Department of Earth System Science and Technology, Interdisciplinary Graduate School of Engineering Sciences, Kyushu University

Fluid Dynamics for Global Environmental Studies

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Foreword

I am very pleased and proud to announce the publication of the second volume in the Green Asia Lecture Book Series, following the first one, entitled *Mathematical Analysis of Environmental System*, published in 2014.

Green Asia (Global Strategy for Green Asia) is one of the Programs for Leading Graduate Schools promoted by the Ministry of Education, Culture, Sports and Technology in Japan, in which we aim to establish a science and engineering leadership training program that promotes environmental and energy innovation to reach out from Asia to the rest of the world. One of the outstanding features of Green Asia as an advanced educational program is that all lectures are offered in English. Hence, the publication of cutting-edge books is one of the most important, visible, and tangible outputs of the Green Asia Program. Each of the volumes in the series deals with essential theories, fundamentals, practical applications, or upcoming topics, all of which are actually used in the program lectures.

This volume was originally published in Japanese and established an excellent reputation from many students for many years. It is because our excellent professors working for the Department of Earth System Science and Technology, Interdisciplinary Graduate School of Engineering Sciences, Kyushu University, were responsible for intelligibly describing key points of fluid dynamics and its application. The book has been carefully translated into English by the professors so as to maintain the flavor of the original. I am confident that this book can serve as a lighthouse for beginning students as well as for engineers and scientists.

It will be wonderful if our publication project can disseminate all of the brilliant content and approaches produced in the Green Asia Program to a worldwide audience.

Prof. Jun Tanimoto, Dr. Eng.
Director of Green Asia Education Center
and Head Coordinator of Advanced
Graduate Program in Global Strategy for Green Asia
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Preface to the English Edition

This book is an English edition of a Japanese book with the same title. A new chapter, "Space Plasma Environment," written by Prof. Tohru Hada, has been added in this edition to cover the plasma flow influenced by the Lorentz force that was not included in the Japanese edition.

This English edition is supported by the advanced graduate program in Global Strategy for Green Asia at Kyushu University. We would like to express our sincere gratitude to all those people who have supported this publication.

April 2015

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Preface to the Original Japanese Edition

This book introduces the fundamental concepts of environmental fluid dynamics. It is intended for use by students, researchers, and specialists working in the research fields of geophysical fluids (such as atmosphere or ocean) and general fluid environment. Because the Earth is covered by atmosphere and oceans and is exposed to solar wind, the knowledge of fluid motion is essential for tackling its environmental issues. Although fundamental fluid mechanics is found in many existing textbooks, few of them clearly describe all the essential ideas, from the fundamentals of fluid dynamics to environmental science, with a careful explanation of the governing mathematics. This book has been developed to bridge that gap.

In recent years, with an improvement in the educational setting for environmental science, many students graduating from various university departments choose a career in environment-related fields and undertake specialized researches. Under these circumstances, many students have probably never acquired a fundamental knowledge of fluid mechanics before engaging in such research. The present book will be an invaluable resource for such students. Meanwhile, students who have previously studied fluid dynamics will come to realize that certain aspects of environmental fluid dynamics, such as stratification or rotation effects, are not truly covered by general fluid mechanics. Furthermore, general fluid dynamics treats flows around objects, whereas in environmental fluid dynamics, flows inside boundaries are also important. To accommodate these needs, fluid dynamics "repackaged for environmental sciences" should be learned again. Considering these issues, this book takes the reader from the derivation of the fundamental fluid dynamics equations through to environmental science in eight chapters. Intermediate calculations are carefully demonstrated, and fundamental concepts are explained as comprehensively as possible. Using this book, readers will acquire plentiful knowledge about the dynamics of fluid motion, which will assist them in their more advanced research of environmental science and technology. A home page related to this book will be established at http://www.esst.kyushu-u.ac.jp/textbook/, which allows the interaction of the authors and the readers of this book.

This book is based on a series of actual lectures, which are compulsory for obtaining a master's degree in the Department of Earth System Science and

Technology, Kyushu University. This fact is reflected in the selection of contents. Gravitational and Coriolis forces are given as early examples of external forces, although advanced electromagnetic flow under the Lorentz force is excluded in the present edition. Each chapter was written by the corresponding course instructor in FY2000. During the editing process, predicates and symbols, which may vary across disciplines, have been unified wherever possible. References are provided as footnotes within the text, while more general materials accessed while writing the text are compiled as a bibliography at the end of the book.

In addition to sincerely thanking the authors of the book, we offer sincere thanks to many staff who participated in the planning and operating of these lectures. We also express the heartfelt thanks to Seizando-Shoten Publishing Co., Ltd., for their valuable support in publishing this book.

