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Francisco S.N. Lobo *Editor*

Wormholes, Warp Drives and Energy Conditions

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Wormholes, Warp Drives and Energy Conditions

Springer

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Preface

The General Theory of Relativity is an extremely successful theory, with a well-established experimental footing, at least for weak gravitational fields. Its predictions range from the existence of black holes, gravitational radiation (now confirmed) to the cosmological models, predicting a primordial beginning, namely the big-bang. All these solutions have been obtained by first considering a plausible distribution of matter, and through the Einstein field equation, the spacetime metric of the geometry is determined. However, one may solve the Einstein field equation in the reverse direction, namely one first considers an interesting and exotic spacetime metric and then finds the matter source responsible for the respective geometry. In this manner, it was found that some of these solutions possess a peculiar property, namely "exotic matter," involving a stress-energy tensor that violates the null energy condition. These geometries also allow closed timelike curves, with the respective causality violations. It is thus perhaps important to emphasize that these solutions are primarily useful as "gedanken-experiments" and as a theoretician's probe of the foundations of general relativity, and include traversable wormholes and superluminal "warp drive" spacetimes. This book, in addition to extensively exploring interesting features, in particular, the physical properties and characteristics of these "exotic spacetimes," is meant to present a state of the art of wormhole physics, warp drive spacetimes and recent research on the energy conditions. The ideal audience is intended for undergraduate and postgraduate students, with a knowledge of general relativity, and researchers in the field, who are interested in exploring new avenues of research in these topics.

More specifically, in this book, general relativistic rotating wormhole solutions, supported by a phantom scalar field, are presented. The properites of these rotating wormhole solutions including their mass, angular momentum, quadrupole moment, and ergosphere are discussed, and the stability issues are explored. Concerning the astrophysical signatures, physical properties and characteristics of matter forming thin accretion disks in wormhole geometries are analyzed. It is shown that specific signatures appear in the electromagnetic spectrum of thin disks around wormhole spacetimes, thus leading to the possibility of distinguishing these geometries by using astrophysical observations of the emission spectra from accretion disks.

Explicit examples of globally regular static, spherically symmetric solutions in general relativity are also constructed with scalar and electromagnetic fields, describing traversable wormholes with flat and AdS asymptotics and regular black holes, in particular, black universes. (A black universe is a regular black hole with an expanding, asymptotically isotropic spacetime beyond the horizon.) Such objects exist in the presence of scalar fields with negative kinetic energy ("phantoms," or "ghosts"), which are not observed under usual physical conditions. To account for that, "trapped ghosts" (scalars whose kinetic energy is only negative in a strong-field region of spacetime) are considered, as well as "invisible ghosts," i.e., phantom scalar fields sufficiently rapidly decaying in the weak-field region. Self-sustained traversable wormholes, which are configurations sustained by their own gravitational quantum fluctuations, are also considered. The investigation is evaluated by means of a variational approach with Gaussian trial wave functionals to one loop, and the graviton quantum fluctuations are interpreted as a kind of *exotic energy*. It is shown that for every framework, the self-sustained equation will produce a Wheeler wormhole of Planckian size. Some consequences on topology change are discussed together with the possibility of obtaining an enlarged wormhole radius.

In the context of modified theories of gravity, it is shown that the higher-order curvature terms, interpreted as a gravitational fluid, can effectively sustain wormhole geometries, while the matter threading the wormhole can be imposed to satisfy the energy conditions. In this context, a systematic analysis of static spherically symmetric solutions describing a wormhole geometry in a Horndeski model with Galileon shift symmetry is presented. In addition to this, working in a metric-affine framework, explicit models are explored in four and higher dimensions. It is shown that these solutions represent explicit realizations of the concept of geon introduced by Wheeler, interpreted as topologically nontrivial self-consistent bodies generated by an electromagnetic field without sources. Several of their properties are discussed. Furthermore, using exactly solvable models, it is shown that black hole singularities in different electrically charged configurations can be cured. These solutions describe black hole spacetimes with a wormhole giving structure to the otherwise point-like singularity. It is shown that geodesic completeness is satisfied despite the existence of curvature divergences at the wormhole throat. In some cases, physical observers can go through the wormhole, and in other cases, the throat lies at an infinite affine distance. The removal of singularities occurs in a nonperturbative way.

Quantum field theory violates all the classical energy conditions of general relativity. Nonetheless, it turns out that quantum field theories satisfy remnants of the classical energy conditions, known as quantum energy inequalities (QEIs), that have been developed by various authors since the original pioneering work of Ford in 1978. Here, an introduction to QEIs is introduced, as well as to some of the techniques of quantum field theory in curved spacetime (particularly, the use of microlocal analysis together with the algebraic formulation of QFT) that enable rigorous and general QEIs to be derived. Specific examples are computed for the free scalar field, and their consequences are discussed. QEIs are also derived for the class of unitary, positive energy conformal field theories in two spacetime dimensions. In that setting, it is also possible to determine the probability distribution for individual measurements of certain smearings of the stress-energy tensor in the vacuum state. Semiclassical quantum effects also typically violate the energy conditions. The characteristics of a nonlinear energy condition and the flux energy condition (FEC) are also studied, and a quantum version of this energy condition (QFEC) is presented, which is satisfied even in more situations of physical interest. Other possible nonlinear energy conditions are introduced, namely the "trace-of-square" (TOSEC) and "determinant" (DETEC) energy conditions.

While General Relativity (GR) ranks undoubtedly among the best physical theories ever developed, it is also among those with the most striking implications. In particular, GR admits solutions that allow faster-than-light motion and consequently allow closed timelike curves, with the respective causality violations, such as warp drive spacetimes. The basic definition and interesting aspects of these spacetimes are extensively discussed, such as the violation of the energy conditions associated with these spacetimes, the appearance of horizons for the superluminal case, and the possibility of using a warp drive to create closed timelike curves. Applying linearized gravity to the weak-field warp drive, it is found that the energy condition violations in this class of spacetimes are generic to these geometries and are not simply a side effect of the superluminal properties. Furthermore, a "preemptive" chronology protection mechanism is considered that destabilizes superluminal warp drives via quantum matter back-reaction and hence forbids even the conceptual possibility to use these solutions for building a time machine. This result will be considered both in standard quantum field theory in curved spacetime and in the case of a quantum field theory with Lorentz invariance breakdown at high energies. Some lessons and future perspectives will be finally discussed.

