = \lambda x + E_1$.

The dimension of the space E/E_1 is called the *codimension* of the subspace E_1 in the space E. A *hypersubspace* in a vector space E is its subspace G with $\dim E/G = 1$, i.e., there exists a nonzero vector v such that every vector in E is a linear combination of v and some vector in G. In this case, we say that the codimension of G in E is one.

A subset Γ in a vector space E is called a *hyperplane* if E contains a hypersubspace G and an element a such that $a + G = \Gamma$ (then $a \in \Gamma$). In other words, a subset Γ in vector space E is called a hyperplane precisely when for some (hence for any) element $b \in \Gamma$ the set $\Gamma - b$ is a hypersubspace (a hypersubspace is a hyperplane passing through the origin).

A subset A in a vector space E is called an *affine subspace* or a *linear manifold* if it is nonempty and for all $a, b \in A$ and every $t \in \mathbb{K}$ one has the inclusion $ta + (1-t)b \in A$. The set $\{ta + (1-t)b\colon\ t \in \mathbb{K}\}$ is (for $a \neq b$) the straight line passing through a and b. In other words, the set A is an affine subspace if it has the form $a + L$ for some vector subspace L and some (then for any) element $a \in A$.

The *linear span* of A is the smallest linear subspace containing A.

1.1.2. Definition. *A set V in a real or complex vector space is called convex if $tu + (1-t)v \in V$ for all $u, v \in V$ and $t \in [0,1]$.*

In other words, a set is convex if along with every two its points it contains the interval joining them. The interval $[a,b]$ with the endpoints a and b is defined by the equality

$$[a,b] := \{x\colon\ x = ta + (1-t)b, t \in [0,1]\}.$$

Set also

$$(a,b) := [a,b]\backslash\{a,b\}, \quad [a,b) := [a,b]\backslash\{b\}, \quad (a,b] := [a,b]\backslash\{a\}.$$

The *convex hull* (or *convex envelope*) of a nonempty set A in a real or complex vector space E is the intersection $\operatorname{conv} A$ of all convex sets containing A.

Thus, the convex hull of the set A is the smallest convex sets containing A. It is readily verified that it consists of all possible sums of the form $t_1 a_1 + \cdots + t_n a_n$, where $a_i \in A$, $t_i \geqslant 0$, $t_1 + \cdots + t_n = 1$.

1.1.3. Definition. *A set M is called circled or balanced if $\lambda x \in M$ for all $x \in M$ and $|\lambda| \leqslant 1$.*

A convex circled set is also called absolutely convex.

The *circled* and *convex circled* (or *absolutely convex*) hulls of a set A in a linear space are, respectively, the smallest circled set and the smallest convex circled set $\operatorname{abs\,conv} A$ containing A.

1.1.4. Definition. *If A and B are sets in a linear space E, then we say that A absorbs B (or that the set B is absorbed by the set A) if there exists a number $r > 0$ such that $kB \subset A$ whenever $|k| < r$, $k \in \mathbb{K}$.*

A set in E is called absorbing (*or absorbent*) *if it absorbs every singleton* (*and then every finite set*) *in E.*

A simple example of a set which does not absorb itself is $\mathbb{K} \setminus \{0\}$; every balanced set absorbs itself (take $r = 1$). If a normed field $\mathbb{K}$ is discrete, then the property to absorb, although formally meaningful, has no useful content since then $\{0\}$ absorbs every set with $r = 1$.

For nonempty sets A and B in a vector space and a scalar λ, we set

$$A + B := \{a + b\colon\ a \in A, b \in B\}, \quad \lambda A := \{\lambda a\colon\ a \in A\};$$

$A + B$ is the *algebraic* (*vector*) *sum* of sets. Further,

$$A - B = A + (-B) = \{a - b\colon\ a \in A, b \in B\}.$$

Throughout we also use two set-theoretic concepts: the equivalence relation and the partial ordering relation.

Suppose that a certain set $\mathcal{R}$ if pairs of elements in the set X is given, i.e., a subset $\mathcal{R} \subset X \times X$. We say that $\mathcal{R}$ defines an *equivalence relation* on the set X and write $x \sim y$ for $(x, y) \in \mathcal{R}$ if

(i) $x \sim x$ for all $x \in X$,

(ii) if $x \sim y$, then $y \sim x$,

(iii) if $x \sim y$ and $y \sim z$, then $x \sim z$.

The reader can easily see by simple examples that these three conditions are independent.

The equivalence relation splits X into disjoint *equivalence classes* consisting of pairwise equivalent elements. For example, if $x \sim y$ only when $x = y$, then every class consists precisely of a single element; if, in the opposite direction, all elements are equivalent, then we obtain only one equivalence class. Yet another example: let $x \sim y$ for $x, y \in \mathbb{R}^1$, if $x - y \in \mathbb{Q}$. Then the equivalence classes are countable. If is often useful to be able to choose a representative in every equivalence class. It turns out that for realizing this, at the first glance absolutely innocent desire, a special axiom is required.

The axiom of choice. *If we are given a collection of nonempty pairwise disjoint sets, then there exists a set containing precisely one element from each of these sets.*

The use of this axiom is essential for many issues in functional analysis, and without this axiom at least for countable collections very little remains from the continuous mathematics at all. Nevertheless, it is useful to remember that this is indeed an axiom that does not follow from the so-called naive set theory.

We say that a set X is equipped with a *partial order* or *partial ordering* if a certain collection $\mathcal{P}$ of pairs $(x, y) \in X \times X$ is distinguished, for which we write $x \leqslant y$, such that (i) $x \leqslant x$, (ii) if $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$. If $x \leqslant y$, then we also write $y \geqslant x$. Note that we do not include the equality $x = y$ in the case where $x \leqslant y$ and $y \leqslant x$, unlike some other authors (though, one can pass to this case by identifying such elements). For example, our definition applies to the relation $f \leqslant g$ almost everywhere for measurable functions on an interval.

It is not required that all elements be pairwise comparable. For example, the plane $\mathbb{R}^2$ can be partially ordered in the following way:

$x = (x_1, x_2) \leqslant y = (y_1, y_2)$ if $x_1 \leqslant y_1$ and $x_2 \leqslant y_2$.

If all elements of X are pairwise comparable, then X is called *linearly ordered*.

For example, the real line with its usual ordering is linearly ordered, and the aforementioned coordinate-wise ordering of the plane is not linear. However, the plane can be naturally linearly ordered: take the so-called *lexicographic order*, when $x \leqslant y$ if either $x_1 < y_1$ or $x_1 = y_1$ and $x_2 \leqslant y_2$.

In a partially ordered set some parts can be linearly ordered. Such parts are called *chains*. For example, the real line as a part of the plane with the coordinate-wise order is a chain.

If X is a partially ordered set and $M \subset X$, then an element $\mu \in X$ is called an *upper bound* of the set M if $m \leqslant \mu$ for all $m \in M$. If m is an upper bound of M such that $m \leqslant \widehat{m}$ for every other upper bound $\widehat{m}$ of the set M, then m is called the *least upper bound* M. An element $m \in X$ is called *maximal* if there is no element $m' \in X$ such that $m \leqslant m'$. It is not required that all elements of X be smaller than m. For example, if $x \leqslant y$ only when $x = y$, then every element is maximal. Similarly one defines a *lower bound*, *the greatest lower bound* and a *minimal* element.

A linearly ordered set X is called *completely ordered* if every nonempty part of X has a minimal element.