February 2001

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The original version of the book was revised: Belated corrections in Copyright page of Frontmatter and Chapter 9 have been updated. The erratum to the book is available at 10.1007/978-4-431-56499-7_10

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Contents

Chapter 1
Fundamental Equations in Fluid Dynamics

Fluid dynamics is a study of the movement of gases and liquids. It has an unbelievable range of applications. The Earth's atmosphere and oceanic movements are within the scope of fluid dynamics. This book mainly aims to describe dynamics of rotating and stratified fluids, as well as environmental fluid dynamics. This chapter is allocated for the explanation of fundamental equations in fluid dynamics. First, stress and stress tensors will be discussed, and then fluids will be characterized on their basis. Next, the mass conservation law, equation of motion (momentum conservation law), and energy equations as well as the specifications of fluid motion will be discussed. To complete the equation of motion for a viscous fluid, it is necessary to relate the deformation of fluid elements due to fluid motion with stress. Furthermore, the concept of vorticity and its governing equations will be discussed. In this book, incompressible flow will be considered in most cases; however, this chapter has been made as general as possible. In this book, appropriate vector notation and tensor notation will be used where necessary.

1.1 Fluid and Fluid Dynamics

Liquids and gases are generally called **fluids**. Compared with solids, which resist operations that try to deform them, fluids cannot resist such operations and become deformed without limit, although they do resist compression. This nature is a characteristic of fluids, and it also links them to the phenomenon of "flowing." In the next section we will adopt this as the mechanical definition of fluids. Whether something can be treated as a fluid depends on the time and space scales under consideration. For example, although the Earth's mantle is usually classified as a solid, when considering mantle convection in which deformation occurs over extremely long time and space scales, it can be thought of as a fluid.

Fluid dynamics is a study that assumes fluid as a continuum and discusses its macroscopic motion. Microscopically, substances have discontinuous structures composed of molecules. Substances generally comprise a vast number, for example

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10^{23}, of molecules; hence, it is not realistic to explain the macroscopic motion of substances on the basis of the motion of individual molecules. In fluid dynamics, the microscopic structure of fluid substances is ignored and treated as a continuum, and its macroscopic movement is discussed. Phenomena are described using a small number of **macroscopic variables** such as velocity, density, pressure, and temperature. Considering fluid substances as a continuum in this way, there is also the benefit that air and water, whose microscopic structures are very different, can be treated in a unified manner. Fluid dynamics has a vast range of applications such as in applied engineering, biology, atmosphere, ocean, and space partly because fluids have a unified framework, which is not dependent on individual fluid molecules.

Let us consider the conditions for regarding an actual fluid as a continuum. Under the continuum hypothesis, we consider, for example, a fluid's density $\rho(\boldsymbol{r}, t)$ as a continuous function of position vector $\boldsymbol{r}$[1] and time t (usually including partial derivatives of an appropriate order). Let us look at gas density on a molecular basis. At time t, if the number of molecules within volume δV of length scale $\delta\ell$ with point $\boldsymbol{r}$ as the center is N and the molecular mass is m, the average density is $Nm/\delta V$. The limiting value of this average density when $\delta\ell \to 0$ is the density in the continuum approximation. However, it is clear that such a limiting value does not physically exist. This is because if $\delta\ell$ is decreased and becomes comparable with the size of a molecule, then the discontinuity of the substance structure is directly reflected, and the average density becomes discontinuous. In contrast, if $\delta\ell$ is increased and δV starts to contain plenty of gas molecules, the average density will begin to maintain a fixed value. This value is the actual local value of density. If $\delta\ell$ is increased further and becomes a size similar to the length of the scale of the macroscopic phenomenon under consideration L_{macro} (assuming that the density changes macroscopically), then the average density is influenced by the macroscopic change. Furthermore, if there is a sufficiently large range of $\delta\ell$ for which the average density takes a fixed value such as that mentioned above, then it can be said that an actual continuum approximation is valid.

Moreover, we usually assume that fluids are in a **locally thermal equilibrium** or a state close to it and for a volume δV, which is negligibly small relative to $(L_{\text{macro}})^3$ but still contains a large number of molecules, relations in equilibrium thermodynamics can be applied to thermodynamic quantities such as temperature.[2] For this purpose, molecules need to make sufficiently frequent collisions within a time period much shorter than the time scale T_{macro} of macroscopic phenomenon under consideration.

The following two conditions must be satisfied for these conditions to be true.

1. When the mean free path of the substance molecules, i.e., the average distance a molecule moves until a molecule collides with another molecule, is λ,

$$\lambda \ll L_{\text{macro}} .$$

[1] It will be stated simply as point $\boldsymbol{r}$ hereinafter.