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Francisco S.N. Lobo

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Acronyms

AFEC	Averaged flux energy condition
AGN	Active galactic nuclei
ANCC	Averaged null convergence condition
ANEC	Averaged null energy condition
ASEC	Averaged strong condition
ATCC	Averaged timelike convergence condition
AWEC	Averaged weak energy condition
CCC	Closed chronological curve
CFT	Conformal field theory
CTC	Closed timelike curve
DEC	Dominant energy condition
DETEC	Determinant energy condition
FEC	Flux energy condition
FTL	Faster than light
GR	General Relativity
NCC	Null convergence condition
NEC	Null energy condition
NED	Nonlinear electrodynamics
QDEC	Quantum dominant energy condition
QDEI	Quantum Dominated Energy Inequality
QEI	Quantum Energy Inequality
QFEC	Quantum flux energy condition
QFT	Quantum field theory
QG	Quantum gravity
QNEI	Quantum Null Energy Inequality
QWEC	Quantum weak energy condition
QWEI	Quantum Weak Energy Inequality
RSET	Renormalized stress-energy tensor
SAWEC	Spacetime averaged weak energy condition
SEC	Strong energy condition

SET	Stress-energy tensor
TCC	Timelike convergence condition
TM	Time machine
TOSEC	Trace-of-square energy condition
UV	Ultra-violet
WEC	Weak energy condition

Chapter 1
Introduction

Francisco S.N. Lobo

1.1 Historical Background

Traversable wormholes and "warp drive" spacetimes are solutions to the Einstein field equation that violate the classical energy conditions and are primarily useful as "gedanken-experiments" and as a theoretician's probe of the foundations of general relativity. They are obtained by solving the Einstein field equation in the reverse direction, namely, one first considers an interesting and exotic spacetime metric, then finds the matter source responsible for the respective geometry. It is interesting to note that they allow "effective" superluminal travel, although the speed of light is not surpassed *locally*, and generate closed timelike curves, with the associated causality violations.

Wormhole physics can originally be tentatively traced back to Flamm in 1916 [1, 2], where his aim was to render the conclusions of the Schwarzschild solution in a clearer manner. Recall that Schwarzschild published two remarkable papers in 1916, where the first is related to the exterior static and spherically symmetric vacuum solution [3], and the second to the interior solution of a general relativistic incompressible fluid [4]. Flamm in his paper showed through sketches of an equatorial plane that the spatial sections of Schwarzschild's interior solution possess the geometry of a portion of a round sphere. Furthermore, he showed that the surface of revolution is isometric to a planar section of the Schwarzschild exterior solution. Now, he considered that the meridional curve is a parabola, where the surface of revolution joins two asymptotically flat sheets, which in a modern terminology can be considered as a tunnel. However, we emphasize that he was not contemplating the possibility of bridge-like, or wormhole-like, solutions [2].

It was only in 1935, that specific wormhole-type solutions were considered by Einstein and Rosen [5]. Their motivation was to construct an elementary particle

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model represented by a "bridge" connecting two identical sheets. This mathematical representation of physical space being connected by a wormhole-type solution was subsequently denoted as an "Einstein–Rosen bridge". In fact, the neutral version of the Einstein–Rosen bridge is an observation that a suitable coordinate change seems to make the Schwarzschild (coordinate) singularity disappear, at $r = 2M$. In particular, Einstein and Rosen discovered that certain coordinate systems naturally cover only two asymptotically flat regions of the maximally extended Schwarzschild spacetime. Thus, the key ingredient of the bridge construction is the existence of an event horizon, and the Einstein–Rosen bridge is a coordinate artefact arising from choosing a coordinate patch, which is defined to double-cover the asymptotically flat region exterior to the black hole event horizon.

The field lay dormant for approximately two decades after the work by Einstein and Rosen, and it was only in 1955 that John Wheeler began to be interested in topological issues in General Relativity [6]. More specifically, in a multiply-connected spacetime, where two widely separated regions were connected by a tunnel, and taking into account the coupled Einstein–Maxwell field equations, Wheeler constructed hypothesized "geon" solutions. These denote a "gravitational-electromagnetic entity" and in modern language, the geon may be considered as a hypothetical "unstable gravitational-electromagnetic quasisoliton" [7]. Building on this work, in 1957, Misner and Wheeler presented an extensive analysis, where Riemannian geometry of manifolds of nontrivial topology was investigated with an ambitious view to explain all of physics [8]. Their objective was essentially to use the source-free Maxwell equations, coupled to Einstein gravity, in the context of nontrivial topology, to build models for classical electrical charges and all other particle-like entities in classical physics. Indeed, this work was one of the first uses of abstract topology, homology, cohomology, and differential geometry in physics [7] and their point of view is best summarized by the phrase: "Physics is geometry". It is interesting to note that this is also the first paper [8] that introduces the term "wormhole". In fact, Misner and Wheeler considered that the existing well-established "already unified classical theory" allows one to describe in terms of empty curved space [8] the following concepts: *gravitation without gravitation*; *electromagnetism without electromagnetism*; *charge without charge*; and *mass without mass* (where around the mouth of the "wormhole" lies a concentration of electromagnetic energy that gives mass to this region of space).

Despite of the fact that considerable effort was invested in attempting to understand the "geon" concept, the geonlike-wormhole structures seem to have been considered a mere curiosity and after the solutions devised by Wheeler and Misner, there is a 30-year gap between their original work and the 1988 Morris–Thorne renaissance of wormhole physics [9]. However, isolated pieces of work appeared in the 1970s, such as the Homer Ellis' drainhole [10, 11] concept and Bronnikov's tunnel-like solutions [12]. It is only in 1988 that a renaissance of wormhole physics took place, through the seminal paper by Morris and Thorne [9]. In 1995, Matt Visser wrote a full-fledged treatise on wormhole physics and we refer the reader to [13] for a more recent review on wormhole physics and warp drive spacetimes.