For example, the set of natural numbers with its natural order is completely ordered, but the sets of rational and real numbers are not.

The axiom of choice is equivalent to the following assertion (if we accept it as an axiom, then the axiom of choice becomes a theorem); for a proof, see Kolmogorov, Fomin [**284**], Kurosh [**306**].

The Zermelo theorem. *Every nonempty set can be completely ordered.*

Let us give yet another corollary of the axiom of choice (equivalent to it).

The Zorn (or Kuratowski–Zorn) lemma. *If every chain in a partially ordered set has a majorant, then this set contains a maximal element.*

We recall that a maximal element need not be unique. Let us give an example of how Zorn's lemma works.

1.1.5. Proposition. *Every real or complex linear space possesses an algebraic basis. Moreover, any two such bases have the same cardinality. In addition, any algebraic basis in a linear subspace can be complemented to an algebraic basis of the whole space.*

PROOF. We assume that our space X contains nonzero vectors. Then X contains systems of algebraically independent vectors. Let Λ denote the collection of all such systems and let us introduce in Λ the following relation: $\lambda_1 \leqslant \lambda_2$ if $\lambda_1 \subset \lambda_2$. Clearly, we obtain a partial order. We have to show that the set Λ has a maximal element, i.e., a system λ of algebraically independent vectors that is not a proper subset of any other system of independent vectors. Such a maximal system will be a basis, since the existence of a vector v not representable as a linear combination of vectors in λ would mean that the system $\lambda \cup v$ is also independent, which contradicts the maximality of λ. The existence of a maximal element follows from Zorn's lemma, but in order to apply it we have to show that every chain Λ_0 in Λ has an upper bound. In other words, having a set Λ_0 of independent collections of vectors such that every two collections are comparable (i.e., one of the two is contained in the other), we have to find an independent system of vectors containing all systems from Λ_0. For such a system we simply take the union Λ_1 of all systems from Λ_0. The fact that the obtained system is independent is clear from the following. If vectors $v_1, \ldots, v_n$ belong to Λ_1, then there exist systems $\lambda_1, \ldots, \lambda_n \in \Lambda_0$ such that $v_i \in \lambda_i$ for $i = 1, \ldots, n$. Since the systems λ_i are pairwise comparable, among them there is the largest one λ_{i_0}. Then all v_i belong to λ_{i_0} and hence are linearly independent.

A minor modification of this reasoning enables us to complement algebraic bases in a subspace to bases of the whole space: it suffices to take for elements Λ independent systems containing a fixed basis in the given subspace. By the way, this reasoning is true for every field.

Finally, the assertion about the equicardinality of algebraic bases in the space X in the case of a finite-dimensional space is known from linear algebra. If X is infinite-dimensional and γ_1 and γ_2 are two its algebraic bases, then the cardinality of γ_2 does not exceed the cardinality of γ_1. Indeed, every element $v \in \gamma_2$ corresponds to a finite set of elements $S \subset \gamma_1$ through which it is linearly expressed. This finite set S is associated to at most finitely many elements in γ_2 (not more than the cardinality of S, since by using k vectors we cannot linearly express more than k linearly independent vectors). Hence the cardinality of γ_2 does not exceed the cardinality of the set of finite subsets of γ_1, which has the same cardinality as γ_1 (see Hrbacek, Jech [**229**, p. 136]). Thus, the cardinality of γ_2 does not exceed the cardinality of γ_1, and the opposite inequality is true. □

With the aid of this result any linear mapping T defined on a linear subspace E_0 of a vector space E and taking values in a vector space F can be extended to a linear mapping of the whole space E to F. It suffices to complement an algebraic basis in E_0 to a basis of the whole space E, define T by zero on the additional elements of the basis and then define by linearity on all vectors.

Let us now proceed to necessary topological concepts. For more details, see Arkhangel'skiĭ, Ponomarev [**22**], Bogachev, Smolyanov [**72**], Edwards [**150**], Engelking [**154**], and Kelley [**268**].

A *topology* on a set X is a family τ of subsets of this set possessing the following properties:

(i) $X, \varnothing \in \tau$;

(ii) if $V_1, V_2 \in \tau$, then $V_1 \cap V_2 \in \tau$;

(iii) the union of every collection of sets from τ belongs to τ.

Hence the minimal topology is $(X, \varnothing)$ and the maximal topology is 2^X, the class of all subsets of X.

A *topological space* is a pair (X, τ), where X is a set, called the set of elements of this topological space, and τ is a topology on X. The elements of τ are called *open subsets* of the topological space X.

A subset of a topological space is called *closed* if its complement is open. The topology in X can be also defined by introducing the class $\mathcal{F}$ of all closed sets, which must satisfy the following conditions:

(i) $X, \varnothing \in \mathcal{F}$;

(ii) if $F_1, F_2 \in \mathcal{F}$, then $F_1 \cup F_2 \in \mathcal{F}$;

(iii) the intersection of every collection of sets from $\mathcal{F}$ belongs to $\mathcal{F}$.

An important subclass of the class of topological spaces is formed by metric spaces. Although a grasp knowledge of them is assumed, we recall that a *metric space* (M, d) is a pair, where M is a set and $d\colon\ M \times M \to [0, +\infty)$ is a function, called a *metric*, that satisfies the following conditions:

(i) $d(a, b) = d(b, a)$, in addition, $d(a, b) = 0$ if and only if $a = b$,

(ii) $d(a, c) \leqslant d(a, b) + d(b, c)$ (*the triangle inequality*).

A linear space with a norm $\|\cdot\|$ (a *normed space*) is a metric space with the metric $d(x, y) = \|x - y\|$.

Let $a \in M$ and $r > 0$. The set

$$\{x \in M\colon\ d(x, a) < r\}$$

is called the *open ball* with the center a and radius r. If we define open sets in M as arbitrary unions of open balls (with arbitrary centers and radii) and the empty set, then we obtain a topological space (a simple verification is left as an exercise). Here any open ball will be an open set (which is easily verified with the aid of the triangle inequality). The *closed ball* with the center a and radius r is defined as the set

$$\{x \in M\colon\ d(x, a) \leqslant r\}$$

The topological space is called *metrizable* if its topology is obtained in the indicated way from some metric on it. Different metrics can generate the same topology. For example, the usual metric on the real line generates the same topology as the bounded metric $d(x, y) = \min(1, |x - y|)$. Below we encounter many examples of nonmetrizable spaces, so we do not give artificial examples of this sort. The *discrete topology* on X is $\tau = 2^X$.

We assume that the concept of a complete metric space is known (anyway, we recall it in § 1.7).

A *pseudometric* on a set M is a function $\varrho\colon\ M \times M \to [0, \infty)$ with the following properties:

(1) $\varrho(x, x) = 0$;

(2) $\varrho(x, y) = \varrho(y, x)$;

(3) $\varrho(x, y) \leqslant \varrho(x, z) + \varrho(z, y)$.

If the triangle inequality (3) is written as

(3′) $\varrho(x, y) \leqslant \varrho(x, z) + \varrho(y, z)$,

then conditions (2) and (3) together will be equivalent to the pair of conditions (2) and (3′), but (2) will follow from (1) and (3′) by replacing z in (3′) with x.

The pseudometric ϱ on a nonempty set M generates a topology on this set in the same way as a metric: a set $V \subset M$ is called open in the topology generated by the pseudometric ϱ if for each $x \in V$ there is $\varepsilon > 0$ such that $\{z\colon\ \varrho(z,x) < \varepsilon\}$ is contained in V. In addition, any pseudometric generates a metric on the set of equivalence classes if we set $x \sim y$ when $d(x,y) = 0$.

