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Jacques Sesiano

Magic Squares in the Tenth Century

Two Arabic Treatises by Anṭākī and Būzjānī

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Part I

Tenth-century construction methods

Introduction

§1. General notions on magic squares

A *magic square* is a square divided into a square number of cells in which natural numbers, all different, are arranged in such a way that the same sum is found in each horizontal row, each vertical row, and each of the two main diagonals.

A square with n cells on each side, thus n^2 cells altogether, is said to have the *order* n. The constant sum to be found in each row is called the *magic sum* of this square.

Usually what is written in a square of order n, thus with n^2 cells, are the first n^2 natural numbers. Since the sum of all these numbers equals

$$\frac{n^2\,(n^2+1)}{2},$$

the sum in each row, thus the magic sum for such a square, will be

$$M_n = \frac{n(n^2+1)}{2}.$$

I. A square displaying this magic sum in the $2n+2$ aforesaid rows is an *ordinary magic square* (Fig. 1). It meets the minimal number of required conditions, and such a square can be constructed for any given order $n \geq 3$ (a magic square of order 2 is not possible with different numbers).

1	**31**	**22**	**15**	**30**	**12**
2	**32**	**14**	**23**	**29**	**11**
3	**33**	**24**	**13**	**28**	**10**
34	**4**	**16**	**21**	**9**	**27**
35	**5**	**17**	**20**	**8**	**26**
36	**6**	**18**	**19**	**7**	**25**

Fig. 1

But there are magic squares which display further properties.

II. A *bordered magic square* is one where removal of its successive borders leaves each time a magic square (Fig. 2). With an odd-order square, after removing each (odd-order) border in turn, we shall finally reach the smallest possible square, that of order 3. With an even-order square, that will be one of order 4 (the border cannot be removed since there is no magic square of order 2). For order $n \geq 5$, bordered squares are always possible.

92	17	4	95	8	91	12	87	16	83
99	76	31	22	77	26	73	30	69	2
1	20	64	41	36	63	40	59	81	100
3	19	67	58	47	51	46	34	82	98
96	80	33	52	45	57	48	68	21	5
7	78	35	49	56	44	53	66	23	94
90	27	62	43	54	50	55	39	74	11
13	72	42	60	65	38	61	37	29	88
86	32	70	79	24	75	28	71	25	15
18	84	97	6	93	10	89	14	85	9

Fig. 2

As seen above, the magic sum for a square of order n filled with the n^2 first natural numbers is

$$M_n = \frac{n(n^2+1)}{2}.$$

Clearly, the *average* sum in each case is $\frac{n^2+1}{2}$. Accordingly, for m cells, this average sum should be m times that quantity; this will be called the *sum due* for m cells. Thus, the mth border ($m \geq 5$) of a bordered square of order n will contain in each row its sum due, namely

$$M_n^{(m)} = \frac{m(n^2+1)}{2}$$

if the main square is filled with the n^2 first natural numbers. The sum in a row in one border will therefore differ from the next one by n^2+1. With this in mind, we clearly see the structure of bordered squares: the elements of the same border which are opposite (horizontally, vertically, or diagonally for the corner cells) are what we shall call *complements*, that is they add up to n^2+1 if the main square has the order n. For example, in each border of the above figure, the sum of opposite elements is 101.

III. In both the above cases the main diagonals must always contain the magic sum. But there are magic squares in which broken diagonals also make the magic sum.[1] Such squares are called *pandiagonal* (Fig. 3). They

[1] A broken diagonal is a pair of diagonal rows, on either side of and parallel to a main diagonal, comprising n cells altogether.

are possible for any odd order from 5 on, and for any even order *divisible by 4*, from 4 on. Thus, in this figure, each of the two broken diagonals starting with 22 —namely 22, 2, 43, ... , 33, 55 and 22, 38, 41, ... , 19— makes the sum 260.

11	22	47	50	9	24	45	52
38	59	2	31	40	57	4	29
18	15	54	43	20	13	56	41
63	34	27	6	61	36	25	8
3	30	39	58	1	32	37	60
46	51	10	23	48	49	12	21
26	7	62	35	28	5	64	33
55	42	19	14	53	44	17	16

Fig. 3

In such squares any lateral row can be moved to the other side: the main diagonals of the new square will be magic since they were already magic as broken diagonals. By repeating such vertical and/or horizontal moves we are able to place any element in any given cell of the square.

IV. Even more particular are the *composite* squares: the main square comprises subsquares which, taken individually, are also magic (Fig. 4). The possibility of such an arrangement depends on the divisibility of the order of the main square.

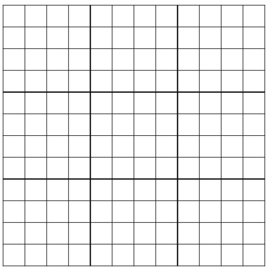

Fig. 4

By *general construction methods* we mean ways of directly constructing a square of given order and given type, that is, belonging to one of the aforementioned types: once the empty square is drawn, a few, easily remembered instructions will enable us to place the sequence of consecutive numbers without any computation or recourse to trial and error.

Now, considering one of the above types of magic square, there exists no general method applicable to any order. General methods are applicable at most to one of three categories of order, which are:

— The squares of *odd* orders, also called *odd squares*, thus with $n = 2k+1$, of which the smallest is the square of order 3.

— The squares of *evenly even* orders, also called *evenly even squares*, thus with $n = 4k$, of which the smallest is the square of order 4.

— The squares of *evenly odd* orders, also called *evenly odd squares*, thus with $n = 4k + 2$, of which the smallest is the square of order 6.

These methods are, by definition, applicable whatever the size of the order; but they may require some adapting for squares of lower orders, like 3 or 4, sometimes also 6 and 8.