1.2 State of the Art: Wormhole Geometries and Warp Drive Spacetimes

The purpose of the present book is to provide an update on the state of the art on several topics of research in wormhole physics and warp drive spacetimes. Although rather incomplete in all the existing topics, we present the relevant fields of modern research in this interesting topic throughout the book.

In Chap. 2, the basics of wormhole physics are briefly reviewed, where the interesting properties and characteristics of static spherically symmetric traversable wormholes are considered, such as, the mathematics of embedding, equations of structure for the wormhole, the traversability conditions, the necessity of exotic matter to support these geometries and wormhole solutions in modified gravity. Furthermore, recent advances are presented on dynamic spherically symmetric thin-shell traversable wormholes. More specifically, a novel approach is considered in the stability analysis of thin-shell wormholes, by reversing the logic flow and the surface mass is determined as a function of the potential. This procedure implicitly makes demands on the equation of state of the matter residing on the transition layer, and demonstrates in full generality that the stability of thin-shell wormholes is equivalent to choosing suitable properties for the material residing on the thin shell.

In Chap. 3, rotating wormholes in General Relativity are presented in four and five dimensions. Their nontrivial topology is supported by a phantom field, and it is shown that the wormhole solutions depend on three parameters, which are associated with the size of the throat, the magnitude of the rotation, and the symmetry of the two asymptotic regions. The physical properties of these wormholes are discussed in detail. Their global charges are derived, including the mass formulae for the symmetric and nonsymmetric cases, and their geometry is discussed, a definition of their throat is presented in the nonsymmetric case, and their ergoregions are investigated. Furthermore, the existence of limiting configurations are demonstrated, which correspond to extremal rotating vacuum black holes. Since a stability analysis of rotating wormholes in four dimensions is very involved, a stability analysis of five-dimensional rotating wormholes is performed, with equal magnitude of the angular momenta only, where the investigation is restricted to the unstable radial modes. It is interesting to note that when the rotation is sufficiently fast, the radial instability disappears for these five dimensional wormholes.

In Chap. 4, the observational and astrophysical features are considered and the physical properties of matter forming thin accretion disks in static spherically symmetric and stationary axially symmetric wormhole spacetimes are discussed. The time averaged energy flux, the disk temperature and the emission spectra of the accretion disks are obtained for these exotic geometries, and are compared with the Schwarzschild and Kerr solutions, respectively. For static and spherically symmetric wormholes it is shown that more energy is emitted from the disk than in the case of the Schwarzschild potential and the conversion efficiency of the accreted mass into radiation is more than a factor of two higher for wormholes than for static black holes. For axially symmetric wormhole spacetimes, by comparing the mass accretion with the one of a Kerr black hole, it is verified that the intensity of the flux emerging from the disk surface is greater for wormholes than for rotating black holes with the same geometrical mass and accretion rate. Furthermore, it is shown that the rotating wormholes provide a much more efficient engine for the transformation of the accretion mass into radiation than the Kerr black holes. It is then concluded that specific signatures appear in the electromagnetic spectrum, thus leading to the possibility of distinguishing wormhole geometries by using astrophysical observations of the emission spectra from accretion disks.

In Chap. 5, wormhole geometries in modified gravity are considered, in particular, a systematic analysis of static spherically symmetric solutions describing a wormhole geometry in a specific Horndeski model with Galileon shift symmetry is presented. The Lagrangian of the theory contains the term $(\varepsilon g^{\mu\nu} + \eta G^{\mu\nu})\phi_{,\mu}\phi_{,\nu}$ and represents a particular case of the general Horndeski lagrangian, which leads to second-order equations of motion. The Rinaldi approach is used to construct analytical solutions describing wormholes with nonminimal kinetic coupling. It is shown that wormholes exist only if $\varepsilon = -1$ (phantom case) and $\eta > 0$. The wormhole throat connects two anti-de Sitter spacetimes. The wormhole metric has a coordinate singularity at the throat. However, since all curvature invariants are regular, there is no curvature singularity there.

In Chap. 6, *self sustained* traversable wormholes are considered, which are configurations sustained by their own gravitational quantum fluctuations. The analysis is evaluated by means of a variational approach with Gaussian trial wave functionals to one loop, and the graviton quantum fluctuations are interpreted as a kind of *exotic energy*. Since these fluctuations usually produce ultra-violet divergences, two procedures to keep them under control are introduced. The first consists of a zeta function regularization and a renormalization process that is introduced to obtain a finite one loop energy. The second approach considers the case of *distorted gravity*, namely, when either Gravity's Rainbow or a noncommutative geometry is used as a tool to keep under control the ultra-violet divergences. In this context, it is shown that for every framework, the self-sustained equation will produce a Wheeler wormhole of Planckian size. Some consequences on topology change are discussed together with the possibility of obtaining an enlarged wormhole radius.

Chapter 7 reviews the properties of static and spherically symmetric configurations of general relativity with a minimally coupled scalar field ϕ, whose kinetic energy is negative in a restricted (strong-field) region of space and positive outside it. This "trapped ghost" concept may, in principle, explain why no ghosts are observed under usual weak-field conditions. The configurations considered are wormholes and regular black holes without a center in particular, black universes (black holes with an expanding cosmology beyond the event horizon). Spherically symmetric perturbations of these objects are considered, and it is stressed that, due to the universal shape of the effective potential near a transition surface from canonical to phantom behavior of the scalar field, such surfaces restrict the possible perturbations and play a stabilizing role.

In Chap. 8, an explicit implementation of geons in the context of gravitational theories extending General Relativity is discussed in detail. Such extensions are formulated in the *Palatini* approach, where the metric and affine connection are regarded as independent entities. This formulation is inspired on the macroscopic description of the physics of crystalline structures with defects in the context of solid state physics. Several theories for the gravitational field are discussed, including additional contributions of the Ricci tensor in four and higher dimensions. As opposed to the standard metric approach, which generically develops higher order derivative field equations and ghost-like instabilities, the Palatini formulation generates ghost-free and second-order equations that reduce to the general relativistic equations in vacuum. In this context, static and spherically symmetric solutions with electric fields generate a plethora of wormhole solutions satisfying the classical energy conditions, and whose properties allow to identify them with the concept of the geon, originally introduced by Wheeler. These solutions provide new insights on the avoidance of spacetime singularities in classical effective geometries.