Every subset X_0 of a topological space X becomes itself a topological space if we define open sets in X_0 to be the sets of the form $U \cap X_0$, where U is open in X. Certainly, such sets need not be open in X (if X_0 itself was not open in X). This topology on X_0 is called the *induced* topology.

An *open neighborhood of a point* x is any open set containing x. Sometimes it is useful to employ a broader concept of a *neighborhood* of a point (not necessarily open!) as an arbitrary set containing some open neighborhood of this point. For example, it becomes possible to speak of closed neighborhoods in this sense.

A *base of the topology* (*topology base*) is any collection of open sets with the property that all possible unions of the elements of this collection give all nonempty open sets.

A *base of the topology at a point* x or a *fundamental system of neighborhoods of the point* x is any collection of open neighborhoods of the point x with the property that every neighborhood of x contains an element of this collection. Sometimes, similarly to neighborhoods, bases of not necessarily open neighborhoods are used. A *prebase* of neighborhoods of a point in a topological space is a family of neighborhoods of this point finite intersections of elements of which form a base of its neighborhoods.

A point x in a topological space X is called a *limit point* of a set $A \subset X$ (or an *accumulation point* of A) if every neighborhood of x contains a point of A distinct from x. If every neighborhood of x intersects A, then x is called a *cluster point* of A. The closure $\overline{A}$ of a set A (the intersection of all closed sets containing A) is exactly the set of all its cluster points. The points of A that are not limit are called *isolated*.

If $X = \overline{A}$, then A is called *everywhere dense* in X. If X contains an at most countable everywhere dense set, then X is called *separable*.

If we are given a collection of nonempty topological spaces X_t, where $t \in T$, then the product $\prod_{t\in T} X_t$ is equipped with the *Tychonoff* product topology, in which open sets are all possible unions of the sets of the form $\prod_{t\in T} U_t$, where every U_t is open in X_t, but only for finitely many indices t the set U_t differs from X_t. See Exercise 2.10.26 about the competing box topology.

A mapping $f\colon X \to Y$ between topological spaces is called *continuous* if, for every open set V in the space Y, the set $f^{-1}(V)$ is open in X. The continuity at a point $x_0 \in X$ is defined as follows: for every open set V containing $f(x_0)$, there exists an open set U containing x_0 such that $f(U) \subset V$.

The continuity of f is equivalent to the continuity at every point. Indeed, if f is continuous and V is an open set containing $f(x_0)$, then $U = f^{-1}(V)$ is open, $x_0 \in U$ and $f(U) \subset V$. Conversely, let f be continuous at every point x and let $V \subset Y$ be open. For every point $x \in U := f^{-1}(V)$ there exists an open set $U_x \ni x$ such that $f(U_x) \subset V$. Then the set $W := \bigcup_{x \in U} U_x$ is open. Since $f(W) \subset V$ and $U \subset W$, we have $W = U$.

If X and Y are topological spaces, then a mapping $F\colon X \to Y$ is called a *homeomorphism* if it is one-to-one, $F(X) = Y$ and both mappings F and F^{-1} are continuous; in this case X and Y are called *homeomorphic*.

Let us introduce certain separation properties. A topological space (X, τ) is called a *Kolmogorov* space or a T_0-space if, for every two its different points, there exists an open set containing precisely one of them; (X, τ) is called a T_1-space if, for every two different points a, b in X, there are open sets $A, B \in \tau$ such that $a \in A \setminus B$ and $b \in B \setminus A$; (X, τ) is called a *Hausdorff* or *separated* space (or a T_2-space) if, for every two different points $a, b \in X$, there are open sets $A, B \in \tau$ such that $A \cap B = \varnothing$, $a \in A$, $b \in B$; a *regular* space (or a T_3-space) is a T_1-space every point of which possesses a base of closed neighborhoods. In a Hausdorff space every point is closed. A *completely regular* space is a space with the following property: for every closed set $F \subset X$ and every point $x \notin F$, there is a continuous function $g\colon X \to [0, 1]$ such that $g(x) = 0$ and $g = 1$ on F. A *Tychonoff* space (or a $T_{3\frac{1}{2}}$-space) is a Hausdorff completely regular space. If a pseudometric is not a metric, then the topology generated by it is not Hausdorff. We shall see below that topological vector spaces are completely regular.

A *cover* of a set is any collection of sets whose union contains this set.

1.1.6. Definition. *A subset of a topological space X is called compact if every cover of this set by open sets contains a finite subcover. If this is true for the whole space X, then X is called a compact or a compact space.*

A topological space is called *locally compact* if every point in it possesses a fundamental system of neighborhoods consisting of compact sets.

A subset of a topological space is called *relatively compact* if its closure is compact. This is equivalent to the following: this subset is contained in a compact set.

A topological space is called *connected* if it cannot be represented as a union of two disjoint nonempty open sets, or, which is the same, it cannot be represented as a union of two disjoint nonempty closed sets.

Useful tools for working with topological spaces are the concepts of a net and a filter. For the reader's convenience, we briefly explain these concepts, which are sometimes used below.

1.1.7. Definition. *A partially ordered set T is called directed if, for every two elements $t, s \in T$, there exists an element $\tau \in T$ such that $t \leqslant \tau$ and $s \leqslant \tau$.*

A net in a given set X is a family $\{x_t\}_{t \in T}$ of its elements indexed by a directed set T.

For example, the plane with the lexicographic order and the set of all neighborhoods of a given point in a topological space partially ordered by the inverse inclusion are directed sets. The set of all nonempty open subsets of the real line, partially ordered by the inverse inclusion, is not directed (two disjoint open sets have no common upper bound).

In a somewhat peculiar way (as compared to sequences) one introduces the notion of a subnet $\{y_s\}_{s\in S}$ of a net $\{x_t\}_{t\in T}$: it is required that there is a mapping $F\colon S\to T$ such that $y_s = x_{F(s)}$ for all $s\in S$ and, for each $t\in T$, there is $s_t\in S$ such that $F(s) > t$ whenever $s > s_t$ (certainly, a subsequence in a sequence satisfies this condition).

For example, in the countable net $\mathbb{Z}$ of integer numbers (indexed by the same set $\mathbb{Z}$ with its usual order) the subset of negative numbers is not a subnet, but the subset of natural numbers is a subnet. A subnet of a countable sequence can be uncountable.

Not every countable net is isomorphic to a sequence indexed by natural numbers with its usual order (say, it can occur in a countable net that for every index there are infinitely many smaller indices).

1.1.8. Definition. *A net $\{x_t\}_{t\in T}$ in a topological space X converges to a point x if, for every neighborhood U of the point x, there exists an index $\tau\in T$ such that $x_t\in U$ whenever $t\geqslant\tau$.*

Note that the set of indices $t\in T$ such that $x_t\notin U$ can be infinite. Hence even for countable sets T convergence of nets does not reduce to convergence of sequences. For example, if on $\mathbb{N}$ we introduce the order such that all odd numbers are smaller than 2 and on even and odd numbers separately the usual order is kept, then we obtain a directed countable set; the net $\{x_n\}$ in which $x_n = 0$ for even n and $x_n = 1$ for odd n converges to zero. One can give an example of a convergent countable net in a topological space that does not contain convergent subsequences (Exercise 1.12.27).