[2] This will be explained with a little more detail in Sect. 1.9.

2. When the mean free time of the substance molecules, i.e., the average time until a molecule collides with another molecule, is τ,

$$\tau \ll T_{\text{macro}} .$$

In air at 0 °C and 1 atmospheric pressure, $\lambda \sim 6 \times 10^{-6}$ cm and $\tau \sim 10^{-10}$ s; hence, everyday phenomena satisfy these conditions easily. Moreover, whether it can be deemed as a continuum depends on what type of phenomenon is considered. Thus, even for a collection of stars such as the Galaxy, if a phenomenon of a scale much greater than the average distance between stars is considered, then it can be treated as a continuum.

As mentioned above, physically, there is a lower limit of $\delta\ell$ for which the average density maintains as a fixed value; however, a continuum is a hypothetical medium supposing that $\delta\ell$ satisfying this quality can be considered to be as small as desired. Hereafter, fluids are assumed to be a continuum, and macroscopic variables that describe fluids, such as density, velocity, pressure, and temperature, are assumed to be continuous functions of a point in the flow region $\boldsymbol{r}$ and time t.

1.2 Stress and Stress Tensors

1.2.1 Body Force and Surface Force

Generally, two types of forces work on fluids. One is **body force** such as gravity, electromagnetic forces, and inertial forces. If the force per unit mass is $\boldsymbol{K}(\boldsymbol{r}, t)$, and the fluid density is $\rho(\boldsymbol{r}, t)$, then the force working on a body element δV at point $\boldsymbol{r}$ is

$$\boldsymbol{K}(\boldsymbol{r}, t)\rho\delta V . \tag{1.1}$$

The other is **surface force** that is acted by fluids on either side of a surface toward one another through the surface, and the force per unit surface area is called **stress**. In contrast to body force, which is generally a long-range force, surface force is a short-range force originating from a molecular basis. The mechanics of the molecular basis of surface force differ between air and water; however, as a continuum, such differences can be ignored and both the surface forces can be expressed in the same form.

1.2.2 Stress and Stress Tensors

Let us consider stress. Generally, stress is dependent on the direction of the surface through which it is acting on. As shown in Fig. 1.1, consider a small surface with

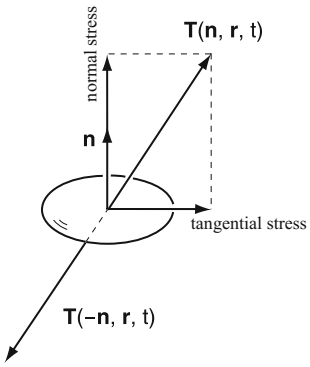

Fig. 1.1 Direction of surface and stress

area δS and unit normal vector $\boldsymbol{n}$ at point $\boldsymbol{r}$ in the fluid at time t. The force acting on the fluid on the reverse by the fluid on the obverse side (define the obverse side of the surface as the side to which $\boldsymbol{n}$ points) through this surface is written as

$$\boldsymbol{T}(\boldsymbol{n}, \boldsymbol{r}, t)\delta S, \tag{1.2}$$

and the force per unit area $\boldsymbol{T}(\boldsymbol{n}, \boldsymbol{r}, t)$ is called stress acting on that surface at time t and point $\boldsymbol{r}$. The component of $\boldsymbol{T}(\boldsymbol{n}, \boldsymbol{r}, t)$ in the direction $\boldsymbol{n}$ is called normal stress, and the component parallel to the surface (more precisely, a tangential plane) is called tangential stress. $\boldsymbol{T}(\boldsymbol{n}, \boldsymbol{r}, t)$ is the stress exerted on the fluid on the reverse side by the fluid on the obverse side, and the stress exerted on the fluid on the obverse side by the fluid on the reverse side is expressed as $\boldsymbol{T}(-\boldsymbol{n}, \boldsymbol{r}, t)$ by replacing $\boldsymbol{n}$ with $-\boldsymbol{n}$. Therefore, by Newton's third law of motion, the following relation is established:

$$\boldsymbol{T}(-\boldsymbol{n}, \boldsymbol{r}, t) = -\boldsymbol{T}(\boldsymbol{n}, \boldsymbol{r}, t)\,. \tag{1.3}$$

Hence, if normal stress is positive, the fluids will be pulling against one another across the surface, and if it is negative, they will be pushing against one another. Hereafter, $\boldsymbol{r}$ and t in $\boldsymbol{T}(\boldsymbol{n}, \boldsymbol{r}, t)$ will be omitted.