The standard energy conditions of classical general relativity are (mostly) linear in the stress–energy tensor, and have clear physical interpretations in terms of geodesic focussing, but suffer the significant drawback that they are often violated by semi-classical quantum effects. In contrast, it is possible to develop non-standard energy conditions that are intrinsically non-linear in the stress–energy tensor, and which exhibit much better well-controlled behavior when semi-classical quantum effects are introduced, at the cost of a less direct applicability to geodesic focussing. In Chap. 9, a review of the standard energy conditions and their various limitations is presented. (Including the connection to the Hawking–Ellis type I, II, III, and IV classification of stress-energy tensors). One then turns to the averaged, nonlinear, and semi-classical energy conditions, and see how much can be done once semi-classical quantum effects are included.

Chapter 10 surveys the violation of classical energy conditions in quantum field theory (QFT) and the theory of Quantum Energy Inequalities (QEIs). The latter QEIs are lower bounds on local averages of energy densities and related quantities in QFT. They replace the classical energy conditions of classical general relativity. In particular, (a) the main properties of QEIs are indicated using the example of a free scalar field in Minkowski spacetime; (b) a rigorous derivation of a QEI for scalar fields in general curved spacetimes is given; (c) the resulting QEI is evaluated explicitly in some specific cases; (d) further recent developments, including QEIs for conformal field theories and an integrable QFT are presented, along with work on the probability distribution for measurements of averaged energy densities; (e) the status of QEIs in interacting models is discussed; (f) various applications of the QEIs are presented.

Moving on to "warp drive" spacetimes, the basic definition is considered and interesting aspects of these spacetimes are explored, in Chap. 11. In particular, the violation of the energy conditions associated with these spacetimes is discussed, as well as some other interesting properties such as the appearance of horizons for the superluminal case, and the possibility of using a warp drive to create closed timelike curves. Applying linearized gravity to the weak-field warp drive, it is found that the energy condition violations in this class of spacetimes is generic to the form of the geometry under consideration and is not simply a side-effect of the "superluminal" properties. Fundamental limitations of "warp drive" spacetimes are also found, by proving extremely stringent conditions placed on these geometries.

An interesting aspect of the warp drive resides in the fact that points on the outside front edge of a superluminal bubble are always spacelike separated from the centre of the bubble. This implies that an observer in a spaceship cannot create nor control on demand an Alcubierre bubble. However, causality considerations do not prevent the crew of a spaceship from arranging, by their own actions, to complete a round trip from the Earth to a distant star and back in an arbitrarily short time, as measured by clocks on the Earth, by altering the metric along the path of their outbound trip. Thus, Krasnikov introduced a metric with an interesting property that although the time for a one-way trip to a distant destination cannot be shortened, the time for a round trip, as measured by clocks at the starting point (e.g., Earth), can be made arbitrarily short. Interesting properties of this solution, denoted as the Krasnikov tube are presented such as its four-dimensional generalization, the violations of the energy condition, among other features. Finally, the generation of closed timelike curves are considered in the warp spacetime and the Krasnikov tube.

Faster than light travel and time machines are among the most tantalizing possibilities allowed for by Einstein's General Relativity. In Chap. 12, the main features of these phenomena are reviewed, namely, in which spacetimes they appear to be realized, and it is explained why they are interconnected with the Einsteinian framework. The paradoxes related to the possibility of time travel of the proposed solutions are then briefly discussed. Finally, an explicit example is provided where a purely semi-classical gravity framework seems sufficient to prevent the stability of a spacetime allowing faster than light propagation. It is argued that this supports a sort of "pre-emptive" chronology protection that forbids the generation of the very spacetime structures which could lead to the construction of time machines.

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Part I
Traversable Wormholes

Chapter 2
Wormhole Basics

Francisco S.N. Lobo

2.1 Static and Spherically Symmetric Traversable Wormholes

2.1.1 *Spacetime Metric*

Throughout this book, unless stated otherwise, we will consider the following spherically symmetric and static wormhole solution [1]

$$ds^2 = -e^{2\Phi(r)}\, dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2\, (d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{2.1}$$

The metric functions $\Phi(r)$ and $b(r)$ are arbitrary functions of the radial coordinate r. As $\Phi(r)$ is related to the gravitational redshift, it has been denoted the redshift function, and $b(r)$ is called the shape function, as it determines the shape of the wormhole [1–3], which will be shown below using embedding diagrams. The radial coordinate r is non-monotonic in that it decreases from $+\infty$ to a minimum value r_0, representing the location of the throat of the wormhole, where $b(r_0) = r_0$, and then increases from r_0 to $+\infty$. Although the metric coefficient g_{rr} becomes divergent at the throat, which is signalled by the coordinate singularity, the proper radial distance $l(r) = \pm \int_{r_0}^{r} [1 - b(r)/r]^{-1/2}\, dr$ is required to be finite everywhere. The proper distance decreases from $l = +\infty$, in the upper universe, to $l = 0$ at the throat, and then from zero to $-\infty$ in the lower universe. One must verify the absence of horizons,

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in order for the wormhole to be traversable. This condition must imply that $g_{tt} = -e^{2\Phi(r)} \neq 0$, so that $\Phi(r)$ must be finite everywhere.[1]

Another interesting feature of the redshift function is that its derivative with respect to the radial coordinate also determines the "attractive" or "repulsive" nature of the geometry. In order to verify this, consider the four-velocity of a static observer given by $U^\mu = dx^\mu/d\tau = (e^{-\Phi(r)}, 0, 0, 0)$. The observer's four-acceleration is $a^\mu = U^\mu{}_{;\nu}\, U^\nu$, which has the following components:

$$a^t = 0\,, \qquad a^r = \Phi' \left(1 - \frac{b}{r}\right)\,, \tag{2.2}$$

where the prime denotes a derivative with respect to the radial coordinate r. Now, note that from the geodesic equation, a radially moving test particle which starts from rest initially has the equation of motion

$$\frac{d^2 r}{d\tau^2} = -\Gamma^r_{tt} \left(\frac{dt}{d\tau}\right)^2 = -a^r\,. \tag{2.3}$$

Here, a^r is the radial component of proper acceleration that an observer must maintain in order to remain at rest at constant $r,\ \theta,\ \phi$, so that from Eq. (2.2), a static observer at the throat for generic $\Phi(r)$ is a geodesic observer. In particular, for a constant redshift function, $\Phi'(r) = 0$, static observers are also geodesic. Thus, a wormhole is "attractive" if $a^r > 0$, i.e. observers must maintain an outward-directed radial acceleration to keep from being pulled into the wormhole. If $a^r < 0$, the geometry is "repulsive", i.e. observers must maintain an inward-directed radial acceleration to avoid being pushed away from the wormhole. Indeed, this distinction depends on the sign of Φ', as is transparent from Eq. (2.2).