If b is an accumulation point of the set A, then there exists a net $\{a_t\}$ of elements of A converging to b. Indeed, let T be the collection of all neighborhoods of b, partially ordered by the inverse inclusion. Every such neighborhood of t by condition contains a point $a_t\in A$. The obtained net converges to b, since for every fixed neighborhood of $\tau\in T$ we have $a_t\in t\subset\tau$ if $t\geqslant\tau$.

1.1.9. Definition. *A filter on a set X is any nonempty set Φ of nonempty subsets of X satisfying the following conditions:*

(i) *if $A, B\in\Phi$, then $A\cap B\in\Phi$,*

(ii) *if $B\in\Phi$ and $B\subset C$, then $C\in\Phi$.*

A filter basis (base) on a set X is any nonempty set $\mathcal{B}$ of nonempty subsets of X such that the following condition is fulfilled: for every pair of sets $B_1, B_2\in\mathcal{B}$, there exists $B_3\in\mathcal{B}$ with $B_3\subset B_1\cap B_2$.

A filter Φ is majorized by a filter Ψ if $\Phi\subset\Psi$.

A filter basis is an integral term (not assuming any filter).

Among all filters containing a given filter basis $\mathcal{B}$ there exists the (unique) minimal filter $\Phi_{\mathcal{B}}$, which is called the filter generated by the filter basis $\mathcal{B}$. Then $\mathcal{B}$ is called a basis of the filter $\Phi_{\mathcal{B}}$. The minimal filter is the intersection of all filters containing $\mathcal{B}$ (such filters exist, for example, the class of all sets containing at least one set from $\mathcal{B}$). If τ is a topology on the set X and $x\in X$, then the set of all (not necessarily open) neighborhoods of the point x is a filter on X, called the filter of neighborhoods of this point with respect to τ and denoted by the symbol Φ_τ^x. Thus, a fundamental system of neighborhoods of zero is a basis of the filter of all neighborhoods of zero.

1.1.10. Definition. *A filter in X is called converging to a point x in the topology τ if it majorizes the filter of neighborhoods of this point.*

Maximal elements of the system of all filters on the set X, partially ordered by the relation of majorizing by inclusion, are called *ultrafilters* on X. One can deduce from the axiom of choice that every filter on X is majorized by some ultrafilter on X. A filter Ψ on X is an ultrafilter precisely when the conditions $A \cup B = X$ and $A \notin \Psi$ imply that $B \in \Psi$. As a simple example of application of filters we mention the following assertions, the proof of which we leave as an exercise.

1.1.11. Proposition. *A mapping f of topological spaces is continuous at a point x precisely when, for every filter Ψ converging to x, the filter generated by the filter basis $f(\Psi)$ converges to $f(x)$.*

Note that the image of a filter need not be a filter, but is always a filter basis. The proof is left as an exercise.

1.1.12. Proposition. *A subset of a topological space is compact if and only if every ultrafilter containing it converges to some element of this subset.*

1.2. Basic definitions

Here we give basic definitions connected with topological vector spaces and prove some simplest facts, but examples will be considered in the next section. Although in our discussion the field $\mathbb{K}$ is usually $\mathbb{R}$ or $\mathbb{C}$ (and occasionally nondiscrete normable), we give a general definition.

1.2.1. Definition. *A topological vector space over a topological field $\mathbb{K}$ is a vector space E over $\mathbb{K}$ equipped with a topology with respect to which the following two mappings are continuous, where $E\times E$ and $\mathbb{K}\times E$ are equipped with the products of the corresponding topologies*: 1) $(x_1, x_2) \mapsto x_1 + x_2,\ E\times E \to E$ (*addition of vectors*), 2) $(k, x) \mapsto kx,\ \mathbb{K}\times E \to E$ (*multiplication of vectors by scalars*).

Such a topology on E is called *compatible with the vector space structure* (or we say that it *agrees with the vector space structure*). A topological vector space E with the topology τ is denoted by the symbol (E, τ). We observe that in the definition of a topological field one requires the same conditions with $\mathbb{K}$ in place of E and the continuity of $k \mapsto k^{-1}$ outside of zero.

Two topological vector spaces over the same field are called *isomorphic* if there exists a continuous linear one-to-one mapping of one of the two spaces onto the other such that the inverse mapping is also continuous (i.e., a linear homeomorphism). The *dimension of a topological vector space* (E, τ) is the dimension of the vector space E.

The continuity of the mapping 1) implies that the topology of any topological vector space (E, τ) is invariant with respect to translations (i.e., for every $a \in E$ the mapping $x \mapsto x + a$ is a homeomorphism of E); hence the topology of a topological vector space can be reconstructed if we know a fundamental system of neighborhoods of zero.

If $\mathcal{U}$ is a base of neighborhoods of zero and $a \in E$, then the collection of sets of the form $a + V$, where $V \in \mathcal{U}$, is a base of neighborhoods of the point a. Thus, for defining a topology of a topological vector space it suffices to define a base of neighborhoods of zero; this is usually done in most of applications of the theory of topological vector spaces. However, not every system of subsets of a vector space can serve as a base of neighborhoods of zero of a topology compatible with the vector space structure; conditions sufficient for this are indicated in Proposition 1.2.7.

Before we proceed to that proposition, it is useful to prove the following result, according to which among fundamental systems of neighborhoods of zero in a topological vector space there are systems with particularly nice properties.

1.2.2. Proposition. (a) *Every base of neighborhoods of zero $\mathcal{U}$ in a topological vector space has the following properties*:

(1) *for every $V \in \mathcal{U}$ there exists a set $W \in \mathcal{U}$ such that $W + W \subset V$*;

(2) *every $V \in \mathcal{U}$ is an absorbent set.*

(b) *In every topological vector space there exists a base of neighborhoods of zero $\mathcal{U}_0$ having also the following properties*:

(3) *every $V \in \mathcal{U}_0$ is a circled closed set*;

(4) *if $V \in \mathcal{U}_0$, then $kV \in \mathcal{U}_0$ for every $k \in \mathbb{K}$, $k \neq 0$.*

PROOF. Let $\mathcal{U}$ be a base of neighborhoods of zero in a topological vector space E. Since the mapping $(x_1, x_2) \mapsto x_1 + x_2$, $E \times E \to E$ is continuous at the point $(0, 0)$ by Axiom 1, we obtain that $\mathcal{U}$ has property (1). Further, by Axiom 2 for every $a \in E$ the mapping $k \mapsto ka$, $\mathbb{K} \to E$ is continuous at the point $0 \in \mathbb{K}$; so if V is a neighborhood of zero in E and $x \in E$, then there exists $\varepsilon > 0$ such that $kx \in V$ whenever $|k| < \varepsilon$, so that an arbitrary neighborhood of zero V in E is an absorbent set. This means that $\mathcal{U}$ possesses property (2). Thus, part (a) of the proposition is proven.