Now let us consider the balance of the forces acting on a small tetrahedron as shown in Fig. 1.2. The forces acting on the tetrahedron are inertial forces, external forces, and surface forces exerted through the four surfaces. Inertia and external forces are body forces, and are of order $(\delta\ell)^3$, where the representative length of the tetrahedron is $\delta\ell$. In contrast, surface forces are of order of $(\delta\ell)^2$; hence, as $\delta\ell \to 0$, the contribution of body forces can be ignored, and the balance equation of the surface forces acting on the four surfaces is obtained:

$$\boldsymbol{T}(\boldsymbol{n})\delta S + \sum_{j=1}^{3} \boldsymbol{T}(-\boldsymbol{e}_j)\delta S_j = 0\,. \tag{1.4}$$

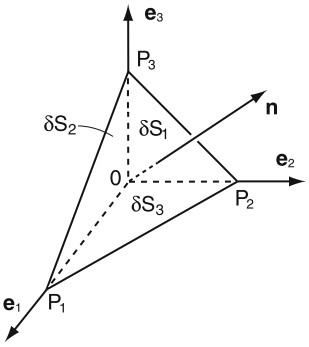

Fig. 1.2 Small tetrahedron with three surfaces perpendicular to the coordinate axes

The inertia force has been taken into account in the balance equation of the force; hence, note that this relationship also holds for moving fluids. Here δS is the area of $\triangle \mathrm{P}_1\mathrm{P}_2\mathrm{P}_3$, and δS_j is the area of the surface perpendicular to the x_j-axis. Moreover, $\boldsymbol{e}_j$ is a unit vector in the x_j-axis direction.[3] The x_i-component of vector $\boldsymbol{T}$ is expressed as T_i, and this can be written as[4]

$$\boldsymbol{T} = (T_1, T_2, T_3)\,.$$

This means that

$$\boldsymbol{T} = T_1\boldsymbol{e}_1 + T_2\boldsymbol{e}_2 + T_3\boldsymbol{e}_3 = \sum_{j=1}^{3} T_j\boldsymbol{e}_j\,. \tag{1.5}$$

Furthermore, by omitting the summation sign in (1.5), it is simplified as

$$\boldsymbol{T} = T_j\boldsymbol{e}_j\,. \tag{1.6}$$

In other words, unless otherwise specified, if the same Latin subscript appears two times, then agree to take a sum from 1 to 3 (Einstein's convention).

If this convention is used, (1.4) can be written as

$$\boldsymbol{T}(\boldsymbol{n})\delta S + \boldsymbol{T}(-\boldsymbol{e}_j)\delta S_j = 0\,. \tag{1.7}$$

If $\delta S_j = (\boldsymbol{n}\cdot\boldsymbol{e}_j)\delta S$ and (1.3) are used, from (1.7), the following is obtained:

$$\boldsymbol{T}(\boldsymbol{n}) = \boldsymbol{T}(\boldsymbol{e}_j)(\boldsymbol{n}\cdot\boldsymbol{e}_j)\,. \tag{1.8}$$

[3] x_1, x_2, x_3 indicates x, y, z, respectively, and $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ refers to the basic vector $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ of the xyz axes, respectively. Hereafter, both will be used as appropriate. Moreover, even if not stated specifically, it will be assumed that an appropriate xyz Cartesian coordinate system is in place.

[4] Under normal circumstances, it should probably be expressed as a column vector because of considerations relating to the amount of paper a row vector will be used. Instead of referring to it as vector with components T_i, it can be referred to simply as vector T_i.

Therefore, if the stresses $\boldsymbol{T}(\boldsymbol{e}_1)$, $\boldsymbol{T}(\boldsymbol{e}_2)$, $\boldsymbol{T}(\boldsymbol{e}_3)$ acting on the surfaces perpendicular to the three coordinate axes are known, then the stress $\boldsymbol{T}(\boldsymbol{n})$ acting on any surface will be known. If the following relation

$$\tau_{ij} = T_i(\boldsymbol{e}_j) \tag{1.9}$$

is defined, then because of $\boldsymbol{n} \cdot \boldsymbol{e}_j = n_j$; (1.8) can be written as