Remark. Speaking of different *methods* suggests that a square of given order may take different aspects. As a matter of fact, the number of possible configurations (excluding mere inversions and rotations of the square) rapidly increases with the size of the order. Whereas there is just one form of the magic square of order 3, there are already 880 for the order 4, as discovered in 1693.[2]

The beginning of the science of magic squares is unknown. No *extant* ancient Greek text deals with magic squares, no allusion whatsoever is made to them in antiquity, and the only Greek text preserved is a Byzantine writing by Manuel Moschopoulos (ca. 1300), who appears to draw his knowledge from some Arabic or Persian text.[3] Furthermore, direct allusions to Greek studies by Arabic authors seem fanciful. The earliest magic square appears in China at the beginning of our era, but it is a square of order 3; higher order squares do not occur there before the 12th century, and are clearly of Arabic or Persian origin. The same holds for Indian magic squares.[4]

In the 7th century the game of chess arrived in Persia from India, where it had been invented sometime before. It appears to be rapidly connected with the construction of magic squares: the early Arabic treatises often describe the successive placing of numbers in small-order squares by means of chess moves.[5] General construction methods existed already by the middle of the 10th century, as attested by our two texts. But they are limited to bordered and composite squares: ordinary magic squares are constructed only for small orders (3, 4, 5, 6), and by methods applicable to *one order only*. (This may be the reason for the early appearance of

[2] By Frénicle de Bessy in his *Table generale des quarrez de quatre*.

[3] Moschopoulos' text has been edited and translated by Tannery; on its origin, see our study of it.

[4] General survey in our *Les carrés magiques*, pp. 8–10 (Russian edition, pp. 15–18).

[5] For our texts, see $\mathcal{A}$.II.37 and $\mathcal{B}$.4, $\mathcal{B}$.5, $\mathcal{B}$6, $\mathcal{B}$.8*vi*, $\mathcal{B}$.9*iii*, $\mathcal{B}$.15*i*, $\mathcal{B}$.26*iii*.

composite squares, as an attempt to reduce the construction of larger, non-bordered squares to that of smaller-order ones.) But the most elaborate result obtained by that time is the construction of bordered squares where the odd numbers appear separated from the even ones, the former being grouped together in a rhomb within the main square. No later treatise till modern times will ever match such a stage of development. Separating in a similar way the numbers according to parity will occur in the 11th century for ordinary magic squares; but there the construction is much simpler. This at least shows that separation by parity has been of permanent interest.[6]

There will be further development in the 11th and early 12th century, with the discovery of general methods for ordinary magic squares, by far the simplest way to construct magic squares. Various constructions of pandiagonal squares also emerge, although what is then seen as characteristic is not the magic property of all diagonals but the possibility of choosing the place of a given number within the square, a mere consequence of the pandiagonality, as already mentioned (p. 5). Squares with non-consecutive numbers also begin to appear. The origin of this has to do with the association of Arabic letters with numerical values —an adaptation of the Greek numerical system (Fig. 5), also used in early Islamic times, before the adoption of the Indian numerals. Since to each letter of a word, or to each word of a sentence, was attributed a numerical quantity, thus to the word or the sentence a certain sum, this word or sentence could be placed as such in the cells of a certain row of a square. The task was then to complete the square numerically so that it would display in each row the sum represented by the word or sentence —a mathematically interesting problem since this is not always possible.[7]

Later centuries in Islamic countries saw mainly improvements and refinements of existing methods. Meanwhile, however, popular use of magic squares as talismans grew apace. Authors of many shorter texts are just not interested in the mathematical properties or, even, the construction of magic squares, their purpose being purely practical if not commercial. They present a few magic squares, most commonly squares of the orders 3 to 9 associated with the seven then known planets (including Moon and Sun) of which they embodied the respective, good or evil, qualities —fully described in these texts. The reader is taught to draw one of these squares on some appropriate material (the nature of which can increase

[6] On these latter constructions, see *Les carrés magiques*, pp. 33–35 (Russian edition, pp. 42–44).

[7] *Les carrés magiques*, pp. 182–220 (Russian edition, pp. 195–233).

its efficacy), if possible at some astrologically predetermined time (which also increases the effect); he will then put it in or near something belonging to the chosen person, beneficiary or victim.[8] It was the arrival of such texts in late mediaeval Europe which first aroused interest in, and study of, such squares there and also gave rise to use of the term ‘magic’ —formerly also ‘planetary’, which we find still employed by Fermat.[9]

$\overline{\alpha}$ ا 1	$\overline{\beta}$ ب 2	$\overline{\gamma}$ ج 3	$\overline{\delta}$ د 4	$\overline{\varepsilon}$ ه 5	ϛ̄ و 6	$\overline{\zeta}$ ز 7	$\overline{\eta}$ ح 8	$\overline{\vartheta}$ ط 9
$\overline{\iota}$ ى 10	$\overline{\varkappa}$ ك 20	$\overline{\lambda}$ ل 30	$\overline{\mu}$ م 40	$\overline{\nu}$ ن 50	$\overline{\xi}$ س 60	$\overline{o}$ ع 70	$\overline{\pi}$ ف 80	ϟ̄ ص 90
$\overline{\rho}$ ق 100	$\overline{\sigma}$ ر 200	$\overline{\tau}$ ش 300	$\overline{\upsilon}$ ت 400	$\overline{\varphi}$ ث 500	$\overline{\chi}$ خ 600	$\overline{\psi}$ ذ 700	$\overline{\omega}$ ض 800	ϡ̄ ظ 900
$,\overline{\alpha}$ غ 1000								

Fig. 5

We thus see that ‘magic’ applications do not seem to have played a significant rôle at the outset. The magic square (*wafq al-a‘dād*: ‘harmonious disposition of numbers’, as it was then called) was originally studied as a branch of number theory, just as were the perfect and amicable numbers so highly esteemed by Pythagoreans and their followers in late antiquity. In short, such research was at the time considered as being perfectly serious and had not yet been tainted with the sad reputation later associated with their practical use. We are nevertheless left with the question of when actual methods of construction first appeared: the two 10th-century texts to be studied here, paradoxically, far from providing an answer, merely complicate matters: at the same time, we witness, reproduced by a very competent author, the first steps in the study of magic squares (this is our text $\mathcal{B}$) and, reported by a mathematician of unequal value, some general methods but also that highly elaborate method never to be surpassed later (this is our text $\mathcal{A}$). In short, the tenth century, for the history of magic squares, gives the impression of being both a beginning and an end.