For the proof of part (b) it suffices to show that the set $\mathcal{U}_0$ of all closed circled neighborhoods of zero in E is a base of neighborhoods of zero in E, since properties (3) and (4) are easily verified. Indeed, it is clear from the definition of the set $\mathcal{U}_0$ that it possesses property (3). By Axiom 2, for every fixed nonzero $k \in \mathbb{K}$, the mapping $x \mapsto kx$ is a linear homeomorphism of E. It follows that if V is a neighborhood of zero in E, then kV, where $k \in \mathbb{K}$, $k \neq 0$, is a neighborhood of zero, moreover, if the set V is closed and balanced, then kV has both properties, so that $\mathcal{U}_0$ possesses property (4). In order to verify that $\mathcal{U}_0$ is a base of neighborhoods of zero in E, we show that every neighborhood of zero in E contains a closed circled neighborhood of zero. Let W be an arbitrary neighborhood of zero in E. By the continuity at zero of the mapping $(x_1, x_2) \mapsto x_1 - x_2$, $E \times E \to E$, which follows from Axioms 1 and 2, there is a neighborhood of zero W_1 such that

$W_1 - W_1 \subset W$. We show that $\overline{W}_1 \subset W$. To this end, we verify that if $x \notin W$, then $x \notin \overline{W}_1$. The set $x+W_1$ is a neighborhood of the point x not intersecting W_1 (if $z \in W_1 \cap (x+W_1)$, then $z = x+y$, $y \in W_1$ and $x = z-y \in W_1 - W_1 \subset W$, but $x \notin W$). The existence of such neighborhood means that $x \notin \overline{W}_1$. Further, by the continuity of the mapping $(k,x) \mapsto kx$, $\mathbb{K}\times E \to E$ at the point $(0,0)$ there exist $\varepsilon > 0$ and a neighborhood of zero W_2 in E such that if $|k| < \varepsilon$ and $x \in W_2$, then $kx \in W_1$. Hence the set $W_3 = \bigcup_{|k|<\varepsilon} kW_2$ is a circled neighborhood of zero in E contained in W_1 (the fact that W_3 is a neighborhood of zero follows from the condition that $\mathbb{K}$ is not discrete, so there exists $k \neq 0$ with $|k| < \varepsilon$). The closure of a circled set is a circled set. Indeed, if $|k| \leqslant 1$ and G is a circled set, then $kG \subset G$, hence $k\overline{G} \subset \overline{kG} \subset \overline{G}$ (if $k \neq 0$, then $k\overline{G} = \overline{kG}$). Hence $\overline{W}_3$ is a closed circled neighborhood of zero and $\overline{W}_3 \subset \overline{W}_1 \subset W$. □

1.2.3. Remark. (i) In the proof we have actually shown that the family of closures of all possible sets in a base of neighborhoods of zero in a topological vector space is again a base of neighborhoods of zero (in fact, this is true for any topological group).

(ii) It has been also shown that every neighborhood of zero in a topological vector space is an absorbent set; this fact will be used all the time below.

(iii) Proposition 1.2.2 remains true if in its formulation in part (b) the word "closed" is replaced by the word "open": in other words, in every topological vector space there exists a base of neighborhoods of zero $\mathcal{U}$ having properties (1) and (4) and the following property (3$'$): every $V \in \mathcal{U}$ is a circled absorbent open set. The proof essentially coincides with the proof of Proposition 1.2.2, but is even simpler. As above, we verify that E has a base consisting of open circled sets. The existence of such a base follows from the fact that for every neighborhood of zero $W \subset E$ there exist $\varepsilon > 0$ and an open neighborhood of zero W_1 such that if $k \in \mathbb{K}$, $|k| < \varepsilon$ and $x \in W_1$, then $kx \in W$. Hence the set $W_2 = \bigcup_{|k|<\varepsilon} kW_1$ is an open circled neighborhood of zero contained in W.

Let us give some corollaries of the proven proposition.

1.2.4. Corollary. *Each point in a topological vector space possesses a base of neighborhoods consisting of closed sets* (*i.e., every topological vector space is a regular topological space, as well as any topological group*).

PROOF. Indeed, if $\mathcal{U}$ is a base of closed neighborhoods of zero, then $a+\mathcal{U}$ is a base of closed neighborhoods of the point a for every a. □

1.2.5. Corollary. *A topological vector space is a T_3-space* (*hence Hausdorff*) *if and only if it is a T_0-space.*

PROOF. By the previous corollary and the definition of a T_3-space it suffices to show that in a given space E the axiom T_1 is fulfilled. Suppose that $a_1, a_2 \in E$. Since the axiom T_0 holds by assumption, for one of these points — let it be a_1 — there exists a neighborhood of zero W such that $a_1 + W \not\ni a_2$. Then $a_2 - W \not\ni a_1$, since otherwise for some $z \in W$ we have $a_2 = z + a_1$, i.e., $a_2 \in a_1 + W$. Thus, $a_2 - W$ is a neighborhood of the point a_2 not containing a_1. □

Actually, more is true: a Hausdorff topological vector space is completely regular, which will be established in § 1.6.

1.2.6. Corollary. *A topological vector space is Hausdorff if and only if the intersection of all its neighborhoods of zero is the zero element of this space.*

PROOF. The sufficiency follows from the previous corollary; the necessity is obvious. □

The next result can be used for constructing vector topologies.

1.2.7. Proposition. *Let $\mathcal{B}$ be a filter basis in a vector space E consisting of circled sets and having the following properties* (*analogous to properties* 1, 2, 4 *from Proposition* 1.2.2):

$(1)'$ *for every $V \in \mathcal{B}$ there is $W \in \mathcal{B}$ such that $W + W \subset V$*;

$(2)'$ *every $V \in \mathcal{B}$ is an absorbent set*;

$(4)'$ *if $V \in \mathcal{B}$, then $kV \in \mathcal{B}$ for every $k \in \mathbb{K}$, $k \neq 0$.*

Then, there exists a unique topology in E compatible with the vector space structure such that $\mathcal{B}$ is a base of neighborhoods of zero (*not necessarily closed or open*).

PROOF. Let τ be the family of subsets of E defined as follows: $V \in \tau$ precisely when for every $a \in V$ there exists a set W in $\mathcal{B}$ such that $a + W \subset V$. Then τ is a topology in E. Indeed, the inclusions $\varnothing \in \tau$ and $E \in \tau$ and the stability of τ with respect to formation of arbitrary unions follow directly from the definition of τ. We show that τ is stable with respect to finite intersections. Let $V_1, V_2 \in \tau$; we have to show that $V_1 \cap V_2 \in \tau$. Let $a \in V_1 \cap V_2$. Hence there exist sets $W_1, W_2 \in \mathcal{B}$ such that $a + W_i \subset V_i$, $i = 1, 2$. Then $a + (W_1 \cap W_2) \subset V_1 \cap V_2$. Therefore, if $W_3 \subset W_1 \cap W_2$, $W_3 \in \mathcal{B}$ (such W_3 exists, since $\mathcal{B}$ is a filter basis), then $a + W_3 \subset V_1 \cap V_2$.

We show that the topology τ agrees with the vector space structure in E. We first show that $\mathcal{B}$ is a base of neighborhoods of zero in the topology τ. By the definition of the topology τ, if V is an open neighborhood of zero in τ, there is a set $W \in \mathcal{B}$ such that $W \subset V$. Hence it suffices to show that every set that is an element of $\mathcal{B}$ turns out to be a neighborhood of zero in the topology τ. Thus, let $W \in \mathcal{B}$. Let W^0 denote the set defined as follows: $x \in W^0$ if and only if there exists a set $W_1 \in \mathcal{B}$ such that $x + W_1 \subset W$. Since the zero element of the space E is contained in every set from the system $\mathcal{B}$ (these sets are circled), we have $0 \in W^0 \subset W$.