2.1.2 The Mathematics of Embedding

We can use embedding diagrams to represent a wormhole and extract some useful information for the choice of the shape function, $b(r)$. Due to the spherically symmetric nature of the problem, one may consider an equatorial slice, $\theta = \pi/2$, without loss of generality. The respective line element, considering a fixed moment of time, $t =$ const, is given by

$$ds^2 = \frac{dr^2}{1 - b(r)/r} + r^2\, d\phi^2\,. \tag{2.4}$$

---

[1] This follows from a result originally due to C.V. Vishveshwara stated as follows: In any asymptotically flat spacetime with a Killing vector ξ ($\xi = \mathbf{e_0}$ for the metric (2.1)) which (i) is the ordinary time-translation Killing vector at spatial infinity and (ii) is orthogonal to a family of three-dimensional surfaces, the 3-surface $\xi \cdot \xi = 0$, i.e. $\mathbf{e_0} \cdot \mathbf{e_0} = g_{tt} = 0$, is a null surface that cannot be crossed by any outgoing, future-directed timelike curves, i.e. a horizon.

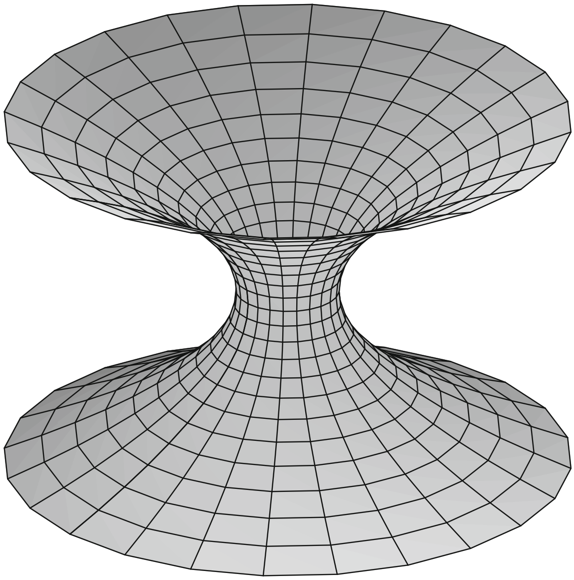

Fig. 2.1 The embedding diagram of a two-dimensional section along the equatorial plane ($t = \text{const}, \theta = \pi/2$) of a traversable wormhole. For a full visualization of the surface sweep through a 2π rotation around the $z-$axis, as can be seen from the graphic on the right

To visualize this slice, one embeds this metric into three-dimensional Euclidean space, in which the metric can be written in cylindrical coordinates, (r, ϕ, z), as

$$ds^2 = dz^2 + dr^2 + r^2\, d\phi^2\,. \tag{2.5}$$

In the three-dimensional Euclidean space the embedded surface has equation $z = z(r)$, so that the metric of the surface can be written as

$$ds^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2\, d\phi^2\,. \tag{2.6}$$

Comparing Eq. (2.4) with (2.6), one deduces the equation for the embedding surface, which is given by

$$\frac{dz}{dr} = \pm\left(\frac{r}{b(r)} - 1\right)^{-1/2}\,. \tag{2.7}$$

To be a solution of a wormhole, the geometry has a minimum radius, $r = b(r) = r_0$, denoted as the throat, at which the embedded surface is vertical, i.e. $dz/dr \to \infty$. Far from the throat, one may consider that space is asymptotically flat, $dz/dr \to 0$ as $r \to \infty$.

To be a solution of a wormhole, one also needs to impose that the throat flares out (see Fig. 2.1 for details). This flaring-out condition entails that the inverse of the embedding function $r(z)$ must satisfy $d^2r/dz^2 > 0$ at or near the throat r_0. Differentiating $dr/dz = \pm(r/b(r) - 1)^{1/2}$ with respect to z, we have

$$\frac{d^2r}{dz^2} = \frac{b - b'r}{2b^2} > 0\,. \tag{2.8}$$

This “flaring-out” condition is a fundamental ingredient of wormhole physics, and plays a fundamental role in the analysis of the violation of the energy conditions. At the throat we verify that the form function satisfies the condition $b'(r_0) < 1$. Note, however, that this treatment has the drawback of being coordinate dependent, and we refer the reader to Refs. [4, 5] for a covariant treatment.

2.1.3 Equations of Structure for the Wormhole

From the metric expressed in the form $ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu$, one may determine the Christoffel symbols (connection coefficients), $\Gamma^\mu{}_{\alpha\beta}$, defined as

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2}\,g^{\mu\nu}\,\left(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}\right)\,, \tag{2.9}$$

which for the metric (2.1) have the following nonzero components:

$$\begin{aligned}
&\Gamma^t{}_{rt} = \Phi'\,, \qquad \Gamma^r{}_{tt} = \left(1 - \frac{b}{r}\right)\,\Phi'\,e^{2\Phi}\,, \qquad \Gamma^r{}_{rr} = \frac{b'r - b}{2r(r-b)}\,,\\
&\qquad\qquad \Gamma^r{}_{\theta\theta} = -r + b\,, \qquad \Gamma^r{}_{\phi\phi} = -(r-b)\,\sin^2\theta\,,\\
&\Gamma^\theta{}_{r\theta} = \Gamma^\phi{}_{r\phi} = \frac{1}{r}\,, \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\,\cos\theta\,, \qquad \Gamma^\phi{}_{\theta\phi} = \tan\theta\,.
\end{aligned} \tag{2.10}$$