[8] Typical examples in our *Magic squares for daily life*.
[9] *Varia opera mathematica*, p. 176; *Œuvres complètes*, II, p. 194.

§2. The two texts from the tenth century

These two texts are, as just hinted at, very different. One, $\mathcal{B}$, is very didactic —indeed, its author explains very well how to construct the magic squares he presents, and mostly justifies his manner of proceeding. It is perfectly adequate for a beginner. Its only weakness is that the knowledge it presents is embryonic: it teaches just *one* general construction method, namely for odd-order bordered squares. The other text, $\mathcal{A}$, is of little use to the reader. For the bordered magic squares of the first orders, it merely indicates the place of each number, without justification and rarely hinting at general procedures; still, the attentive reader will deduce them and thus be able to construct bordered squares of any possible order. On the other hand, $\mathcal{A}$ is unique in that it attests to the surprising results already obtained in the science of magic squares by the tenth century, thereby suggesting a much earlier, possibly Greek time for the first discoveries. These two treatises thus complete each other; they are certainly the most important testimonies on the early history of that science, when it was still in the hands of mathematicians and not yet reduced to popular interest or superstition.

A. Text $\mathcal{A}$

1. Author and manuscript

The University of Ankara, or, more precisely its Faculty of Language, History and Geography (Dil ve Tarih-Coğrafya Fakültesi), was donated by a well-read book collector, Ismail Saib Sencer (d. 1940), more than ten thousand manuscripts divided into two collections. Among the first, one finds the manuscript I, 5311, of 84 leaves, written on paper in the 13th century (7th century of the hegira). It contains ten treatises, all on mathematical subjects (geometry, algebra, magic squares), and copied by the same hand. The first of these (fol. 1^r–36^r), by ʿAlī ibn Aḥmad al-Anṭākī, is our text $\mathcal{A}$. There are between 18 and 24 lines on a page, mostly 20 or 21. The copyist sometimes used red ink, namely for some headings, numerical symbols and framing the figures; he did it later on, and that is why one correction is in red (*l.* 863 in our edition of the text) and some spaces left for numbers were insufficient (*ll.* 1165, 1221, 1317). The writing (*naskhī*) is quite legible, except in some worm-eaten places. The copy is on the whole in good condition, and the copyist is responsible for only a few textual omissions which he partly or completely remedied upon rereading. We may, though, remark that there is some disorder in the Arabic text on the leaves 18 and 19, and that only half of fol. 18^r contains text. It appears that there has been an error in an earlier copy, just reproduced by the present copyist, together with, in red ink, the explanation (fol. 18^{v}, *ll.* 830–831 in our Arabic text): *This part is left blank for the figure, where K is (written) in red on the left-hand page*; now the place referred to is not found in the present copy.

In this copy at least, the text did not arouse much interest: the margins contain no readers' observations, merely, in the copyist's own hand, words or sentences originally omitted. The only extraneous annotations are in a few figures of squares, and are of limited interest, such as calculating the number of cells in a 12×12 square. The ancestors of our manuscript had more success: notes, once written in the margin by readers, have then been incorporated into the text, though mostly in an inappropriate place. But such interventions are mostly devoted to the less difficult parts; the others prompted no comments.

Such is the form in which text $\mathcal{A}$ is preserved. As for its author, al-Anṭākī, we know that he lived in the tenth century, for Ibn al-Nadīm (the writer of a bibliographic work compiled 987/8, known by the name of *Fihrist*, and our main source on the authors of the first three centuries of the hegira) gives the time of his death as 376 of the hegira, thus 987; he also reports the titles of some of his works, including three on reckoning, one on the problem of weights,[10] one on 'the cubes', a commentary on Euclid,[11] and his 'commentary on the Arithmetic' (*tafsīr al-ari<u>th</u>māṭīqī*), that is, on the *Introduction to Arithmetic* by Nicomachos.[12] Only this last work, our $\mathcal{A}$, is known to survive (though incompletely) today.

Opinions about al-Anṭākī's ability as a mathematician are, let us put it this way, somewhat divergent. While the biographer Ibn al-Qifṭī (d. 1248/9) praises it highly, the mathematician al-Nasawī, in the late 10th century, finds one of his writings unclear and verbose.[13] Our text will attest to the plausibility of such a discrepancy: it gives the impression that we have here, on the one hand, a rather weak mathematician (parts I and III) and, on the other hand (in some of part II), a superior mind who very concisely, but correctly, explains the difficult construction of special magic squares (see in particular $\mathcal{A}$.II.12–$\mathcal{A}$.II.35, $\mathcal{A}$.II.44–$\mathcal{A}$.II.54). In such cases, it may safely be assumed that the text reflects the source

[10] Possibly determination of the least number of weights needed to weigh a given integral weight; see our *Récréations mathématiques au Moyen Âge*, pp. 39–47.

[11] His knowledge of Euclid's *Elements* is evident from the first part of the present text; but his comprehension of Euclid seems rather weak.

[12] *Fihrist*, vol. I, p. 284; Suter, *Mathematiker*, p. 64; Sezgin, *Geschichte*, V, p. 310 (and VII, *Nachträge*, p. 407); Woepcke, *Propagation*, p. 493 (or *Etudes*, II, p. 425).

[13] A treatise of his on reckoning is *confus et d'une longueur excessive* (Woepcke, *ibid.*).

from which the author takes his information. This is, for the history of magic squares, excellent news: the most difficult subjects must be transmitted by $\mathcal{A}$ unchanged, as he found them, and thus reproduce what was already known much earlier since he is unlikely to have plagiarized his contemporaries.

The text presented and analyzed here is, as said above, not the whole of al-Anṭākī's commentary on Nicomachos' *Introduction to Arithmetic*. For, as indicated by the title, that is only its *third book*, thus the third part of it, and also the last since the colophon clearly indicates that the work ends there. Since $\mathcal{A}$ comments on Nicomachos' *Introduction to Arithmetic*, and Thābit ibn Qurra's Arabic translation is quite faithful to the Greek original, we shall give a brief outline of it. That will enable us to note points of convergence or divergence, if any, between the original text and its commentary.[14] The reader will keep in mind that, according to its Greek meaning, *arithmetic* has to do with what we could call today number theory, and *not* arithmetical calculations.

2. The *Introduction to arithmetic* by Nicomachos

Nicomachos of Gerasa is a relatively late Greek mathematician (end of the 1st century). His *Introduction to Arithmetic* (Ἀριθμητικὴ Εἰσαγωγή) was widely known during late antiquity, and some Greek commentaries on it have survived: those by Iamblichos, Asclepios of Tralles, J. Philoponos, and Proclos. The *Introduction* had a notable influence later as well. It was first translated into Syriac, then, in the 9th century, into Arabic by Thābit ibn Qurra. Its rôle in mediaeval Europe is important as well. There it was, until the 12th century, the only surviving work of ancient Greek mathematics, through an adaptation by Boëtius (who, in the early 6th century partly summarized the original text and added remarks).[15] In short, by means of Nicomachos' book, ancient Greek, Byzantine, Arabic and mediaeval European readers could learn some fundamentals of number theory.

This notoriety does not go, though, hand in hand with the mathematical level. It is in no way comparable to that of classical authors like Euclid (around −300) and Archimedes (around −250) —or also Diophantos (who, unlike the two others, lived after Nicomachos). With Nicomachos, mathematics seems to take a step backwards: an opinion becomes

[14] We adopt the division of the text used by the modern editor of the Greek text, Hoche; the editor of the Arabic translation, Kutsch, refers page by page to Hoche's edition.

[15] A reported earlier Latin translation by Apuleius is not preserved.

a theorem and a test serves as a demonstration. A typical example is the subject of perfect numbers, which are numbers equal to the sum of their divisors (excluding the number itself). Euclid had demonstrated that, if we take consecutive powers of 2 from 1 on, thus 1, 2, 2^2, …, 2^{n-1}, and their sum, namely 2^n-1, happens to be a prime number, then its product with the last term added, thus $2^{n-1}(2^n-1)$, will be a perfect number. We are informed through Nicomachos that the Greeks knew (at least) the first four ones: 6 $(=1+2+3)$, 28 $(=1+2+4+7+14)$, 496, 8128. Now, observing that each of them occurs between two consecutive powers of 10, and that they end alternately in 6 and 8, Nicomachos asserted that *all perfect numbers* must belong to one, and only one, of the successive decimal orders and must obey the same alternance for the final digit. Of all this, it is only true that they will end (but not alternately) by 6 and 8; as to decimal scale —which has no other origin than our ten fingers— it cannot have any relation to a property inherent in number such as divisibility. In short, Nicomachos' *Introduction* though undoubtedly of use to anyone wishing to acquire elementary notions of number theory, will also delude others who expect thereby to be trained in mathematical thinking. It embodies perfectly what would be called today 'democratization of knowledge'.

Book I begins with some philosophical thoughts. It proceeds with the classification of natural numbers according to kinds of parity. Among odd numbers, there are the *prime and incomposite* (our 'primes') and the *secondary and composite* (our 'composite'); the numbers *secondary and composite in themselves but prime and incomposite with others* are our 'composite and relatively prime'. The even numbers may be *evenly even*, *evenly odd*, *oddly even*, according to whether they have the form 2^n, $2(2k+1)$, $2^m(2k+1)$ (with $m \geq 2$). More practical, perhaps also more useful since becoming thus widespread, are the description of the sieve of Eratosthenes (around -250) for finding the sequence of consecutive prime numbers —of which Euclid had already demonstrated the infinite quantity in his *Elements* (proposition IX, 20)— and how to find the greatest common divisor of two numbers (*Elements*, VII, 1), if any —it will be 1 if the two numbers are relatively prime. Then it defines *abundant*, *deficient* and *perfect* numbers, the sums of the divisors of which (excluding the number itself) are, respectively, greater than, less than, equal to, the number considered. Book I ends with a long section, somewhat tedious and irrelevant to the present study, on the different kinds of numerical ratios.

Book II discusses in particular geometric progressions. One then finds the theory of (so-called) figured numbers, first plane then solid (a natural number is represented, according to the quantity of its units, by dots arranged so as to represent a regular geometric figure), and the properties inferred thereby. It is also observed that, considering the sequence of consecutive odd numbers starting with 1, the first, the sum of the next two, the sum of the next three, and so on, give the sequence of consecutive integral cubes. Book II ends with an enumeration of the different kinds of mean proportionals —another somewhat tedious conclusion.

3. Contents of text $\mathcal{A}$

Book III of al-Anṭākī's Commentary consists of three distinct parts without any connection between them.

Part $\mathcal{A}$.I. It consists of an enumeration of definitions, statements of propositions and identities. Definitions and propositions are chiefly taken from Euclid's *Elements*, mostly from the so-called *arithmetical Books* since they have more to do with number theory than geometry:

— from Book VII are taken definitions, namely 8 ($\approx \mathcal{A}$.I.6), 9 ($\approx \mathcal{A}$.I.5′), 11 ($\approx \mathcal{A}$.I.7′), 12 ($\mathcal{A}$.I.1), 15 ($\mathcal{A}$.I.3), 16 ($\mathcal{A}$.I.31), 18 ($\mathcal{A}$.I.90); and propositions, namely 2 corollary ($\mathcal{A}$.I.19), 4 ($\mathcal{A}$.I.21), 5 ($\mathcal{A}$.I.22), 6 ($\mathcal{A}$.I.23), 7 ($\approx \mathcal{A}$.I.26), 8 ($\approx \mathcal{A}$.I.27), 9 ($\mathcal{A}$.I.24), 10 ($\mathcal{A}$.I.25), 11 ($\mathcal{A}$.I.28), 14 ($\mathcal{A}$.I.29), 17-18 ($\mathcal{A}$.I.32), 23 ($\mathcal{A}$.I.33), 24 ($\mathcal{A}$.I.34), 25 ($\mathcal{A}$.I.35), 27 ($\approx \mathcal{A}$.I.36), 29 ($\mathcal{A}$.I.38), 30 ($\mathcal{A}$.I.39), 32 ($\mathcal{A}$.I.37), 34 ($\approx \mathcal{A}$.I.40 & 48), 35 ($\mathcal{A}$.I.50), 36 ($\mathcal{A}$.I.49), 37 ($\mathcal{A}$.I.51);

— from Book VIII are taken propositions 1 ($\mathcal{A}$.I.52), 2 cor. ($\mathcal{A}$.I.53), 3 ($\mathcal{A}$.I.54), 5 ($\mathcal{A}$.I.55), 7 ($\mathcal{A}$.I.56), 8 ($\mathcal{A}$.I.57), 9 ($\mathcal{A}$.I.58), 11 ($\mathcal{A}$.I.59), 12 ($\mathcal{A}$.I.61), 13 ($\mathcal{A}$.I.63), 14 ($\mathcal{A}$.I.64), 15 ($\mathcal{A}$.I.66), 16 ($\mathcal{A}$.I.65), 17 ($\mathcal{A}$.I.67), 18 ($\mathcal{A}$.I.68), 19 ($\mathcal{A}$.I.69), 20 ($\mathcal{A}$.I.70), 21 ($\mathcal{A}$.I.71), 22 ($\mathcal{A}$.I.72), 23 ($\mathcal{A}$.I.73), 24 ($\mathcal{A}$.I.75), 25 ($\mathcal{A}$.I.76), 26 ($\mathcal{A}$.I.77), 27 ($\mathcal{A}$.I.78);

— finally, from Book IX, propositions 1 ($\mathcal{A}$.I.81), 2 ($\mathcal{A}$.I.82), 4–5 ($\mathcal{A}$.I.86), 8 ($\mathcal{A}$.I.91), 9 ($\mathcal{A}$.I.92), 10 ($\mathcal{A}$.I.93), 11 ($\mathcal{A}$.I.41), 12 ($\mathcal{A}$.I.42), 13 ($\mathcal{A}$.I.43), 15 ($\mathcal{A}$.I.45), 16 ($\mathcal{A}$.I.46), 17 ($\mathcal{A}$.I.47), 20 ($\mathcal{A}$.I.44), 24 ($\mathcal{A}$.I.8), 25 ($\mathcal{A}$.I.9), 26 ($\mathcal{A}$.I.10), 27 ($\mathcal{A}$.I.11), 28 ($\mathcal{A}$.I.5), 29 ($\mathcal{A}$.I.7), 30 ($\mathcal{A}$.I.13), 33 ($\mathcal{A}$.I.14), 35 ($\mathcal{A}$.I.95), 36 ($\mathcal{A}$.I.94).

We also find some propositions which are not in the original version of the *Elements* but occur in Greek commentaries (available in Arabic) on it: $\mathcal{A}$.I.12, $\mathcal{A}$.I.74, $\mathcal{A}$.I.80, $\mathcal{A}$.I.83–$\mathcal{A}$.I.85.

Also found, as $\mathcal{A}$.I.124–$\mathcal{A}$.I.133, are the first ten propositions of Book II of the *Elements*, which, though demonstrated geometrically there, in fact correspond to algebraic identities.[16]

Apart from these assertions, there is a set of arithmetical identities ($\mathcal{A}$.I.96–$\mathcal{A}$.I.123), some of which are taken from the algebraic books by al-<u>Kh</u>wārizmī, c. 820 ($\mathcal{A}$.I.96, $\mathcal{A}$.I.105) or Abū Kāmil, c. 880 ($\mathcal{A}$.I.105–$\mathcal{A}$.I.108, $\mathcal{A}$.I.111, $\mathcal{A}$.I.112, $\mathcal{A}$.I.115–$\mathcal{A}$.I.119, $\mathcal{A}$.I.123). When expressed in symbols, they are often banal —though less so for the mediaeval reader faced with them in their verbal form.

This first part sometimes casts doubts on al-Anṭākī's abilities. As pointed out in some of our notes, certain formulations are strange or nonsensical ($\mathcal{A}$.I.18, $\mathcal{A}$.I.20, $\mathcal{A}$.I.26, $\mathcal{A}$.I.115, $\mathcal{A}$.I.119), incomplete ($\mathcal{A}$.I.43, $\mathcal{A}$.I.92, $\mathcal{A}$.I.94) or imprecise ($\mathcal{A}$.I.41, $\mathcal{A}$.I.48, $\mathcal{A}$.I.60, $\mathcal{A}$.I.109, $\mathcal{A}$.I.128, $\mathcal{A}$.I.132); numerical examples are not well chosen ($\mathcal{A}$.I.29, $\mathcal{A}$.I.42, $\mathcal{A}$.I.95); definitions of terms do not appear together with the first use of these terms ($\mathcal{A}$.I.30, $\mathcal{A}$.I.55) or are omitted ($\mathcal{A}$.I.33, $\mathcal{A}$.I.55, $\mathcal{A}$.I.68, $\mathcal{A}$.I.69); a demonstration is wrong ($\mathcal{A}$.I.44), as well as two assertions ($\mathcal{A}$.I.87, $\mathcal{A}$.I.88). Finally, one identity ($\mathcal{A}$.I.117) has been taken without much thought since one condition in Abū Kāmil's problem has been repeated despite being wholly irrelevant to the identity. (We find a similar case in $\mathcal{A}$.I.134.)

Since the items in $\mathcal{A}$.I are, with rare exceptions, merely stated, that did not leave much room for the imagination of early readers; thus, the manuscript reproduces only a few interpolations ($\mathcal{A}$.I.17, $\mathcal{A}$.I.20, $\mathcal{A}$.I.35–$\mathcal{A}$.I.36, $\mathcal{A}$.I.111, $\mathcal{A}$.I.121, $\mathcal{A}$.I.132).

<u>Part $\mathcal{A}$.II</u>. By far more interesting for us is the second part, on the construction of magic squares. Its reading is difficult for two reasons: one of its subjects (the construction of bordered squares with separation by parity) is too advanced for the reader, and not only the average one; the other constructions of magic squares may be understood, but are rendered unnecessarily tedious because of their presentation. A 12th-century reader who had read several treatises on magic squares expresses his opinion as follows: *As for other (authors), among whom al-Anṭākī, I found that they (merely) said to put such and such a number in such and such a cell without explaining the reason for that, although this subject is difficult to understand and requires to be memorized.*[17] Reading al-Anṭākī's construction of usual bordered squares (without separation) fully confirms this reproach. It goes so far that not only does he list all places of lesser numbers but also those of their complements, which is superfluous since

[16] Their algebraic versions are commonly found in mediaeval textbooks; see note 193, p. 140.

[17] See our *Une compilation arabe*, p. 163, lines 239–241 of the Arabic text.

they are anyway placed opposite (see above, p. 4).

Part $\mathcal{A}$.III. The last subject is that of hidden numbers, that is, of guessing a number thought of by a person. The asker has him perform various computations on that number; from the result, communicated by the person, he will be able to find out what number the person had thought of. Of course, there must be several kinds of operation to prevent the person from discovering the link between the instructions and the final result. This part was by far the most successful, as may be inferred from early readers' traces, which often remedy the imprecision of the account. It is indeed not as tedious as the first and more accessible than the second; and its recreational and social purpose would naturally make it more appealing.

Comparing now the content of Nicomachos' work with that of al-Anṭākī's Commentary will lead to a surprising discovery, namely that there is no connection between the two works —at least not in Book III of al-Anṭākī's Commentary since that is all we know. $\mathcal{A}$.I, with its theorems and identities, recalls some other Arabic texts, which begin with such lists;[18] but then Euclid and not Nicomachos would rightly be indicated as the source for the propositions. As for $\mathcal{A}$.III, we know that such problems existed in antiquity —indeed, they left traces in an author of the early Middle Ages, namely Bede (c. 700)[19]— and sections of our text are sometimes found word for word in a writing a century older, the *Treatise on the determination of hidden numbers* by the philosopher and mathematician al-Kindī (d. around 870).[20] But such problems have nothing to do with Nicomachos' *Introduction*. $\mathcal{A}$.II, on magic squares, is also completely foreign to the work by Nicomachos. Finally, we guess from an allusion to the previous part of the work (the only one, in $\mathcal{A}$.I.94) that the subject of perfect numbers was discussed there, which is indeed a subject found in Nicomachos —but also, and more scientifically, in Euclid (see above, p. 12).

From this we learn two things. The first is that al-Anṭākī associates with Greek antiquity —through Nicomachos— three subjects of which two have attested links with Greek mathematics and the third none, as far as we know. The second is that al-Anṭākī takes the essence of his work from other sources, as, by the way, his often weak mathematical knowledge obliges him to.

[18] Characteristically al-Karajī's *Badīʿ*, Book I.

[19] Tropfke, *Geschichte*, p. 643; see below, notes to $\mathcal{A}$.III.14 and $\mathcal{A}$.III.15. A Byzantine example is also appended to Hoche's edition of Nicomachos (pp. 152–153).

[20] *Risāla fī istikhrāj al-aʿdād al-muḍmarra.* Almost identical or very similar are al-Anṭākī's sections $\mathcal{A}$.III.2, 3, 4, 6, (7), 15, 16, (17), 23, 25, 28, 33, (34), (35), 36.

Remark. It has recently been suggested[21] that, first, there could be some allusions by Nicomachos and Iamblichos to magical arrangement in the square of order 3 and, second, that the science of magic squares may have been studied in Jewish circles. As to the first, although these Greek allusions are in themselves not convincing, they can no longer be dismissed *a priori*. As for the Jewish influence, it is hard to believe that, considering the strength of tradition and transmission in the Jewish community, and their fondness for numerology, no trace would have survived. Indeed, wherever magic squares are found in later (mediaeval) Jewish writings, they are obviously copied from earlier Arabic writings.

B. Text $\mathcal{B}$

Abū'l-Wafā' al-Būzjānī (940–997/8) is one of the most famous mathematicians of Islamic civilization. From his work done whilst in Baghdad (his name indicates his Persian origin) part has survived, sufficiently to testify to his broad field of mathematical activity: there are writings on arithmetic, both theoretical (number theory) and practical (reckoning), geometrical constructions, plane and spherical trigonometry as well as their applications to astronomy.[22] Apart from that there is his treatise on magic squares, completely devoted to the subject —unlike $\mathcal{A}$'s single chapter.[23] The manuscript preserving it, of the 13th century as it seems, is in good condition; but the copyist has omitted all figures of magic squares. Fortunately, since the constructions are accurately described in the text, most could be reconstituted. This is, without doubt, a characteristic of the text: everything is thoroughly discussed and in a highly pedagogical way. Indeed, it represents an excellent introduction to the study of magic squares, and if the present reader wishes to look at the translations before reading the commentary, he should begin with text $\mathcal{B}$. Even if general methods are few (presumably because not yet known to the author), he will be quite satisfied with the examples given. In this respect, $\mathcal{B}$ is superior to text $\mathcal{A}$. But, as already said (p. 9), the two treatises complete each other: one because it presents fully developed methods (though either without any justification or only in a extremely concise manner), the other because it presents methods which are not general but shown to have a clear and solid basis.

[21] N. Vinel, (Iamblichi) *In Nicom. Arithmeticam*, pp. 23–35.

[22] Listed e.g. by Suter, *Mathematiker*, pp. 71–72; Sezgin, *Geschichte*, V, pp. 321–325.

[23] Manuscript Istanbul, Aya Sofya 4843, fol. 23^v–31^v, 38^r–56^v. $\mathcal{B}$'s text was edited by us in its entirety some twenty years ago. The most relevant parts are reproduced here.

Text $\mathcal{B}$ is divided into four parts. In the *Introduction* we find various preliminaries on the construction of magic squares and the treatment of the square of order 3. *Chapter I* is devoted to odd-order squares, and ends, after the treatment of a few examples, with a general method inferred from the examples. *Chapter II* treats the even-order squares; it takes them order by order, without distinguishing between the two types of even order, and without proposing any applicable general method. Finally, *Chapter III* presents 'curiosities', which are in fact extensions and particular constructions; we are told how to construct composite squares, and also odd-order bordered squares displaying separation by parity, but this —unlike in $\mathcal{A}$— leaves too much to trial and error.

Remark. The fragments as we present them do not always follow their order in the original text. We have arranged them to suit the order in which subjects are treated in our present study: ordinary magic squares ($\mathcal{B}$.3–$\mathcal{B}$.11), bordered magic squares ($\mathcal{B}$.12–$\mathcal{B}$.19), bordered odd-order squares with separation by parity ($\mathcal{B}$.20–$\mathcal{B}$.23), composite squares ($\mathcal{B}$.24–$\mathcal{B}$.26).

The way $\mathcal{B}$ treats his subject enables us to reconstitute the beginning of the science of magic squares (which in fact concurs with what may safely be conjectured). The construction of bordered squares begins with the smallest possible square (of order 3 or 4), and a border is constructed around this square. Thus, from the square of order 3, we pass to order 5, then to order 7 and so on, from an odd order to the next odd order. From this appears quite naturally a way of placing the numbers for any odd order, border by border. The same is done for the even orders, starting with the smallest possible square, that of order 4. But a general method does not appear as before, the successive (even) orders considered being too few to suggest a treatment for either one of the two types of even order, although $\mathcal{B}$ gives some indications. However, by the 10th century, general methods for bordered squares had already been discovered, as is seen from $\mathcal{A}$.

So much for bordered squares. As for ordinary magic squares, their study is dismissed by $\mathcal{A}$ at the outset as being too complicated ($\mathcal{A}$.II.2). But the form early attempts took appears quite clearly from $\mathcal{B}$. He does not know any general method, and so constructs the squares individually. He does it as was customary for ordinary magic squares before the discovery of general methods; that is, starting with the *natural square* of the corresponding order —thus the square containing the consecutive numbers in natural sequence— and then proceeding with exchanges between opposite rows in order to obtain in each the required sum. Moreover, since $\mathcal{B}$ does not find a method applicable to different orders, he considers in more detail each order and tries to find other configurations —all possible ones, in fact. This will disappear altogether with the discovery, later on, of general methods: authors will no longer concentrate solely on one specific order. From both texts it thus appears that there was no knowledge of general methods for ordinary magic squares in the 10th century. By the way, had the two authors realized how simple their constructions were, they would not have rejected their study as being too complicated for the beginner ($\mathcal{A}$) or bothered themselves with various constructions valid for just one order ($\mathcal{B}$).

Text $\mathcal{B}$ is not seen to have left much trace later. Indeed, since general methods for constructing ordinary magic squares were discovered in the 11th century, and very easy to apply, $\mathcal{B}$ became rapidly outdated. The same happened to his bordered squares constructed order by order — and all the more rapidly in that general methods already existed in his time ($\mathcal{A}$). Of the particular constructions, at most that of the smallest bordered square with separation by parity might have inspired a later reader (al-Kharaqī, our text $\mathcal{C}$, appended to the extracts of $\mathcal{B}$). As for the construction of composite squares, it would be difficult to attribute their later use to $\mathcal{B}$, for at that time they were apparently already in widespread use.

Chapter I. Ordinary magic squares

The two tenth-century texts construct magic squares with the numbers taken in their natural order beginning with 1. But this is merely a particular case since the same construction could be applied to any arithmetical progression with any chosen initial number. This was indeed known then: when $\mathcal{B}$ defines the magic square, he explicitly mentions the general case and gives examples of such squares ($\mathcal{B}$.1).

The construction of ordinary magic squares, which was to attain a remarkable development from the 11th century on, is almost insignificant in the 10th century; as already said, it is on the one hand seen as difficult for the beginner ($\mathcal{A}$.II.2) and, on the other hand, less esteemed than the construction of bordered squares: whereas the arrangement in bordered squares is called 'regular', the ordinary magic arrangement is 'not regular' according to $\mathcal{B}$ ($\mathcal{B}$.2i). To construct such a square, as mentioned by $\mathcal{A}$ when stating its difficulty, we are first to construct the *natural square* of the order considered, that is, the square filled with the numbers to be placed in the magic square, but simply taken in their natural order (Fig. 6); since it appears that the sum in the rows, either horizontal or vertical, is in one half less than the magic sum and correspondingly more in the other, exchanges will be made between opposite rows so as to eliminate these differences. This is what, according to $\mathcal{A}$, is difficult for the beginner. The examples given by $\mathcal{B}$, for the orders 3, 4, 5, 6, 8 indeed confirm that: each order has to be treated separately, and the rules applied to one order will not be valid for another. Once again, we are far from the easy methods to be developed in the 11th century, with the natural sequence of numbers being placed directly in the square to be constructed, thus without resorting to the natural square.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

Fig. 6

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§1. Construction of odd-order squares

A. Particular case of order 3

1. Uniqueness of the square of order 3

Supposing that the square is to be filled with the first nine natural numbers, we immediately know that the magic sum will be $M_3 = 15$ (above, p. 3; or $\mathcal{B}$.2*ii*). Considering that r will occupy the central cell, the sum in each of the four pairs of opposite cells will be $M_3 - r$ (Fig. 7). Thus, the sum on the whole square will be on the one hand $3M_3$, on the other $4(M_3 - r) + r$. Equating these two expressions and putting, as required in our case, $M_3 = 15$, we find $r = 5$. This is indeed the result arrived at in $\mathcal{B}$ ($\mathcal{B}$.3*i*), but rather by postulating that 5 must be in the middle since such is its place in the natural square.[24] Then we are left with pairs of complements adding up to 10 which are to be placed around the centre in opposite cells.

$*$	$\dagger$	$\circ$
$\square$	r	$\square$
$\circ$	$\dagger$	$*$

Fig. 7

2	**9**	**4**
7	**5**	**3**
6	**1**	**8**

Fig. 8

Filling the remainder of the square is not difficult, for it suffices to consider the place 1 must occupy. It can be only in the middle cell of a lateral row or in a corner, and (to make the sum 15) 9 will face it. If, first, it is in a corner cell, we are to find two pairs of numbers making the complement, thus 14; if, second, it is in a middle cell, only one. Now of the still available numbers only one pair, namely 6 and 8, makes 14. Thus 1 must be in a middle cell with 6 and 8 as its neighbours, say 6 on the left (the other choice just inverts the figure). That enables us to determine the occupants of the other cells (Fig. 8). This is exactly the reasoning found in $\mathcal{B}$.3*ii–iii*. Incidentally, this deduction of the construction shows that the magic square of order 3 with the first nine numbers admits of only one form, others being just rotations or inversions of it.

2. Construction of the square of order 3 ‘by displacement’

This logical deduction in $\mathcal{B}$ is followed by two ways of obtaining the same magic square but this time *by displacement*, thus by moving the numbers in the natural square of order 3 ($\mathcal{B}$.4).

[24] With odd orders n, the median number is always in the central cell for bordered squares, often for ordinary squares.

(*a*) Consider the natural square of order 3 (Fig. 9). Leaving the occupant of the central cell where it is, we move those of the border. First, we move the numbers in the corner cells to their respective knight's cell by turning in the same direction, namely two cells towards a corner and one cell sideways (Fig. 10). Then we take the elements formerly in those cells and transfer them to the next corner, moving this time in the opposite direction (Fig. 11). The result is the square already seen, but inverted.

1	2	3
4	5	6
7	8	9

Fig. 9

	$\underline{9}$	
$\underline{3}$	5	$\underline{7}$
	$\underline{1}$	

Fig. 10

$\underline{4}$		$\underline{2}$
$\underline{8}$		$\underline{6}$

Fig. 11

(*b*) Considering again the natural square, we move each number in the border to the next cell by turning in the same direction (Fig. 12). Since then the sum of the occupants of two corners in any row and the occupant of the middle cell in the opposite row makes 15, we shall just exchange the occupants of opposite middle cells (Fig. 13).

$\underline{4}$	$\underline{1}$	$\underline{2}$
$\underline{7}$	5	$\underline{3}$
$\underline{8}$	$\underline{9}$	$\underline{6}$

Fig. 12

4	$\underline{9}$	2
$\underline{3}$	5	$\underline{7}$
8	$\underline{1}$	6

Fig. 13

3. A construction 'without displacement'

$\mathcal{B}$ adds to these two ways a method which avoids using the natural square ($\mathcal{B}$.5): we are told to place 5 in the central cell, 1 in the middle of a lateral row, 2 in the knight's cell of 1, and 3 in the queen's cell of 1 (the next diagonally), namely, $\mathcal{B}$ adds, in the knight's cell of 2 (this determines the cell). Having placed these four numbers, completing the square is straightforward.

As to $\mathcal{A}$, he also tells us where to place the numbers; but, unlike $\mathcal{B}$, he does it for *all* occupants, without mentioning that the places of some may determine those of others ($\mathcal{A}$.II.3). Such tedious instructions are common with him, and he was blamed for that by a later reader (above, p. 14).

B. Square of order 5

The two subsequent examples are historically quite interesting since (apart from the particular case of order 3) they are the only ones where a displacement method valid for just one odd order is fully described. They have a common feature, made evident by the author ($\mathcal{B}$.6*i*), which will