We now show that W^0 is open in the topology τ; this will mean that W is a neighborhood of zero in this topology. It suffices to be able, for every given $a \in W^0$, to find a set $W_2 \in \mathcal{B}$ such that $a + W_2 \subset W^0$. Let $a \in W^0$. Then by the definition of W^0 there exists a set $W_1 \in \mathcal{B}$ such that $a + W_1 \in W$; by property (1) there exists $W_2 \in \mathcal{B}$ such that $W_2 + W_2 \subset W_1$, i.e., $(a + W_2) + W_2 \subset W$. This means that $a + W_2 \subset W^0$.

Since by the very definition the topology τ is translation invariant, for every vector $a \in E$ the family of the sets of the form $a + V$, where $V \in \mathcal{B}$, is a base of neighborhoods of the point a. Hence for the proof of continuity in τ of the operation of addition (i.e., verification of Axiom 1 in Definition 1.2.1) it suffices to show that if $a = x_1 + x_2$ and $W \in \mathcal{B}$, then there exists a set $W_1 \in \mathcal{B}$ such that

$$(x_1 + W_1) + (x_2 + W_1) \subset a + W.$$

By Axiom $1'$ there exists W_1 such that $W_1 + W_1 \subset W$; this W_1 satisfies the necessary relation.

Let us proceed to the proof of continuity of the operation of multiplication (i.e., verification of Axiom 2 from Definition 1.2.1). Let $a \in E$, $k \in \mathbb{K}$ and $W \in \mathcal{B}$ be given. We have to prove the existence of $W_1 \in \mathcal{B}$ and $\varepsilon > 0$ such that if $a_1 \in a + W_1$ and $|k_1 - k| < \varepsilon$, then $k_1 a_1 \in ka + W$. Since the addition operation is continuous, as we have just proved, there exists a set $W_2 \in \mathcal{B}$ such that $W_2 + W_2 + W_2 \subset W$. Since

$$k_1 a_1 - ka = (k_1 - k)a + (k_1 - k)(a_1 - a) + k(a_1 - a),$$

the required properties hold for the set W_1 and the number $\varepsilon > 0$ such that the inclusion $a_1 \in a + W_1$ and the inequality $|k_1 - k| < \varepsilon$ imply that

$$(k_1 - k)a \in W_2,\ (k_1 - k)(a_1 - a) \in W_2,\ k(a_1 - a) \in W_2.$$

Since the set $W_2 \in \mathcal{B}$ is circled, the relations $|k_1 - k| < 1$, $a_1 - a \in W_2$ yield that $(k_1 - k)(a_1 - a) \in W_2$; since the set W_2 is absorbing, there exists $\varepsilon_1 \in (0, 1)$ such that $(k_1 - k)a \in W_2$ if $|k_1 - k| < \varepsilon_1$. Finally, if $k = 0$, then we can take $W_1 = W_2$; if $k \neq 0$, then we find a neighborhood $W_1 \in \mathcal{B}$ such that $W_1 \subset W_2 \cap k^{-1}W_2$. Thus, in both cases $W_1 \in \mathcal{B}$, and at the same time the inclusion $a_1 - a \in W_1$ yields that $k(a_1 - a) \in W_2$. Therefore, we can set $\varepsilon = \varepsilon_1$.

Let us verify the uniqueness of the indicated topology. Let t be yet another topology in E compatible with the vector space structure for which $\mathcal{B}$ is a base of neighborhoods of zero. Then all sets of the form $x + W$, where $x \in E$ and $W \in \mathcal{B}$, give a base of both topologies, whence $t = \tau$. □

1.2.8. Remark. The last but one paragraph of this proof yields that condition $(4)'$ of the proven proposition can be replaced with the following weaker condition: for every $s \in \mathbb{K} \setminus \{0\}$ and every $V \in \mathcal{B}$ there exists a set $V_1 \in \mathcal{B}$ such that $V_1 \subset sV$. However, in the case where $\mathbb{Q} \subset \mathbb{K} \subset \mathbb{C}$, this last condition is a corollary of Axiom 1 along with the fact that all sets in $\mathcal{B}$ are circled. Indeed, Axiom 1 implies that for any natural number n and a set $W \in \mathcal{B}$, there exists $V \in \mathcal{B}$ such that

$$2^n V \subset \underbrace{V + V + \cdots + V}_{2^n \text{times}} \subset W,$$

i.e., that $V \subset 2^{-n}W$. Hence if we are given $\varepsilon > 0$ and $W \in \mathcal{B}$ and a number $n \in \mathbb{N}$ is such that $2^{-n} < \varepsilon$, there exists $V \in \mathcal{B}$ such that $V \subset 2^{-n}W \subset \varepsilon W$ (the last inclusion follows from the fact that the set W is circled). Thus, if $\mathbb{K} \subset \mathbb{C}$, then for the validity of the conclusion of the proposition above it suffices to require that $\mathcal{B}$ be a filter basis in E possessing properties (1) and (2) and consisting of circled sets.

1.2.9. Corollary. *Let E be a vector space over a field $\mathbb{K}$ and let τ be a translation invariant topology in E possessing a basis of neighborhoods of zero $\mathcal{B}$ consisting of circled sets and having properties* $(1)'$, $(2)'$, $(4)'$ *from Proposition* 1.2.7 (*and in case where $\mathbb{K} \subset \mathbb{C}$, just properties* (1) *and* (2)). *Then τ is compatible with the vector structure in the space E.*

PROOF. By Proposition 1.2.7 and the previous remark in this case in E there exists a topology τ_1 compatible with the vector structure such that $\mathcal{B}$ is a base of neighborhoods of zero. Since τ_1 is translation invariant, we have $\tau = \tau_1$. □

Among topological vector spaces over the fields of real and complex numbers, the most important for applications class is formed by locally convex spaces, the definition of which will be now given.

Note that the closure $\overline{A}$ of a convex subset A in a topological vector space is convex, since by the continuity of the vector operations we have

$$t\overline{A} + (1-t)\overline{A} \subset \overline{tA} + \overline{(1-t)A} \subset \overline{tA + (1-t)A} \subset \overline{A}.$$

Further, the convex hull $\operatorname{conv} W$ of an open set W is again open: this follows from the fact that $\operatorname{conv} W$ is the union of all possible sets of the form $\sum_{k=1}^n \alpha_k W$, where $n \in \mathbb{N}$, $\alpha_k \geqslant 0$, $\sum_{k=1}^n \alpha_k = 1$, each of which is open by the continuity of the operations of addition and multiplication by scalars.

In addition, the interior $\breve{A}$ of any convex subset A in a topological vector space is convex. Indeed, if $a, b \in \breve{A}$, then $\breve{A}$ is a neighborhood of the points a and b, and the set $t\breve{A} + (1-t)\breve{A}$ is an open neighborhood of the point $ta + (1-t)b$ contained in A for every $t \in [0,1]$ (see also Proposition 1.4.2).

1.2.10. Definition. *A locally convex topological vector space is a topological vector space over $\mathbb{R}$ or $\mathbb{C}$ possessing a base of convex neighborhoods of zero.*

In place of the term a "locally convex topological vector space" it is customary to use the term a *locally convex space* or the abbreviation *LCS*. A topology τ in a vector space E (over $\mathbb{R}$ or $\mathbb{C}$) is called *locally convex* if the space (E, τ) is locally convex. In the definition of a locally convex space the Hausdorff separation property is often included, but we do not do this, although in most of the results in this book we shall consider separated spaces.