$$T_i(\boldsymbol{n}) = \tau_{ij}\, n_j\,. \tag{1.10}$$

The quantity with these nine quantities τ_{ij} as components is known as a **stress tensor**.[5] By definition, τ_{ij} is the x_i-axis component of the force per unit area $\boldsymbol{T}(\boldsymbol{e}_j)$ that acts on the surface perpendicular to the x_j-axis in the direction from the side whose x_j is greater to that whose x_j is smaller. τ_{11}, τ_{22}, τ_{33} are normal stresses, and τ_{ij} $(i \neq j)$ are tangential stresses. The stress tensor is independent of $\boldsymbol{n}$ and is determined by $\boldsymbol{r}$ and t alone. (1.10) shows that the (second-order) tensor serves as a linear operator to make the vector $\boldsymbol{n}$ correspond to the vector $\boldsymbol{T}(\boldsymbol{n})$ (or make the vector $\boldsymbol{n}\delta S$ correspond to the vector $\boldsymbol{T}(\boldsymbol{n})\delta S$ by multiplying both sides of (1.10) by δS). Moreover, the above mentioned definition of τ_{ij} shows that τ_{ij} is the very component of the representation matrix[6] in the Cartesian coordinate system of the linear operator. In reality, the right-hand side of (1.10) is a product of the matrix τ_{ij} and vector n_i. Both vector and tensor are physical quantities and are independent of the coordinate system. Therefore, their components are transformed according to a specific transformation rule for the rotation of the coordinate system. Vector and tensor can also be defined from this point of view (see Appendix A.1).

Note 1.1 Let us provide a little supplement regarding (1.4). $\boldsymbol{T}(\boldsymbol{n})$ in (1.4) is strictly a value at some point in $\triangle \mathrm{P_1P_2P_3}$. The same is true for $\boldsymbol{T}(-\boldsymbol{e}_j)$. This can be seen by applying the mean value theorem to the surface integrals: originally each term of (1.4) is a surface integral. If the representative point of the tetrahedron is O, $\boldsymbol{T}(\boldsymbol{n})$ and $\boldsymbol{T}(-\boldsymbol{e}_j)$ are expanded around O, and $\delta S_j = \boldsymbol{n}_j \delta S$ is used, the left-hand side of (1.4) can be written as

$$(\boldsymbol{T}(\boldsymbol{n}, \boldsymbol{r}, t) - \boldsymbol{T}(\boldsymbol{e}_j, \boldsymbol{r}, t)\, n_j)\, \delta S + O((\delta\ell)^3)\,. \tag{1.11}$$

$\boldsymbol{r}$ is the position vector of the point O. If the tetrahedron is contracted to the point O while maintaining a similar shape, then the first term of (1.11) is $O((\delta\ell)^2)$. This is because the coefficient of δS is constant. The second term represents a difference from the first term and is $O((\delta\ell)^3)$. On the other hand, inertia and external forces are $O((\delta\ell)^3)$; hence, the coefficient of δS in (1.11) has to be 0, and (1.8) is obtained.

[5] More precisely, this is a second-order tensor. Moreover, a tensor with τ_{ij} as components may instead be referred to simply as tensor τ_{ij}.

[6] Matrix vertically lining up vectors $\boldsymbol{T}(\boldsymbol{e}_1)$, $\boldsymbol{T}(\boldsymbol{e}_2)$, $\boldsymbol{T}(\boldsymbol{e}_3)$, which are mappings of the basic vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, respectively.

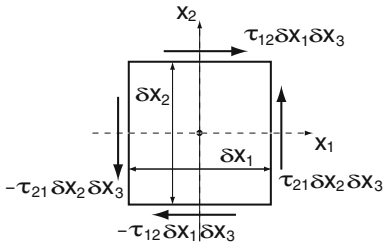

Fig. 1.3 Surface force due to tangential stress acting on a small parallelepiped parallel to the coordinate axes (cross section perpendicular to the x_3-axis)

The balance of the remaining $O((\delta\ell)^3)$ term becomes the fluid equation of motion, which will be discussed later.

A stress tensor is a symmetric tensor; in other words, $\tau_{ij} = \tau_{ji}$. This can be shown from the moment balance of the surface forces around the central axis of the small parallelepiped such as the one in Fig. 1.3, $\delta x_2(\tau_{12}\delta x_1\delta x_3) = \delta x_1(\tau_{21}\delta x_2\delta x_3)$. This is because even in this case, if the size of the parallelepiped is infinitesimally reduced, the angular momentum term and moment term due to external forces can be ignored relative to the moments of the surface forces. In this case, normal stress does not contribute to the moment, thus was omitted in Fig. 1.3.

Now, suppose that the stress acting on the surface with normal vector $\boldsymbol{n}$ satisfies the following relation:

$$\boldsymbol{T}(\boldsymbol{n}) = \lambda \boldsymbol{n}\,. \tag{1.12}$$

In this case, only a normal stress of magnitude $|\lambda|$ acts on its surface. If (1.12) is rewritten using (1.10), it becomes