The Riemann tensor is defined as

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + \Gamma^\alpha{}_{\lambda\gamma}\Gamma^\lambda{}_{\beta\delta} - \Gamma^\alpha{}_{\lambda\delta}\Gamma^\lambda{}_{\beta\gamma}\,. \tag{2.11}$$

However, the mathematical analysis and the physical interpretation is simplified using a set of orthonormal basis vectors. These may be interpreted as the proper reference frame of a set of observers who remain at rest in the coordinate system (t, r, θ, ϕ), with (r, θ, ϕ) fixed. Denote the basis vectors in the coordinate system as $(\mathbf{e}_t, \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$. Thus, the orthonormal basis vectors are given by

$$\begin{cases}
\mathbf{e}_{\hat{t}} = e^{-\Phi}\;\mathbf{e}_t\\
\mathbf{e}_{\hat{r}} = (1 - b/r)^{1/2}\;\mathbf{e}_r\\
\mathbf{e}_{\hat{\theta}} = r^{-1}\;\mathbf{e}_\theta\\
\mathbf{e}_{\hat{\phi}} = (r\sin\theta)^{-1}\;\mathbf{e}_\phi
\end{cases}\,. \tag{2.12}$$

The nontrivial Riemann tensor components, given in the orthonormal reference frame, take the following form:

$$R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} = -R^{\hat{t}}{}_{\hat{r}\hat{r}\hat{t}} = R^{\hat{r}}{}_{\hat{t}\hat{t}\hat{r}} = -R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} = \left(1-\frac{b}{r}\right)\left[-\Phi'' - (\Phi')^2 + \frac{b'r-b}{2r(r-b)}\Phi'\right], \quad (2.13)$$

$$R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} = -R^{\hat{t}}{}_{\hat{\theta}\hat{\theta}\hat{t}} = R^{\hat{\theta}}{}_{\hat{t}\hat{t}\hat{\theta}} = -R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} = -\left(1-\frac{b}{r}\right)\frac{\Phi'}{r}, \quad (2.14)$$

$$R^{\hat{t}}{}_{\hat{\phi}\hat{t}\hat{\phi}} = -R^{\hat{t}}{}_{\hat{\phi}\hat{\phi}\hat{t}} = R^{\hat{\phi}}{}_{\hat{t}\hat{t}\hat{\phi}} = -R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} = -\left(1-\frac{b}{r}\right)\frac{\Phi'}{r}, \quad (2.15)$$

$$R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{\theta}} = -R^{\hat{r}}{}_{\hat{\theta}\hat{\theta}\hat{r}} = R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} = -R^{\hat{\theta}}{}_{\hat{r}\hat{r}\hat{\theta}} = \frac{b'r-b}{2r^3}, \quad (2.16)$$

$$R^{\hat{r}}{}_{\hat{\phi}\hat{r}\hat{\phi}} = -R^{\hat{r}}{}_{\hat{\phi}\hat{\phi}\hat{r}} = R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} = -R^{\hat{\phi}}{}_{\hat{r}\hat{r}\hat{\phi}} = \frac{b'r-b}{2r^3}, \quad (2.17)$$

$$R^{\hat{\theta}}{}_{\hat{\phi}\hat{\theta}\hat{\phi}} = -R^{\hat{\theta}}{}_{\hat{\phi}\hat{\phi}\hat{\theta}} = R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} = -R^{\hat{\phi}}{}_{\hat{\theta}\hat{\theta}\hat{\phi}} = \frac{b}{r^3}, \quad (2.18)$$

where, as before, a prime denotes a derivative with respect to the radial coordinate r.

The Ricci tensor, $R_{\hat{\mu}\hat{\nu}}$, is given by the contraction $R_{\hat{\mu}\hat{\nu}} = R^{\hat{\alpha}}{}_{\hat{\mu}\hat{\alpha}\hat{\nu}}$, and the nonzero components are the following:

$$R_{\hat{t}\hat{t}} = \left(1-\frac{b}{r}\right)\left[\Phi'' + (\Phi')^2 - \frac{b'r-3b+4r}{2r(r-b)}\Phi'\right], \quad (2.19)$$

$$R_{\hat{r}\hat{r}} = -\left(1-\frac{b}{r}\right)\left[\Phi'' + (\Phi')^2 + \frac{b-b'r}{2r(r-b)}\Phi' + \frac{b-b'r}{r^2(r-b)}\right], \quad (2.20)$$

$$R_{\hat{\theta}\hat{\theta}} = R_{\hat{\phi}\hat{\phi}} = \left(1-\frac{b}{r}\right)\left[\frac{b'r+b}{2r^2(r-b)} - \frac{\Phi'}{r}\right]. \quad (2.21)$$

The curvature scalar or Ricci scalar, defined by $R = g^{\hat{\mu}\hat{\nu}}\, R_{\hat{\mu}\hat{\nu}}$, is given by

$$R = -2\left(1-\frac{b}{r}\right)\left[\Phi'' + (\Phi')^2 - \frac{b'}{r(r-b)} - \frac{b'r+3b-4r}{2r(r-b)}\Phi'\right]. \quad (2.22)$$

Thus, the Einstein tensor, given in the orthonormal reference frame by $G_{\hat{\mu}\hat{\nu}} = R_{\hat{\mu}\hat{\nu}} - \frac{1}{2}R\, g_{\hat{\mu}\hat{\nu}}$, yields for the metric (2.1), the following nonzero components:

$$G_{\hat{t}\hat{t}} = \frac{b'}{r^2}, \quad (2.23)$$

$$G_{\hat{r}\hat{r}} = -\frac{b}{r^3} + 2\left(1-\frac{b}{r}\right)\frac{\Phi'}{r}, \quad (2.24)$$

$$G_{\hat{\theta}\hat{\theta}} = \left(1-\frac{b}{r}\right)\left[\Phi'' + (\Phi')^2 - \frac{b'r-b}{2r(r-b)}\Phi' - \frac{b'r-b}{2r^2(r-b)} + \frac{\Phi'}{r}\right], \quad (2.25)$$

$$G_{\hat{\phi}\hat{\phi}} = G_{\hat{\theta}\hat{\theta}}, \quad (2.26)$$

respectively.