1.2.11. Proposition. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.*

(i) *Every locally convex space E over $\mathbb{K}$ has a base of neighborhoods of zero consisting of closed circled convex absorbent sets and invariant with respect to multiplication by nonzero numbers from $\mathbb{K}$.*

(ii) *If τ is a topology in a vector space E over $\mathbb{K}$ invariant with respect to translations and possessing a base of neighborhoods of zero that consists of circled convex absorbent sets and contains along with each set V the set $2^{-1}V$, then E is a locally convex space.*

(iii) *If $\mathcal{B}$ is a filter basis in a vector space over $\mathbb{K}$ consisting of circled convex absorbent sets and containing along with each set V the set $2^{-1}V$, then in E there is a unique topology making E a locally convex space such that $\mathcal{B}$ is a base of neighborhoods of zero.*

PROOF. If $\mathcal{B}$ is some base of neighborhoods of zero in E consisting of convex sets, then every set of the form $W_V = V \cap (-V)$, where $V \in \mathcal{B}$, is a convex circled neighborhood of zero in E in case of $\mathbb{K} = \mathbb{R}$; in case of $\mathbb{K} = \mathbb{C}$ for a convex circled neighborhood $W_V \subset V$ (for $V \in \mathcal{B}$) we take $W_V = \bigcap_{|z|=1} zV$. This is indeed a neighborhood (not necessarily open), since there are a neighborhood U and $\varepsilon > 0$ such that $kU \subset V$ if $|k| \leqslant \varepsilon$, whence $\varepsilon U \subset W_V$. In both cases the collection of all sets W_V is a base of neighborhoods of zero in E; by Remark 1.2.3 the same is true for the collection $\mathcal{B}$ of their closures, which are again convex and circled. Hence the family of all sets of the form kV, where $V \in \mathcal{B}$, $k \in \mathbb{K}$, $k \neq 0$, is a base of neighborhoods of zero in E, the existence of which is asserted in (i) (as already noted, every neighborhood of zero is an absorbent set).

The remaining assertions (ii) and (iii) follow from Proposition 1.2.7 and Corollary 1.2.9. It suffices to verify that the sets $\mathcal{B}$ mentioned in these assertions have property (1) from Proposition 1.2.7. Let $V \in \mathcal{B}$. Then $2^{-1}V \in \mathcal{B}$. By the convexity of V we have $2^{-1}V + 2^{-1}V = V$. □

1.2.12. Remark. Similarly one can prove that every locally convex space has a base of neighborhoods of zero consisting of open circled convex absorbent sets and invariant with respect to multiplication by nonzero numbers in $\mathbb{K}$. Indeed, let $\mathrm{int} A$ be the interior of A.

If V is a convex neighborhood of zero and $W \subset V$ is an open neighborhood of zero, then its convex hull $\operatorname{conv} W$ is open and is contained in V by the convexity of V. Since $W \subset \operatorname{conv} W$, the set $\operatorname{conv} W$ is an open convex neighborhood of zero contained in V and the set $W_0 = \operatorname{conv} W \cap (-\operatorname{conv} W)$ in the real case and the set $W_0 = \mathrm{int} \bigcap_{|z|=1} \operatorname{conv}(zW)$ in the complex case is an open convex circled neighborhood of zero (observe that there are $\varepsilon > 0$ and an open neighborhood of zero W_1 such that if $k \in \mathbb{C}$ and $|k| \leqslant \varepsilon$, then $kW_1 \subset W$; so $\varepsilon W_1 \subset W_0$), and we also have $W_0 \subset V$. Hence the family $\mathcal{U}_0$ of all such neighborhoods of zero is a base of neighborhoods of zero. The same is true for the family of all sets of the form kV, where $k \in \mathbb{K}$, $k \neq 0$, $V \in \mathcal{U}_0$.

1.3. Examples

Here we present a rich collection of model examples.

1.3.1. Example. Every algebraic field $\mathbb{K}$ is a one-dimensional vector space over $\mathbb{K}$ with respect to the operations of addition and multiplication in $\mathbb{K}$; this one-dimensional vector space over $\mathbb{K}$ is denoted by $\mathbb{K}^1$. If $\mathbb{K}$ is a topological field with respect to a topology τ, then $\mathbb{K}^1$ is a one-dimensional topological vector space over $\mathbb{K}$ with respect to the same topology; it is denoted again by $\mathbb{K}^1$ or $\mathbb{K}$.

1.3.2. Example. Let $\mathbb{K}$ be an arbitrary topological field, let T be a nonempty set, and let $\mathbb{K}^T$ be the vector space over $\mathbb{K}$ that is the product of T copies of $\mathbb{K}$ equipped with the product topology; so $\mathbb{K}^T$ is the set of all functions $x\colon T \to \mathbb{K}$ with the topology of pointwise convergence whose base consists of the sets

$$U_{x_0,t_1,\ldots,t_n,V} = \{x\colon\ x(t_i) - x_0(t_i) \in V, i = 1,\ldots,n\},$$

where $x_0 \in \mathbb{K}^T$, $t_i \in T$ and V is a neighborhood of zero in $\mathbb{K}$. Then $\mathbb{K}^T$ is a topological vector space. More generally, the product of any family of topological vector spaces over the field $\mathbb{K}$ is again a topological vector space over $\mathbb{K}$ with respect to the product topology of the factors.

For $T = \mathbb{N}$ and $\mathbb{K} = \mathbb{R}$ we obtain $\mathbb{R}^\infty$, the *space of all real sequences* with the topology of coordinate-wise convergence; it can be defined by the metric

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \min(|x_n - y_n|, 1), \quad \text{where } x = (x_n),\ y = (y_n).$$

1.3.3. Example. If the topology of a topological field $\mathbb{K}$ is discrete, then every vector space E over $\mathbb{K}$ equipped with a topology compatible with the structure of its additive group (this means the continuity of the mapping $(x_1, x_2) \mapsto x_1 - x_2$, $E \times E \to E$) and invariant with respect to the operation of multiplication by nonzero elements of $\mathbb{K}$ is a topological vector space over $\mathbb{K}$ (in particular, this condition is fulfilled for the discrete topology on E). Topological vector spaces over fields with the discrete topology are called *topological vector groups*.

Throughout we assume that the field $\mathbb{K}$ is not discrete. In most of the examples $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

1.3.4. Example. Let E be a vector space over a nondiscrete normed field $\mathbb{K}$ and let $\mathcal{P}$ be a family of seminorms on E. The *open ball of radius $r > 0$ with the center at zero with respect to a seminorm p on E* is the set $\{x \in E\colon\ p(x) < r\}$. The collection of the intersections of all possible finite families of open balls of positive radii with respect to seminorms from the family $\mathcal{P}$ is a base of neighborhoods of zero for some topology $\tau_{\mathcal{P}}$ in E that agrees with the vector structure; it is customary to say that this *topology is given* (*or defined*) *by the family of seminorms $\mathcal{P}$*. Thus, the collection of open balls of all possible positive radii (for all given seminorms) is a pre-base of neighborhoods of zero in the topology $\tau_{\mathcal{P}}$. Note that all seminorms in $\mathcal{P}$ are continuous in this topology. If $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, the topology $\tau_{\mathcal{P}}$ is locally convex, since the sets $\{x\colon\ p(x) < r\}$ are convex; it is shown in § 1.4 that the topology of every locally convex space (over $\mathbb{R}$ or $\mathbb{C}$) is defined by some family of seminorms. A topological vector space is called *normable* if its topology can be given by a single norm. A *Banach* space is a normed space that is complete with respect to the metric generated by the norm (the concept of completeness is recalled in § 1.7). A *Hilbert* space is a complete Euclidean space. A criterion of normability of a topological vector space over $\mathbb{R}$ or $\mathbb{C}$ (discovered by A.N. Kolmogorov) will be given in § 1.5.