$$\tau_{ij} n_j = \lambda\, \delta_{ij} n_j\,. \tag{1.13}$$

δ_{ij} is the Kronecker delta and defined as $\delta_{11} = \delta_{22} = \delta_{33} = 1$, $\delta_{ij} = 0$ $(i \neq j)$. (1.13) is an eigenvalue problem for the symmetric matrix τ_{ij}, and these are three real eigenvalues τ_1', τ_2', τ_3', if multiplicities are counted. The corresponding eigenvectors $\boldsymbol{e}_1'$, $\boldsymbol{e}_2'$, $\boldsymbol{e}_3'$ can be selected to be mutually orthogonal to form a right-handed system. If these eigenvectors are selected as the basis of the new Cartesian coordinate system $x_1'x_2'x_3'$, only normal stress acts on each of the surfaces perpendicular to the coordinate axes. Hence, the representation matrix of the stress tensor in the new system becomes

$$\mathrm{diag}\,(\tau_1',\ \tau_2',\ \tau_3')\,. \tag{1.14}$$

Here $\mathrm{diag}(a, b, c)$ is the diagonal matrix with diagonal components a, b, c, and $\tau_1', \tau_2', \tau_3'$ are called the **principal stresses**. Moreover, these new coordinate axes are called the principal axes of the stress tensor.

As is commonly known, the sum of the diagonal components of the matrix τ_{ij} remains unchanged with respect to the rotation of the Cartesian coordinate system.[7] In other words,

$$\tau_{ii} \equiv \tau_{11} + \tau_{22} + \tau_{33} = \tau_1' + \tau_2' + \tau_3' \tag{1.15}$$

holds true. Furthermore, in the principal axes system, the stress acting through the surface element with normal vectors (n_1', n_2', n_3') is

$$(\tau_1' n_1',\ \tau_2' n_2',\ \tau_3' n_3')\,. \tag{1.16}$$

Exercise 1.1 Show that $\delta S_j = (\boldsymbol{n} \cdot \boldsymbol{e}_j)\delta S$ holds true for the tetrahedron in Fig. 1.2.

1.2.3 Stress Tensors in Stationary Fluids

Now, let us characterize fluids using stress. The characteristic of fluids is that they easily deform relative to solids; however, they resist compression. Hence, a fluid is defined as a substance that cannot resist any type of action trying to deform it without change in volume. The stress tensor (1.14) expressed in the principal axes system will be decomposed into the following two terms (see Appendix A.1):

$$\mathrm{diag}\left(\frac{1}{3}\tau_{ii},\ \frac{1}{3}\tau_{ii},\ \frac{1}{3}\tau_{ii}\right) \tag{1.17}$$

and

$$\mathrm{diag}\left(\tau_1' - \frac{1}{3}\tau_{ii},\ \tau_2' - \frac{1}{3}\tau_{ii},\ \tau_3' - \frac{1}{3}\tau_{ii}\right). \tag{1.18}$$

The contribution of the tensor (1.17) to the stress acting through the surface element with normal vector $\boldsymbol{n}'$ is $(\tau_{ii}/3)\boldsymbol{n}'$. That is, regardless of which way the surface faces, only the normal stress of the same magnitude operates. This type of tensor is called an isotropic tensor. The components of (1.17) can be expressed, using the Kronecker delta, as $(\tau_{ii}/3)\,\delta_{ij}$; actually, any second-order isotropic tensor can be written in the form of a scalar multiple of δ_{ij} (see Appendix A.1). Tensor (1.17) corresponds to states such as that shown in Fig. 1.4. Fluids resist this type of compression (usually $\tau_{ii}/3 < 0$ in fluids) and can maintain a stationary state. On the other hand, (1.18) represents a difference from the isotropic stress tensor, and the sum of the diagonal elements turns out to be 0 from (1.15). Therefore, at least one of the diagonal elements is positive, and one is negative, and is a stress state that tries to purely deform the fluid element without changing volume, as shown in Fig. 1.5. Fluids are unable to resist this stress state. Since body forces are infinitesimal of higher order relative to the surface forces, body force cannot resist this stress state.

[7] Can be obtained from the invariance against the coordinate system rotation of the characteristic polynomial obtained from (1.13).

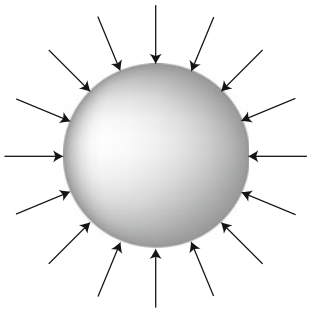

Fig. 1.4 Isotropic compression

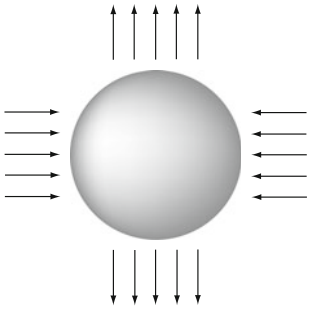

Fig. 1.5 Stress state that tries to deform without changing volume

Therefore, if this sort of stress state exists, fluids cannot maintain the stationary state. Hence, when fluids are stationary,

$$\tau_1' = \tau_2' = \tau_3' = \frac{1}{3}\,\tau_{ii} \tag{1.19}$$

must hold. As a result, in a stationary fluid, the stress tensor can be expressed as

$$\tau_{ij} = -\,p\,\delta_{ij}\,, \tag{1.20}$$

where $p\ (= -\tau_{ii}/3)$ is a function of $\boldsymbol{r}$ only and is called **pressure**.

It is customary, as (1.20), to insert a negative sign on the right-hand side. This is because fluids are usually under a pressured state, and hence, p is positive under the pressured state. (1.20) implies that the force per unit surface area through the surface element with a normal vector $\boldsymbol{n}$ is $-p\boldsymbol{n}$. Moreover, this pressure p in the stationary fluid is thermodynamic pressure (represent this as p_e) itself.

Now, suppose that tangential stress is always zero regardless of whether it is in a stationary state or in motion. Then normal stress is constant regardless of the selection of the surface. In other words, even in this situation, the stress tensor is in the form (1.20). This is because if (1.16) is to be parallel to an arbitrary normal vector $\boldsymbol{n}'$, (1.19) needs to be true.

Generally, a fluid in motion generates tangential stress through viscosity (internal friction). For example, a fluid flowing along a fixed plate exerts tangential stress in the direction of the flow on the plate, and the plate exerts tangential stress on the fluid in the direction opposite the flow. This type of fluid that generates tangential stress through viscosity is called a **viscous fluid**. In contrast, a hypothetical fluid that does not generate tangential stress even in a state of motion is considered and is called an **inviscid** or **perfect fluid**. In a perfect fluid, the stress tensor is in the same form (1.20) as a stationary fluid even when in motion. In other words, in perfect fluids, only isotropic pressure acts as stress. However, generally, the value of pressure p when fluids are in motion is different from that when they are stationary. Usually, in perfect fluids, it is assumed that a locally thermal equilibrium is set up, and hence, the pressure $p(\boldsymbol{r}, t)$ in (1.20) is equivalent to the local thermodynamic pressure $p_e(\boldsymbol{r}, t)$.

Generally, the reason for considering such a hypothetical fluid despite viscosity acting on fluids is because the mathematical treatment of this fluid is far easier than that of viscous fluids; furthermore, flows in the hypothetical fluid frequently express real flows with good approximation.

Note 1.2 When the stress tensor is in the form (1.20), p can be referred to as **hydrostatic pressure**. However, if so, even the pressure of the perfect fluid in relative motion expressed in exactly the same form has to also be referred to as hydrostatic pressure and is inconvenient. Hence, in this book, out of the pressure in (1.20), the words hydrostatic pressure will only be used to refer to the pressure of fluids that are stationary under the influence of external forces (usually gravity).

The (mechanical) pressure on viscous fluids in relative motion is defined by a form that is an extension of (1.20), as shown in Sect. 1.7. This pressure is generally not equal to thermodynamic pressure; however, the difference can be ignored in the phenomenon discussed in this book. Therefore, beyond Chap. 2, all pressure may be considered to be equal to locally thermodynamic pressure.

Exercise 1.2 Water is stationary in a uniform gravitational field. Suppose that atmospheric pressure p_0 is acting on the water surface, obtain the pressure at depth h from the water surface. Here, water density ρ is assumed to be fixed.

1.3 Specifications of Fluid Motions

1.3.1 Lagrangian and Eulerian Specifications

There are two ways of describing fluid motion: the **Lagrangian specification** and the **Eulerian specification**.

In the Lagrangian specification, the movement over time of each point (we will refer to these as **fluid particles**)[8] constructing the fluid as a continuum, as fluid is in

[8]The word fluid particle is often used to indicate an infinitesimal fluid element. Here it is considered to be a "point" that makes up the fluid substance as a continuum.

motion, are investigated. To perform this, it is convenient to mark the fluid particles. The mark can be anything as long as they can be individually identified; however, it is usually the position vector of each fluid particle $\boldsymbol{r}_0 = (x_0, y_0, z_0)$ at initial time $t = 0$.[9] If the fluid particle position vector (position coordinate) at time t is $\boldsymbol{r} = (x, y, z)$, then

$$\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{r}_0, t) \tag{1.21}$$

where $\boldsymbol{r}_0 = \boldsymbol{r}(\boldsymbol{r}_0, 0)$. (1.21) shows that, in this specification, fluid motion can be observed as one of the point transformations with time t as a parameter.

In this specification, the fluid particle's initial position vector $\boldsymbol{r}_0$ and time t are independent variables, and its position vector $\boldsymbol{r}$ is one of the dependent variables. Fixing $\boldsymbol{r}_0$ and partially differentiating $\boldsymbol{r}$ with respect to t means focusing on the fluid particle initially present at $\boldsymbol{r}_0$ and obtaining the rate of change over time of that position vector, thereby calculating the velocity of the fluid particle $\boldsymbol{u} = (u, v, w)$. If this is partially differentiated with respect to t once more, then acceleration is obtained. In other words,

$$\boldsymbol{u} = \frac{\partial \boldsymbol{r}}{\partial t}, \quad \text{or} \quad (u,\ v,\ w) = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right), \tag{1.22}$$

$$\boldsymbol{a} = \frac{\partial \boldsymbol{u}}{\partial t} = \frac{\partial^2 \boldsymbol{r}}{\partial t^2}. \tag{1.23}$$

Generally, to know the flow, not only the change over time of the fluid particle position but also the density $\rho(\boldsymbol{r}_0, t)$, pressure $p(\boldsymbol{r}_0, t)$, and temperature $T(\boldsymbol{r}_0, t)$ that accompany each particle are required.

In the Eulerian specification, the spatial coordinates $\boldsymbol{r} = (x, y, z)$ and time t are independent variables, and velocity $\boldsymbol{u} = (u, v, w)$ and density ρ, etc. are dependent variables; they can be expressed as

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{r}, t)\,. \tag{1.24}$$

Moreover, if t is fixed, the state of the instantaneous whole flow field becomes apparent, and if $\boldsymbol{r}$ is fixed, the time development of the flow at that point becomes apparent. Generally, one fluid particle after another passes through this one point over time. $\boldsymbol{u}(\boldsymbol{r}, t)$ is called the velocity field, and $\rho(\boldsymbol{r}, t)$ can be called the density field.

Even in the Eulerian specification, there are often situations when, focusing on one fluid particle, time change of the physical quantities of that particle becomes necessary. Time derivative showing the change over time focusing on fluid particles in this way is called the **Lagrangian derivative** or **material derivative** and is denoted as D/Dt.[10] In the Lagrangian specification, this is denoted as $\partial/\partial t$.

[9] Sometimes, $\boldsymbol{r}_0 = (x_0, y_0, z_0)$ can be referred to as a material coordinate.

[10] Can also be called a substantial derivative.

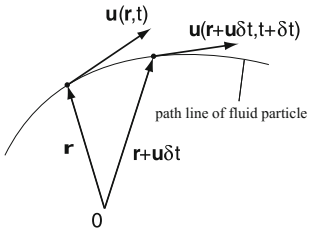

Fig. 1.6 Velocity of fluid particle at time t and $t + \delta t$

Now, let us consider the physical quantity F. In the Eulerian specification, it is a function of $\boldsymbol{r} = (x, y, z)$ and t. If the fluid particle presently under consideration is at position $\boldsymbol{r} = (x, y, z)$ at time t, then after an infinitesimal time δt at time $t + \delta t$, that fluid particle will be at position $\boldsymbol{r} + \boldsymbol{u}\delta t = (x + u\delta t, y + v\delta t, z + w\delta t)$ (Fig. 1.6). Therefore, the increment δF of F in that time is

$$\begin{aligned}\delta F &= F(x + u\delta t, y + v\delta t, z + w\delta t, t + \delta t) - F(x, y, z, t) \\ &= \frac{\partial F}{\partial t}\delta t + \frac{\partial F}{\partial x}u\delta t + \frac{\partial F}{\partial y}v\delta t + \frac{\partial F}{\partial z}w\delta t + O((\delta t)^2)\,.\end{aligned}$$

Hence, the rate of F's change when focusing on specific fluid particle is

$$\frac{DF}{Dt} = \lim_{\delta t \to 0}\frac{\delta F}{\delta t} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z}\,. \tag{1.25}$$

The physical quantity F can be arbitrary; hence, if it is expressed as a differential operator, it becomes

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + u_j\frac{\partial}{\partial x_j} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\,, \tag{1.26}$$

where ∇ expresses the differential operator:

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} \tag{1.27}$$

and is called the nabla.

In particular, if x, y, z are taken as F, then the following relation holds

$$\frac{Dx}{Dt} = \frac{\partial x}{\partial t} + u\frac{\partial x}{\partial x} + v\frac{\partial x}{\partial y} + w\frac{\partial x}{\partial z} = u,\quad \frac{Dy}{Dt} = v,\quad \frac{Dz}{Dt} = w\,.$$

Hence, the result can clearly be expressed as expected by

$$\frac{D\boldsymbol{r}}{Dt} = \boldsymbol{u}\,. \tag{1.28}$$