2.1.4 Stress–Energy Tensor

Through the Einstein field equation, $G_{\hat{\mu}\hat{\nu}} = 8\pi T_{\hat{\mu}\hat{\nu}}$, one verifies that the stress–energy tensor $T_{\hat{\mu}\hat{\nu}}$ has the same algebraic structure as $G_{\hat{\mu}\hat{\nu}}$, Eqs. (2.23)–(2.26), and the only nonzero components are precisely the diagonal terms $T_{\hat{t}\hat{t}}$, $T_{\hat{r}\hat{r}}$, $T_{\hat{\theta}\hat{\theta}}$ and $T_{\hat{\phi}\hat{\phi}}$. Using the orthonormal basis, these components carry a simple physical interpretation, i.e.

$$T_{\hat{t}\hat{t}} = \rho(r)\,, \qquad T_{\hat{r}\hat{r}} = -\tau(r)\,, \qquad T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}} = p(r)\,, \tag{2.27}$$

where $\rho(r)$ is the energy density, $\tau(r)$ is the radial tension, with $\tau(r) = -p_r(r)$, i.e. it is the negative of the radial pressure, $p(r)$ is the pressure measured in the tangential directions, orthogonal to the radial direction.

Thus, the Einstein field equation provides the following stress–energy scenario:

$$\rho(r) = \frac{1}{8\pi}\,\frac{b'}{r^2}\,, \tag{2.28}$$

$$\tau(r) = \frac{1}{8\pi}\left[\frac{b}{r^3} - 2\left(1-\frac{b}{r}\right)\frac{\Phi'}{r}\right], \tag{2.29}$$

$$\begin{aligned} p(r) = \frac{1}{8\pi}\left(1-\frac{b}{r}\right)\Bigg[\Phi'' + (\Phi')^2 - \frac{b'r-b}{2r^2(1-b/r)}\Phi' \\ -\frac{b'r-b}{2r^3(1-b/r)} + \frac{\Phi'}{r}\Bigg]. \end{aligned} \tag{2.30}$$

Note that one now has three equations with five unknown functions of the radial coordinate. Several strategies to solve these equations are available, for instance, one can impose an equation of state [6–10] and consider a specific choice of the shape function or of the redshift function.

Note that the sign of the energy density depends on the sign of $b'(r)$. One often comes across the misleading statement, in the literature, that wormholes should necessarily be threaded by negative energy densities, or negative matter; however, this is not necessarily the case. Note, however, that due to the flaring-out condition, observers traversing the wormhole with sufficiently high velocities, $v \to 1$, will measure a negative energy density. This will be shown below. Furthermore, one should perhaps correctly state that it is the radial pressure that is necessarily negative at the throat, which is transparent for the radial tension at the throat, which is given by $p_r(r) = -\tau(r_0) = -(8\pi r_0^2)^{-1}$.

By taking the derivative with respect to the radial coordinate r, of Eq. (2.29), and eliminating b' and Φ'', given in Eqs. (2.28) and (2.30), respectively, we obtain the following equation:

$$\tau' = (\rho - \tau)\Phi' - \frac{2}{r}(p+\tau)\,. \tag{2.31}$$

Equation (2.31) is the relativistic Euler equation, or the hydrostatic equation for equilibrium for the material threading the wormhole, and can also be obtained using the conservation of the stress–energy tensor, $T^{\hat{\mu}\hat{\nu}}{}_{;\hat{\nu}} = 0$, inserting $\hat{\mu} = r$.

The effective mass, $m(r) = b(r)/2$ contained in the interior of a sphere of radius r, can be obtained by integrating Eq. (2.28), which yields

$$m(r) = \frac{r_0}{2} + \int_{r_0}^{r} 4\pi\, \rho(r')\, r'^2\, dr' \,. \tag{2.32}$$

Therefore, the form function has an interpretation which depends on the mass distribution of the wormhole.

2.1.5 *Exotic Matter and Modified Gravity*

2.1.5.1 Exoticity Function

To gain some insight into the matter threading the wormhole, Morris and Thorne defined the dimensionless function $\xi = (\tau - \rho)/|\rho|$ [1], which taking into account Eqs. (2.28) and (2.29) yields

$$\xi = \frac{\tau - \rho}{|\rho|} = \frac{b/r - b' - 2r(1 - b/r)\Phi'}{|b'|} \,. \tag{2.33}$$

Combining the flaring-out condition, given by Eq. (2.8), with Eq. (2.33), the exoticity function takes the form

$$\xi = \frac{2b^2}{r|b'|}\,\frac{d^2 r}{dz^2} - 2r\left(1 - \frac{b}{r}\right)\frac{\Phi'}{|b'|} \,. \tag{2.34}$$

Now, taking into account the finite character of ρ, and consequently of b', and the fact that $(1 - b/r)\Phi' \to 0$ at the throat, we have the following relationship:

$$\xi(r_0) = \frac{\tau_0 - \rho_0}{|\rho_0|} > 0 \,. \tag{2.35}$$

The restriction $\tau_0 > \rho_0$ is a somewhat troublesome condition, depending on one's point of view, as it states that the radial tension at the throat should exceed the energy density. Thus, Morris and Thorne coined matter constrained by this condition "exotic matter" [1]. We shall verify below that this is defined as matter that violates the null energy condition (in fact, it violates all the energy conditions) [1, 2].

Exotic matter is particularly troublesome for measurements made by observers traversing through the throat with a radial velocity close to the speed of light. Consider a Lorentz transformation, $x^{\hat{\mu}'} = \Lambda^{\hat{\mu}'}{}_{\hat{\nu}}\, x^{\hat{\nu}}$, with $\Lambda^{\hat{\mu}}{}_{\hat{\alpha}'}\, \Lambda^{\hat{\alpha}'}{}_{\hat{\nu}} = \delta^{\hat{\mu}}{}_{\hat{\nu}}$ and $\Lambda^{\hat{\mu}}{}_{\hat{\nu}'}$ defined as

$$(\Lambda^{\hat{\mu}}{}_{\hat{\nu}'}) = \begin{bmatrix} \gamma & 0 & 0 & \gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma v & 0 & 0 & \gamma \end{bmatrix} . \tag{2.36}$$

The energy density measured by these observers is given by $T_{\hat{0}'\hat{0}'} = \Lambda^{\hat{\mu}}{}_{\hat{0}'}\ \Lambda^{\hat{\nu}}{}_{\hat{0}'}\ T_{\hat{\mu}\hat{\nu}}$, i.e.

$$T_{\hat{0}'\hat{0}'} = \gamma^2 \left(\rho_0 - v^2 \tau_0\right) , \tag{2.37}$$

with $\gamma = (1 - v^2)^{-1/2}$. For sufficiently high velocities, $v \to 1$, the observer will measure a negative energy density, $T_{\hat{0}'\hat{0}'} < 0$.