1.3.5. Example. Let $n \in \mathbb{N}$. The topology in $\mathbb{K}^n$ is generated by the norm given by the equality $\|(x_1, \ldots, x_n)\| = \max_{i=1,\ldots,n} |x_i|$, where the symbol $|\cdot|$ denotes the norm in $\mathbb{K}$. We could also take here $\sum_{i=1}^n |x_i|$ or $\left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$, but below we need the fact that the set of values of the norm $\max_{i=1,\ldots,n} |x_i|$ coincides with the set of values of the norm $k \mapsto |k|$. It will be shown in § 1.5 that if the field $\mathbb{K}$ is complete, then every n-dimensional separated topological vector space over $\mathbb{K}$ is isomorphic to the space $\mathbb{K}^n$ (for $n = 1$ this is also true without the assumption of completeness of $\mathbb{K}$), and if the field $\mathbb{K}$ is locally compact, then a separated topological vector space over $\mathbb{K}$ is finite-dimensional precisely when it possesses a precompact neighborhood of zero. The first part of the last sentence means that in a n-dimensional real or complex topological vector space there exists precisely one separated topology compatible with the vector structure; this topology will be further called *standard*.

1.3.6. Example. Let $\mathbb{Q}$ be the field of rational numbers (with its usual topology defined by the norm equal to the absolute value of a number) and let α be an irrational real number. The set $\{\alpha q_1 + q_2\colon\ q_1, q_2 \in \mathbb{Q}\}$ in $\mathbb{R}$ with the topology induced by the usual topology of the real line is a two-dimensional topological vector space over $\mathbb{Q}$ that is not isomorphic to the topological vector space $\mathbb{Q}^2$ (Exercise 1.12.26).

1.3.7. Definition. *A pseudonorm on a vector space E is a nonnegative function $p\colon\ E \to [0,\infty)$ such that*

(1) $p(0) = 0$, (2) $p(-x) = p(x)$, (3) $p(x_1 + x_2) \leqslant p(x_1) + p(x_2)$.

The pseudonorm p is called nondegenerate if $p(x) = 0$ only for $x = 0$.

Note that our definition differs from the one given in the book Schaefer [**436**], where the nondegeneracy is required along with the estimate $p(\lambda x) \leqslant p(x)$ whenever $|\lambda| \leqslant 1$. Though, for defining vector topologies this distinction is not important (see the end of the proof of Theorem 1.6.1).

Thus, a *seminorm* is a pseudonorm q having the following property, which is more restrictive than (2) and (1) together:

(2′) $q(\alpha x) = |\alpha| q(x) \quad \forall\, \alpha \in \mathbb{K}$.

Unlike a norm, a seminorm can be zero on nonzero elements. For example, the identically zero function is a seminorm.

If p is a pseudonorm on a vector space E, then the equality

$$\varrho(x_1, x_2) = p(x_1 - x_2)$$

defines a pseudometric invariant with respect to translations; this pseudometric, in turn, generates a topology on E compatible with the structure of an additive group of the space E; the pseudometric ϱ becomes a metric precisely when $p(x) = 0$ only for $x = 0$.

If E is a topological vector space the topology τ of which is metrizable, then on E there is a pseudonorm generating this topology in the described way (this will be proved in § 1.6). A criterion of metrizability of a topological vector space will be also given in § 1.6.

Note also that if p is an arbitrary pseudonorm on a vector space E, then the topology generated by p need not be compatible with the vector structure (give an example); in order it be compatible with the vector structure, it suffices (and is obviously necessary) that the pseudonorm p have the following additional properties:

(4) if $x_n \in E$, $t \in \mathbb{K}$, $p(x_n) \to 0$, then $p(tx_n) \to 0$;

(5) if $x \in E$, $t_n \in \mathbb{K}$, $t_n \to 0$, then $p(t_n x) \to 0$;

(6) if $x_n \in E$, $t_n \in \mathbb{K}$, $t_n \to 0$, $p(x_n) \to 0$, then $p(t_n x_n) \to 0$.

Property (6), as one can show, follows from properties (4) and (5); we leave the proof to the reader; these properties hold if and only if the operation of multiplication by scalars is continuous with respect to the topology generated by the pseudonorm p.

1.3.8. Definition. *A quasi-norm is a pseudonorm possessing properties* (4) *and* (5) (*hence also property* (6)). *Thus, a pseudonorm p defining the topology of a metrizable topological vector space is automatically a quasi-norm* (*having the property $p(x) \neq 0$ for all $x \neq 0$*).

1.3.9. Example. Let E be a vector space and let $\mathcal{P}$ be a family of quasi-norms on E. The *open ball of radius $r > 0$ with the center at zero with respect to the quasi-norm $p \in \mathcal{P}$* is the set $\{x \in E\colon\ p(x) < r\}$; the collection of all open balls of all possible positive radii with respect to the quasi-norms in $\mathcal{P}$ is a prebase of neighborhoods of zero of some topology in E compatible with the vector structure; this topology is called the topology generated by the family $\mathcal{P}$ of quasi-norms. It will be shown in § 1.6 that the topology of every topological vector space can generated by a suitable family of quasi-norms. Note that all quasi-norms of a family defining the topology are continuous in this topology.

1.3.10. Example. Let (E, τ) be a topological vector space, let $E_1 \subset E$ be a vector subspace, and let τ_1 be the topology induced in E_1 by the topology τ. The topology τ_1 agrees with the vector structure. The topological vector space (E_1, τ_1) is called a *topological vector subspace* of the topological vector space E. If $\mathcal{U}$ is a base (or prebase) of neighborhoods of zero in (E, τ), then the family $\{V \cap E_1\colon\ V \in \mathcal{U}\}$ is a base (respectively, a prebase) of neighborhoods of zero in the space (E_1, τ_1). If (E, τ) is Hausdorff (or metrizable, or locally convex), then (E_1, τ_1) has the respective property. If the topology τ is given by some set of seminorms (or pseudonorms), then the topology τ_1 is defined by their restrictions to the subspace E_1.

The following sufficient condition for the closedness of E_1 as a subset in the topological vector space E is useful.

1.3.11. Lemma. *Let a vector subspace F in a Hausdorff topological vector space E be complete with respect to some metric defining the topology of this subspace. Then F is closed in E.*

PROOF. A very short proof of a generalization of this lemma can be given by means of the concept of a Cauchy filter (see Proposition 1.7.8); here we give a direct justification, which will be used also in the first proof of Theorem 1.5.1. We show that every point y in the closure $\overline{F}$ of the subspace F in E actually belongs to F. Let $\{V_j\colon\ j \in \mathbb{N}\}$ be a base of neighborhoods of zero in the metric topology of the subspace F. For every $j \in \mathbb{N}$, let W_j and W_j' be neighborhoods of zero in E such that $V_j = W_j \cap F$ and $W_j' - W_j' \subset W_j$ and also $W_{j+1}' \subset W_j'$. Then, for every $j \in \mathbb{N}$, we obviously have

$$\big((y + W_j') \cap F\big) - \big((y + W_j') \cap F\big) \subset W_j \cap F = V_j.$$