This feature also holds for any traversable, nonspherical and nonstatic wormhole. To see this, one verifies that a bundle of null geodesics that enters the wormhole at one mouth and emerges from the other must have a cross-sectional area that initially increases, and then decreases. This conversion of decreasing to increasing is due to the gravitational repulsion of matter through which the bundle of null geodesics traverses.

2.1.5.2 The Violation of the Energy Conditions

The exoticity function (2.33) is closely related to the null energy condition (NEC), which asserts that for any null vector k^μ, we have $T_{\mu\nu} k^\mu k^\nu \geq 0$. For a diagonal stress–energy tensor, this implies $\rho - \tau \geq 0$ and $\rho + p \geq 0$. Using the Einstein field equations (2.28) and (2.29), evaluated at the throat r_0, and taking into account the finite character of the redshift function so that $(1 - b/r)\Phi'|_{r_0} \to 0$, we verify the condition $(\rho - \tau)|_{r_0} < 0$. This violates the NEC. In fact, it implies the violation of all the pointwise energy condition. Although classical forms of matter are believed to obey the energy conditions, it is a well-known fact that they are violated by certain quantum fields, amongst which we may refer to the Casimir effect. Thus, the flaring-out condition (2.8) entails the violation of the NEC, at the throat. Note that negative energy densities are not essential, but negative pressures are necessary to sustain the wormhole throat.

It is interesting to note that the violations of the pointwise energy conditions led to the averaging of the energy conditions over timelike or null geodesics [11]. The averaged energy conditions permit localized violations of the energy conditions, as long on average the energy conditions hold when integrated along timelike or null geodesics. Now, as the averaged energy conditions involve averaging over a line integral, with dimensions (mass)/(area), not a volume integral, they do not provide useful information regarding the "total amount" of energy condition violating matter. In order to overcome this shortcoming, the "volume integral quantifier" was proposed [12]. Thus, the amount of energy condition violations is then the extent that these integrals become negative.

2.1.5.3 Wormholes in Modified Theories of Gravity

Generally, the NEC arises when one refers back to the Raychaudhuri equation, which is a purely geometric statement, without the need to refer to any gravitational field equations. Now, in order for gravity to be attractive, the positivity condition $R_{\mu\nu}k^{\mu}k^{\nu} \geq 0$ is imposed in the Raychaudhuri equation. In general relativity, contracting both sides of the Einstein field equation $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$ (where $\kappa^2 = 8\pi$) with any null vector k^{μ}, one can write the above condition in terms of the stress–energy tensor given by $T_{\mu\nu}k^{\mu}k^{\nu} \geq 0$, which is the statement of the NEC.

In modified theories of gravity the gravitational field equations can be rewritten as an effective Einstein equation, given by $G_{\mu\nu} = \kappa^2 T^{\text{eff}}_{\mu\nu}$, where $T^{\text{eff}}_{\mu\nu}$ is an effective stress–energy tensor containing the matter stress–energy tensor $T_{\mu\nu}$ and the curvature quantities, arising from the specific modified theory of gravity considered [13]. Now, the positivity condition $R_{\mu\nu}k^{\mu}k^{\nu} \geq 0$ in the Raychaudhuri equation provides the *generalized* NEC, $T^{\text{eff}}_{\mu\nu}k^{\mu}k^{\nu} \geq 0$, through the modified gravitational field equation.

Therefore, the necessary condition to have a wormhole geometry is the violation of the generalized NEC, i.e. $T^{\text{eff}}_{\mu\nu}k^{\mu}k^{\nu} < 0$. In classical general relativity this simply reduces to the violation of the usual NEC, i.e. $T_{\mu\nu}k^{\mu}k^{\nu} < 0$. However, in modified theories of gravity, one may in principle impose that the matter stress–energy tensor satisfies the standard NEC, $T_{\mu\nu}k^{\mu}k^{\nu} \geq 0$, while the respective generalized NEC is necessarily violated, $T^{\text{eff}}_{\mu\nu}k^{\mu}k^{\nu} < 0$, in order to ensure the flaring-out condition.

More specifically, consider the generalized gravitational field equations for a large class of modified theories of gravity, given by the following field equation: [13]

$$g_1(\Psi^i)(G_{\mu\nu} + H_{\mu\nu}) - g_2(\Psi^j)\, T_{\mu\nu} = \kappa^2\, T_{\mu\nu}\,, \tag{2.38}$$

where $H_{\mu\nu}$ is an additional geometric term that includes the geometrical modifications inherent in the modified gravitational theory under consideration; $g_i(\Psi^j)$ ($i = 1, 2$) are multiplicative factors that modify the geometrical sector of the field equations, and Ψ^j denote generically curvature invariants or gravitational fields such as scalar fields; the term $g_2(\Psi^i)$ covers the coupling of the curvature invariants or the scalar fields with the matter stress–energy tensor, $T_{\mu\nu}$.

It is useful to rewrite this field equation as an effective Einstein field equation, as mentioned above, with the effective stress–energy tensor, $T^{\text{eff}}_{\mu\nu}$, given by

$$T^{\text{eff}}_{\mu\nu} \equiv \frac{1 + \bar{g}_2(\Psi^j)}{g_1(\Psi^i)}\, T_{\mu\nu} - \bar{H}_{\mu\nu}\,, \tag{2.39}$$

where $\bar{g}_2(\Psi^j) = g_2(\Psi^j)/\kappa^2$ and $\bar{H}_{\mu\nu} = H_{\mu\nu}/\kappa^2$ are defined for notational convenience.

In modified gravity, the violation of the generalized NEC, $T^{\text{eff}}_{\mu\nu}k^{\mu}k^{\nu} < 0$, implies the following restriction: