NON-EUCLIDEAN GEOMETRIES

János Bolyai Memorial Volume

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NON-EUCLIDEAN GEOMETRIES

János Bolyai Memorial Volume

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This plaquette depicting János Bolyai was made by Kinga Széchenyi in commemoration of the 200th anniversary of Bolyai's birth.

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PREFACE

János Bolyai is the greatest figure in the history of Hungarian mathematics. He solved the more than two thousand year old problem in connection with Euclid's fifth postulate and discovered non-Euclidean geometry. The glory of the discovery is shared by the Russian Nicolai Ivanovich Lobachevskii but it brought more pain and bitterness than joy to the innovators.

Non-Euclidean geometry fundamentally changed our views about geometry and mathematics, in general. Some historians state that since the time of the ancient Greeks there has never been such a great revolution in mathematics than the one originating in the works of Bolyai and Lobachevskii. It became clear that geometry and reality may be different and geometry does not belong to natural sciences. The same is true for the other branches of mathematics. By the time the famous Greek mathematicians Thales and Pythagoras introduced deductive reasoning into mathematics and their followers, primarily Euclid, systematized mathematical knowledge and clarified which are the assertions that we accept without proof and which are the ones we need to prove, it was only a matter of time to learn that our mathematical way of thinking is based on abstract structures. Mathematics does not address reality in a direct manner but substitutes real life objects by abstract ones, determines their relations to each other and then solves the problems within their structures. These structures or axiomatic systems may or may not adequately describe reality. In successful cases they do and provide us with powerful tools for theoretical and practical problem solutions. For example, non-Euclidean geometrical structures, those created by Bolyai and Lobachevskii as well as more general ones, allowed for the development of modern physical theories in the twentieth century.

By the end of the nineteenth century almost all branches of mathematics became collections of axiomatic systems and the deductive consequences of the statements within. The advent of computers made mathematics more powerful and contributed tremendously to its applicability. Interestingly, applications also began to use axiomatic systems.

In fact, when we start to solve a practical problem first we create a mathematical model, a collection of notions, that represent real life objects, and their relations. Then we elaborate on it, derive its mathematical properties and solve the computational problems.

To commemorate the 200th anniversary of the birth of the great scientist, the Hungarian Academy of Sciences, together with other institutions in Hungary and abroad, organized an international conference on hyperbolic geometry on July 6-12, 2002, in Budapest, at the headquarters of the Academy. Besides the Conference this volume is a tribute to the great scientist and his world-famous scientific achievements.

The 100th, 150th and 175th anniversaries were commemorated in Kolozsvár, Budapest and Budapest, respectively, but only lectures in the Hungarian language were presented. The 200th anniversary is special not only because of the round number of years that have elapsed since 1802 but because the Bolyai research reached a significant stage.

When János Bolyai died he left behind 14,000 pages of manuscript, out of which 3,000 contain his mathematical notes and the rest his utopian ideas about science and society. Some of these pages have been scrutinized earlier, mostly about 100 years ago, and important letters (e.g., ...from nothing I have created a new, different world ...) and theories (e.g., foundations of the theory of complex numbers) have been discovered. However, most of the 14,000 pages remained unread until 1952. In that year Samu Benkő, professor of history in Kolozsvár (Cluj) began to arrange the manuscripts (which were put into chests after János Bolyai's death on order of the commanding officer of the Marosvásárhely garrison). His work lasted sixteen years. During this time he also scrutinized the nonmathematical texts that had remained unread. Similarly, Elemér Kiss, professor of mathematics in Marosvásárhely (Târgu Mureş) studied the mathematical texts during the 1990s. Both scholars have found interesting and important ideas in the manuscripts and presented them to the world.

There are two places, where Bolyai manuscripts, documents and memorabilia are collected: the Teleki Library in Marosvásárhely and the Library of the Hungarian Academy of Sciences in Budapest. Simultaneously with the conference an exhibition of the most important pieces of the latter was organized in the Gallery of the Academy.

At the Conference there were 300 participants from 25 countries. We all were honoured that the most famous geometer of the time, the 95 years old Canadian professor H.S.M. Coxeter, came to Budapest to participate at the Conference and delivered the first plenary talk (the next year we learned the sad news that he had passed away).

On the occasion of the anniversary Kinga Széchenyi made a plaquette of János Bolyai (accepting the relief of the mathematician on the façade of the Palace of Culture of Marosvásárhely to be authentic). Copies of it were given to the main speakers and those who have done outstanding research in connection with János Bolyai. The picture of the plaquette can be seen on page 20. Fig.1 of this volume. A special edition of the Appendix, sponsored by the Hungarian Academy of Sciences, printed in Latin, English and Hungarian, was published and given to all participants. The Hungarian National Bank issued a 3000 HUF face value silver coin designed by György Kiss and the Hungarian Post issued special stamps for the anniversary. The latter could be purchased on site during the conference.

We express our special thanks to the Hungarian Academy of Sciences for allowing the use of the main building of the Academy, together with its equipment. We are also grateful to the Manuscript Section of the Library of the H.A.S., especially to its head Marianne Rozsondai and researchers Béla Mázi and Károly Horányi, for the organization of the exhibition of the "Bolyai Collection" of the Academy. For the excellent organizational work in connection with the Conference our thanks should go to the Conference Organizing Group of the Computing and Automation Institute of the H.A.S., in particular to its head Gusztáv Hencsey and his associate Viktor Richter, who were mainly in charge. Many thanks should go to the members of the Program and Organizing Committees as well as those who contributed to this volume, including Dr Attila Bölcskei and Ms Ildikó Szabó, who made the collection of papers ready for print. Last but not least we express our thanks to Springer Publishers for the publication of this memorial volume.

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I

HISTORY

THE REVOLUTION OF JÁNOS BOLYAI

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1. Summary

János Bolyai is the greatest figure of Hungarian science; many think he is the Copernicus of geometry. In his 26-page work published in 1831 and generally referred to as the Appendix, (which was published as an appendix to Vol. 1 of Tentamen, the two-volume monumental monograph of his father, Farkas Bolyai) he made a revolutionary achievement by the creation of the so-called non-Euclidean geometry. With this work János Bolyai broke the monopoly of Euclidean geometry and paved the way for humanity to think about space in a different fashion. Through his findings in axiomatic thinking Bolyai considerably formed the history of mathematics as a whole. The development of modern mathematics in the 19$^{\text{th}}$ and 20$^{\text{th}}$ centuries can, to a large extent, be attributed to János Bolyai's discovery. However, the importance of his results was recognized only after his death but even then not without resistance. In his lifetime almost no one understood his brilliant ideas, which matured in him by the time he was 21. He presented them with the revolutionary bravery of youth, having no fears for the criticism of the scientific community. Naturally, he exhibited a great degree of naiveté, because he thought that great discoveries, including his, would lead to recognition and fame. But the only individual who understood Bolyai's ideas, Gauss, 'the prince of mathematicians', was unfair to János Bolyai when he formed his opinion of the Appendix in 1832. He wrote in his let-

ter to Farkas Bolyai that he was unable to praise János's work because in so doing he would be praising himself. Gauss reasoned that János Bolyai's way of thinking and results coincided almost entirely with the ideas he had been developing for the last thirty-five years. After Gauss's death in 1855, no written proof of the aforementioned statement was found. Gauss behaved reprehensibly on yet another occasion. When he learned that the Russian Lobachevskii, whose election to be a foreign corresponding member of the Royal Society of Göttingen (arranged in 1842), made the same discovery as János Bolyai, he failed to inform Lobachevskii that there was another person who had achieved almost the same results.

For many years scientists thought that although after his retirement in 1833 János Bolyai produced some work including an important theory on the foundation of complex numbers, the lack of recognition pushed him into a state of depression and he renounced creative mathematical research. It was Elemér Kiss, Professor at Marosvásárhely (now Targu Mureş) who refuted this misconception. Having consulted Bolyai's manuscripts he found significant mathematical 'gems' in them that were new at their birth.

The scientists discovered János Bolyai's greatness first abroad and it was recognized in Hungary later. His work became widely known on the European Continent by the turn of the 19th and 20th centuries. Also, in the Anglo-Saxon countries there were some who knew his work and were enthusiastic about it but they were fewer than those on the Continent. After World War II the world became bipolar. The Russians did not mention János Bolyai much but emphasized the merits of Lobachevskii. In the USA – as has been mentioned above – our scholar was less known.

The year 1977 when the 200th anniversary of Gauss's birth was celebrated all over the world became a turning-point. Although Gauss had always been regarded as the primary discoverer of non-Euclidean geometry by the authors of numerous studies, this tendency became stronger, pushing even Lobachevskii into the background. Russian authors have managed to contest these opinions in the interest of Lobachevskii. We, Hungarians, have the duty to show the rest of the world where János Bolyai's place is in the history of mathematics and universal culture. Therefore, the relevant documents and research results should be presented to the world.

2. Introductory notes

The Hungarian territories, held by the Turks for 150 years were reoccupied by the end of the 17th century. The Treaty of Karlóca, (now

Karlovac) 1699, sealed the new world order. The subsequent Hapsburg attempt to colonize the country was averted by the Rákóczi war of independence but the country's sovereignty remained rather limited in several respects. Transylvania, which had been an autonomous principality during the Turkish rule, did not become again part of Hungary. It continued to be a principality but was treated by the Hapsburg monarch as a province. Maria Teresa promoted it a Grand Duchy in 1765. The Chancellery of Transylvania was in Vienna and, at home there was the Gubernium headed by the governor, a separate General Headquarters (G.H.Q.), the autonomy of the counties and a national assembly. While Croatia belonged to Vienna only through Hungary, Transylvania was subject to the Hapsburgs directly.

However, the existence of a G.H.Q. in Transylvania did not mean that Transylvania was independent in a military sense. In Hungary, too, military affairs came under the authority of the central power, the monarch. János Bolyai was granted the title 'Imperial and Royal Military Engineer' because there was only 'Imperial and Royal Army'.

In the second half of the 18^{th} century both in Transylvania and Hungary middle-class mentality commenced to develop. An excellent book by Kosáry (2001) describes a true picture of the period between the Treaty of Szatmár of 1711 and the Compromise of 1867 and, within this, that of the Bolyais as well. However, in order to do justice to our subject matter, some other details need to be mentioned.

Both Transylvania and Hungary were considerably influenced, from political and cultural points of view, by the movements of the German speaking territories. Most of the German-speaking world belonged to the Holy Roman Empire and only a smaller part of it to the monarch of Vienna. However, from the time of the coronation of Leopold I in 1658 to the fall of the empire, the Austrian duchies, the Hungarian and the Czech kingdoms, the principality of Transylvania, etc. The Holy Roman Empire was a loose political formation. After the Peace Treaty of Westphalia (1648), which marked the end of the Thirty Years' War, the Reichstag, a parliamentary body, without much legislative power over its 143 years of existence, was set up. From the aspect of legislation the individual German states – Baden, Bavaria, Saxony, Prussia etc. – were more important. There were a lot of people among the 17^{th} century Germans who considered the ties to the Empire important. For them the Reich was equal to Germany and they considered themselves not Saxons or Prussians, etc. but, primarily, Germans.

The curricula and teaching methods of German schools and universities were very different from one another. There was no uniform standard. It was not specified how the various levels of education should be built on each other. Few people attended schools for long and even fewer went to universities. The number of students enrolled in universities decreased during the 18^{th} century. Universities were small but their number was not. In 1780 there were six universities in Austria and twenty-four in the other German states. At that time there were 360 students registered in Leipzig, 400 in Halle and 810 in Göttingen (in 1787, fifty years after its foundation) (see Sheehan, 1944). The latter was the most dynamically developing university and, at the same time, it was one of the elite institutions. Noble families sent their sons willingly to this university, which they also supported financially to a great extent. However, most of the students were the sons of lawyers, professors and officials.

In his paper 'Göttingen, Gauss and Erdély' (1979, pp. 294-313) Samu Benkő provides an insight into the relationship between the University of Göttingen and the Transylvanians. Among others this university was famous for the freedom of thought. This is why Transylvanians, who had become used to the freedom of religion at home, were inclined to attend this university. Among the numerous Transylvanian students who studied at Göttingen, Farkas Bolyai was one of the most renowned, spending three years there from 1796 to 1799. Between 1795 and 1798 Gauss, too, studied there and the two became lifelong friends.

Generally, university education was expensive, especially in Göttingen. If a family wanted to have their son educated, they had to be either wealthy, or find a protector who covered the expenses. Historians have also mentioned that drinking, duels, and irresponsible behavior were in fashion at universities, therefore many families sent their sons to universities with reluctance. However, debauchery may have been more moderate in Göttingen where the magistracy kept watch over morals. By the end of the 18^{th} century most states regulated the lives of the universities in some way but they were not allowed to violate academic autonomy.

With the partitioning of Poland in 1772, Prussia became a connected territory. The ruling Hohenzollern dynasty had many talented members. From our point of view, Frederick the Great and his reign (1740-1786) are of special importance. It was at the end of his reign and in the subsequent years that the main works of Immanuel Kant, one of the greatest philosophers of all times, had been published. Kant (1724-1804) was born in Königsberg; he lived and died there. Königsberg was a great city of German culture; many great scholars and artists were born and lived there. The first edition of Kant's greatest work, the 'Critique of Pure Reason' was published in 1781 and the second edition in 1787. He published forty books altogether. In the series entitled 'Critique' there are two more books: the 'Critique of Practical Reason' (1788) and the 'Critique of Judgment' (1790). Kant had an enormous cultural influence. His works were read and studied at all universities of Central Europe; his ideas were discussed in social circles and at dinner tables. Visitors to Königsberg were interested in Kant's newest habits and thoughts or, at least, they wanted to see the famous philosopher.

Kant is a representative of the German Enlightenment, the Auferklärung; Rousseau heavily influenced him in this respect. But having read Hume, he diverged partly from the world of the rationalist philosophy. (As a German representative of this field Kant mentioned C. Wolff.)

With regard to the ideas of the Enlightenment, Kant's critique was related to the tolerant attidude of Frederick the Great. In the preface of the Critique of Pure Reason he explained that his age was the age of critique in which institutions, such as the State and the Church could be criticized freely and ideas could be presented to the public without any restraint. After the death of Frederick the Great in 1786 Frederick William II acceded to the throne. Under his rule the situation changed.

In Kant's philosophy his ideas concerning space are the most important for our purposes. In order to explain Kant's theories we must clarify some of his definitions in advance.

Kant differentiates analytic and synthetic judgments; according to him this difference can be traced in their contents. An analytic judgment only reveals the object of the judgment, while a synthetic one contributes something to it. According to Kant, mathematical judgments are all synthetic. Frege (1884) criticized this distinction claiming that not the content of propositions but their proofs are important. From our point of view, however, this is not significant. Kant formulated his thoughts in the above-mentioned way, primarily, in order to criticize the philosophy of C. Wolff (1679-1754).

Another concept is intuition. Aristotle distinguishes intuitive knowledge from demonstrative knowledge. In his opinion the prime force of scientific knowledge is intuition; it is a direct and immediate cognition, contrary to scientific knowledge, which is acquired through the process of proofs and demonstrations. Yet, the question remains unanswered whether intuition means knowledge inherent in our mind or acquired through experience. Kant combines these two, an example of which is his concept of space expounded in the Chapter 'Transcendental Aesthetics' in the Critique of Pure Reason.

First, Kant states that geometry is a field of science that determines the characteristics of space synthetically and *a priori*. In his opinion the origin of the concept of space should be found in intuition that is *a priori*, i.e., inherent in us before any experience is acquired about an object and it has a coordinating role in the development of the form of our external perception. Space is neither the property of objects, nor is it a determinant of their relations to each other. To put it in a different way, space is in no way attached to objects, i.e., space exists only for man. If we disregard the state of the subject that makes external intuition possible, the representation of space becomes empty. Still, is the geometry of space Euclidean or may it be something else? In his thesis Kant (1770) thinks it possible that the structure of space may be different from Euclidean geometry. Later, however, he abandoned that view and in the Critique of Pure Reason, he maintains that the structure of space is Euclidean. We may observe this in the argumentations in which he discusses the difference between the methods of philosophy and those of mathematics.

Now the question should be posed: What was the mathematics of the 18^{th} century, especially in the second half of it, like? Was it similar, in style, to today's mathematics or that of the 19^{th} and 20^{th} centuries? Was Herman Hankel right when he said (Hankel, 1884): 'In most sciences newer generations destroy what the former built and discard their theories. It only occurs in mathematics that a new generation erects a new story on the old structure'. Hankel's statement is only partially true. The most appropriate contrast to this, which will be treated more thoroughly later, is provided by geometry. A further example can be found when the level of exactness of mathematics in the 18^{th} century and that of the subsequent centuries are studied. The mathematicians of the 18^{th} century: Leibniz, the members of the Bernoulli family, Euler, Taylor and Lagrange were not particulary concerned about the exactness of their results. For them, results were more important than precise proofs. That had been typical not only of the mathematics of the 18^{th} century but also of earlier times, starting from the Renaissance (Grabiner, 1974).

This result-oriented period that lasted till the end of the 18^{th} century, more or less, came after the discovery of the formula of the roots of the cubic equation in 1545. The discovery of differential and integral calculus as well as probability theory took place during that period. Although new results in their first forms often lack a high degree of exactness, the founders of the above sciences did not even think to strive for it.

What made mathematicians turn their attention to exactness from the turn of the 18^{th} and 19^{th} centuries? Two causes may be mentioned. One of them is that by the end of the 18^{th} century the pace of new mathematical discoveries slowed down considerably. By that time, a great number of results had been accumulated. They had to be systematized, unified and this could not be achieved at the former level of exactness. The other reason is that since the French Revolution the financial state of the rulers and patrons were shaken, and mathematicians were obligated to teach. In order to earn a living, however, only logically constructed and attractive material can be taught with success. By the end of the 18^{th} century one of the manifestations of the increasing demand for exactness was that in 1784 the Academy of Berlin, where Lagrange was the director of the Mathematical Section, offered a prize for the clear and precise foundation of mathematical infinity (Grattan-Guinness, ed., 1980). The prize was awarded to Simon L'Huilier in 1786.

Cauchy, Bolzano, Peacock, Babbage and others worked hard to make the theory of functions and analysis exact. In Cambridge, 1813, the latter two with Herschel founded the Analytical Society that, in addition to making analysis exact, set itself the goal to modernize the Newtonian notational system widespread in England.

It was also high time to clarify the axiom of parallels. The scientific world awaited an elegant proof of the statement, i.e., they expected somebody to deduce it from the other axioms. However, something quite different happened. The most brilliant chapter in the history of mathematics in modern times began, and it was shown that what the overwhelming majority of people had expected was impossible. The Hungarian János Bolyai and the Russian Nikolai Ivanovich Lobachevskii achieved the breakthrough. Their works will be analyzed later. Now, in connection with Lobachevskii, we discuss, at some length, the state of Russian universities in the early 19^{th} century.

Alexander I became the czar of Russia in 1801. He made great efforts to develop, among other things, universities. He reopened the university of Dorpat (= Tartu, Estonia) and founded new ones in four cities: Vilna (1802), Kazan (1804), Kharkov (1804) and St. Petersburg (1819). He laid great emphasis on teaching sciences and mathematics. Universities followed the German system; the professors were obliged to do scientific research, and they had to acquire the knowledge of new scientific results and had to include them in their lectures. The number of students was not high: 40 students were enrolled in 1809 at the University of Kazan and 135 students at the University of Moscow. At the new universities as well as at the old ones there were many foreign professors. Their number declined considerably after 1815 because a decree issued in that year stipulated that the language of academic instruction had to be Russian. Simultaneously, it was forbidden for the would-be scholars to study in Germany and, later, it was also forbidden for universities to employ professors who had studied in Germany. The justification for this was that in Germany universities were atheistic and professors made the youth skeptical and hate authorities. As a consequence, foreign professors left the universities voluntarily (Boyer, 1991).

Johann Martin Bartels (1769-1836), a compatriot of Gauss was born in Braunschweig, and was Gauss' instructor at school. Bartels was appointed professor at Kazan, and Lobachevskii became his disciple. Bartels made close friendship with Gauss and may have been aware of the importance of the problem of parallels. It is also possible that Lobachevskii might have learnt about it from Bartels. However, the latter as a foreign professor had to leave Kazan and, thus, could not have been there when Lobachevskii started to work seriously on the solution of the problem in the 1820's. It is also known (see Szénássy, 1977/1980) that Lobachevskii was primarily influenced by Legendre's work (1794). Farkas Bolyai could have learnt about the problem from Kaestner, professor of Göttingen since the latter was an expert of the topic, as his book shows (see Kaestner, 1790).

3. The lives of the Bolyais

We know a lot about the lives of the two Bolyais. It was Ferenc Schmidt, an architect of Temesvár (now Timişoara), and later Budapest, who was the first and the most thorough and devoted researcher of the topic. His father, Antal Schmidt, also an architect of Temesvár, met János Bolyai, as a military engineer working in Temesvár between 1823 and 1826. Thus, Ferenc Schmidt also heard many interesting things about János from his father and he spared no effort to gather all information on the Bolyais. His favorite pastime was to study mathematics and natural sciences. He corresponded with scientists of several Western countries, requesting them to inform him on the newly published scientific books in their countries.

Guillaume Jules Hoüel, a young professor of the history of mathematics in Bordeaux, who became one of the first discoverers of János Bolyai's work, was his French connection. Hoüel translated the Appendix into French and enclosed to it János Bolyai's biography, written by Ferenc Schmidt. The biography also was published in German (see Schmidt, 1868). Ferenc Schmidt provided the bulk of information to Paul Stäckel's two-volume book (1913). According to Barna Szénássy, these pieces of information make up two-thirds of the first volume. Among the first references on the Bolyais, János Bedőházi's book (1897) and articles of Péter Szabó (1910) and Lajos Schlesinger's (1903) should be mentioned. From later literature the books by Lajos Dávid (1922, 1979), Samu Benkő (1968, 1971, 1978, 1979, 2002), Barna Szénássy (1970), Ferenc Kárteszi (1973, 1977), Tibor Weszely (1981, 2002), Tibor Ács

(1997, 2002, 2004) and Elemér Kiss (1999) as well as the papers by Ernő Sarlóska (1965, 1973) and Barna Szénássy (1977, 1980, 1983) may be considered the most important sources.

In this article, it is not our primary goal to elaborate on the lives of the two Bolyais. However, to give a full picture about János Bolyai's life and work, we need to deal with it to some extent.

Farkas Bolyai was born on February 9, 1775 in Bolya near Nagyszeben. He came from a Hungarian family of ancient lineage. The fortified castle of Bolya was given to the family in the early 14th century. Members of the family were gallant soldiers but in the first half of the 17th century another János Bolyai lost the castle while in captivity in Turkey. They became more and more impoverished and Gáspár Bolyai, Farkas' father, inherited only a small estate near Bolya, which belonged to the County of Nagy-Küküllő in those days. A small estate close to Domáld, a village near Marosvásárhely, which comprised the heritage of Krisztina Pávai Vajna, wife of Gáspár Bolyai, was added to their wealth. Between the ages of 6 to 13 Farkas Bolyai was a pupil in the Lutheran and Calvinist College of Nagyenyed. Then Baron Simon Kemény, Sr. hired Farkas as a fellow-student to his son Baron Simon Kemény, Jr. Farkas Bolyai made a lifelong friendship with the young baron. Starting from 1790, they had studied together for five years in the Calvinist School of Kolozsvár. Meanwhile, Farkas' mathematical talent became more and more obvious but he was interested in music, drawing, and acting as well. In the fall of 1795, he set off with Simon Kemény to continue his studies in Göttingen, Germany. However, because of illness, he had to return and was able to join Simon Kemény again only in the spring of 1796. First, they spent some months in Jena, and later enrolled at the University of Göttingen in October. The position as 'a fellow-student' assured the costs of living and the possibility of learning for Farkas. In Göttingen, he made a lifelong friendship with Gauss. After Gauss' death their correspondence became a collection of documents for the history of mathematics. A selection was also published in Hungarian (Bolyai Letters, 1975, The Correspondence of Bolyai and Gauss, 2001). After the years in Göttingen, Farkas went to Kolozsvár in 1799, where he was a family tutor for a short time. He had married Zsuzsanna Árkosi Benkő, the daughter of a chirurgus of Kolozsvár (called 'barber' that time). The newlywed couple moved to Domáld and returned to Kolozsvár only in the fall of 1802, expecting the great event: the birth of János.

Gáspár Bolyai died in 1804. Just before his father's death, Farkas accepted the position of a professor at the Calvinist College of Marosvásárhely where he taught mathematics, physics and chemistry. He held the position until his retirement in 1851. Samu Benkő writes about the process of selecting the best candidate for the job (1979, pp. 155-182). According to him, there were ten applicants, who were supported by a total of twenty-three recommendations. The following three applicants: Pál Sipos, Farkas Bolyai and Mihály Marussi had most supporters. On January 22, 1804, at the consistorial meeting, Farkas Bolyai received eight out of twelve votes and Mihály Marussi received four. Thus, Farkas Bolyai was elected, by a majority vote, and the letter of appointment was prepared on that very day.

The original College of Marosvásárhely was established in the mid-16th century (Koncz, 1887). Its building was erected partly on the ruins of the Holy Church named after St. Nicholas. The church was destroyed by Basta's looting around 1600 (Balázs Orbán, 1868, Vol. 4, p. 134.). The reconstruction of the original school occurred in the early 18th century, when the Calvinist students expelled from Sárospatak were given shelter there.

Two children were born from Farkas Bolyai's first marriage: János and a daughter who died in early childhood. The marriage was unhappy. According to Farkas, his mother-in-law was a troublemaker and wanted to take away Zsuzsanna from him. On the other hand, Zsuzsanna was neurotic; there were signs of the problem already in the first years of the marriage and they grew worse after 1817. She died in 1821 after long suffering.

When Farkas Bolyai was appointed professor, he was paid partly in wheat, wine, salt, pig, lamb, honey, and wood as well as a large house with a garden – and partly in money: his annual salary was 400 Hungarian forints. In four years' time a more spacious and solid house was built for him. That was destroyed in c.100 years, in 1909.

Vásárhely – as Marosvásárhely was called at that time – was a settlement with a local government. It did not belong to the landed aristocracy or the county. It was the largest city of the Székely people, the Hungarians of Eastern Transylvania. Its Gothic castle church dates back to the 15th century and the castle built around it was a reconstructed former monastery of the Black Friars (the Dominicans). In 1571, in the very same church the freedom of religion for Unitarians was proclaimed, strengthening the law on religious tolerance that had been passed under the rule of Prince János Zsigmond in 1560.

Farkas became married for the second time in 1824, to Teréz Somorjai Nagy, who was twenty-two years younger than he was and the daughter of a merchant in Marosvásárhely. Two children, Gergely and Berta were born to them. The latter died in her childhood and the former lived at Bolya as an adult. The second wife, too, was of weak health; she died young in 1833. However, this marriage was more relaxed than the first one.

Farkas Bolyai was a very talented man. As a mathematician he was famous and one of the forerunners of the discovery of non-Euclidean geometry. He devoted his life to proving Euclid's 5^{th} postulate, which, as it is known, is impossible. His main work is the two-volume 'Tentamen' published in 1832/33. It was an outstanding summary on mathematics of the age. Gauss, too, spoke of this work highly, pointing out the author's precise way of discussion. Tentamen served as a textbook for the students of the higher classes in the College of Marosvásárhely but it contained much more than the obligatory curriculum. Farkas Bolyai was elected a corresponding member of the Learned Society (former name of the Hungarian Academy of Sciences) on March 9, 1832 but not in the Department of Mathematics, as has been mentioned in several of his biographies, but in the Department of Natural Sciences. The basis for the election was his book entitled the 'Elements of Arithmetics' published in Hungarian in 1830, and not his Tentamen written in Latin. This is clear from the letter of Secretary Gábor Döbrentei of August 29, 1833 to Farkas Bolyai. The letter states that the Secretary was not able to recommend either Farkas as a full member or János as any kind of member because their works were written in Latin (see Vekerdi, 2001). The Learned Society was founded for the purpose of promoting and developing the Hungarian language. This explains the attitude of the Secretary in this matter.

Farkas Bolyai was not only a very talented mathematician but a many-sided genius. Due to his plays, he acquired a place in the history of Hungarian literature. Another favorite pastime of his was designing stoves and ovens. Having heard that the construction of economical stoves was on the agenda in Vienna, he set out to solve the problem. He invented stoves of different types and had them made or he himself built them. So the Bolyai stoves came into fashion in Transylvania. He had many other inventions, too: a seat on wheels that had to be driven by feet and a stick; 'a coach home' placed on wheels and covered with wooden tiles, which was the forerunner of mobile homes. He gave private music lessons and delivered lectures on the theory of music as well. In addition to Hungarian, he spoke German, Latin and Romanian fluently and knew several other languages. When an opening for the Inspector General of the Forests of the Chamber in Transylvania was advertised, he applied for it in order to lessen his financial troubles. He was not appointed to the job but in an attempt to obtain it he pursued studies in forestry and wrote one of the first books on the topic in Hungarian.

As a witty and good conversationalist, he was a favorite guest of the local high society.

Farkas Bolyai died on November 20, 1856. Pursuant to his will, at his funeral there was no other ceremony but 'the ringing of the school's bell'. Also, his grave was unmarked: only a 'pojnik' apple tree, he introduced in his homeland, was planted on it (Orbán Balázs, 1868, Vol. 4, p. 133), because the famous professor was an expert in horticulture, too. After the funeral, János wrote an essay about his father. There certainly were disputes between father and son but that was not the main characteristic of their relationship. János said his father was remarkably athletic and a man of universal genius. In several of his works, too, he ranks his father among the greatest (see Elemér Kiss, 1999). When discussing János' life, we will see in what a high esteem the father held his son as well.

János Bolyai was born on December 15, 1802 in Kolozsvár where his parents moved from Domáld, to obtain better medical care during the delivery. János was born in a house that had belonged to his mother's family. It is still in existence and there is a memorial tablet on it. In two years' time the family moved to Marosvásárhely when Farkas was appointed professor to the local College.

The genius of János already manifested itself in his childhood. When he was six, he learned to read nearly alone. A year later he learnt German and to play the violin. He was nine years old when his father began to teach him mathematics; at 14, he was well versed in higher mathematics and worked with differential and integral calculus easily and skillfully. This is documented in his father's letter of April 16, 1816 to Gauss. At the same time János made remarkable progress in playing the violin; he already played difficult concert pieces. At 12 he became a regular student of the College. He skipped the first three grades and was enrolled in the fourth grade. This corresponds to the eighth grade of today's elementary school. He passed his final exam in June 1817.

We have already touched on the problem János faced regarding higher education. In Transylvania at that time there were no universities and at the University of Pest and the University of Vienna there were no mathematics professors whose level of instruction would have benefited the young genius. It was clear to Farkas that he had to send János to Gauss in Göttingen. We do not know whether like his father before him, János was offered a contract as a fellow-student to the son of a well-to-do family, which would have provided him with money for living and tuition. At that time, many of the students at German universities led loose lives. Farkas was just aware of that and, perhaps this is why he wanted János to stay in Gauss' house. Note that János was only 15 years old in 1817 and Farkas was 21 when he had gone to Göttingen in 1796. Expecting that his son's higher education was to start in 1817 in Göttingen, Farkas wrote a letter to Gauss on April 10, 1816 in which he asked Gauss to let his son stay in his house for three years, and offered to reimburse him for his expenses. But after this request, he destroyed everything when he asked Gauss to answer the following questions sincerely: '1. Have you not a daughter who may turn (reciproce) to be dangerous...? 2. Are you healthy and not poor? Are you satisfied and not grumbling? And, primarily, is your wife exceptional among women? Is she not more changeable than a weathervane? Is she not unpredictable just like the change of a barometer?' Gauss did not answer this letter.

After this, the possibility that János would study at the Vienna Academy of Military Engineers came up. During his journey to Göttingen, Farkas visited the Academy and fell in love with it so much that he almost stayed there. Thus, he could wholeheartedly recommend it to his son. However, he was unable to raise the necessary money immediately, therefore, he had János enrolled at the Faculty of Arts in the College of Marosvásárhely in 1817. Later, Count Miklós Kemény (1791-1829), the president of the College, together with other benefactors, provided the necessary money for János' education in Vienna. After he passed the admission examination in 1818, János was allowed to begin his studies in the eight-year program of the Imperial and Royal Academy of Military Engineers. One could start either in the fourth year or lower. János was registered in the fourth and he was expected to complete his studies in four years study. Although Count Miklós Kemény and others covered his tuition, there were additional expenses (e.g., for horseback riding); therefore, the father's financial contribution was needed, too. That was not an easy task as the economic situation in Transylvania also began to deteriorate due to the French wars that had been going on since 1792. In 1817, banknotes were devalued to two-fifths. We know that Farkas' annual salary was 200 silver Rhenish Ft around 1820; however, he did not always receive his salary in full or on time. The annual costs of János' education was about 900 Ft. Of this amount, 130 Ft had to be paid in silver. The items János needed to begin his student life cost nearly 220 Ft.

It was Ernő Sarlóska (1965, 1973) who first wrote about János Bolyai's military career. Sarlóska's articles discuss Bolyai's years spent at the Academy of Military Engineers, and his next ten years of his military service. In the books of Tibor Ács (1997, 2002, 2004) there are detailed reports on Bolyai's years at the Academy of Military Engineers. Here, it may be sufficient to mention that János was an excellent student who was ranked first among the students by his professors although his classmates ranked him second. He stayed at that place in the overall rankings. The main reason why he was not ranked first was his performance in drawing: János was just bored by drawing. During the years at the Academy of Military Engineers, from 1820 on, he had been concerned intensively with the research of parallels. He wanted to prove the fifth postulate, which his father had long sought to prove. Farkas warned his son against doing that in his letter of April 4, 1820:

'You must not attempt this approach to parallels: I know this way to its very end. I have traversed this bottomless night, which extinguished all light and joy of my life. For God's sake! I entreat you leave parallels alone, abhor them like indecent talk, they may deprive you [just like me] from your time, health, tranquility and the happiness of your life. That bottomless darkness may devour a thousand of tall towers of Newton and it will never brighten up in the earth.... I thought I would sacrifice myself for the sake of the truth. I was ready to become a martyr who would remove the flaw from geometry and return it purified to mankind. I accomplished monstrous, enormous labors; my creations are far better than those of others and yet I have not achieved complete satisfaction.... I turned back when I saw that no man can reach the bottom of this night. I turned back unconsoled, pitying myself and all mankind.... I have traveled past all reefs of this infernal Dead Sea and have always come back with broken mast and torn sail. The ruin of my disposition and my fall date back to this time. I thoughtlessly risked my life and happiness – *aut Caesar aut nihil.*'

János Bolyai finished his studies at the Academy of Military Engineers in 1822, but he was permitted to stay there to pursue further studies for one more year, as he was one of the two best students.

In early September of 1823 he was nominated sub-lieutenant, and was assigned to the Directorate of Fortification of Temesvár. From here he wrote his father his letter of November 3, 1823, which became widely known all over the world: 'My dear Father! I have so much to write you about my new findings that, for the time being, I cannot help but avoid their discussion here in depth and I am going to write you on a quarto.... I am determined to publish a work on parallels as soon as having arranged and prepared it and, there is an opportunity to do so; for the time being, it is not found out yet but the way I have gone has promised to achieve my goal if possible at all; it is not ready yet but I revealed such superb things that I myself was astonished, and it would mean everlasting shame to let them lost for ever; if you, my dear Father, see them, you will acknowledge them; now I cannot say anything else: from nothing I have created a new different world; all other things that I have sent to you are just a house of cards compared to a tower.'

As it is already known today, this 'new different world' is the magical world of absolute and hyperbolic geometry. At the beginning of 1825, János visited his family at Marosvásárhely. He had great success there. The aristocratic society was fascinated by the personality and violin playing of the elegant officer. His father took delight in his son, primarily because of his mathematical genius. In his letter of February 27, 1825, to Pál Bodor he writes that János is a handsome young man of great and tough nature. In addition, János was an excellent fencer; he had become famous for that in his student years. Once, during his stay in Arad, 13 cavalry officers challenged him to a duel. He accepted the challenge under the condition that after every two duels he might play the violin. He vanquished all thirteen duelers. If this story is true, and if the men used heavy cavalry swords, – the challengers were cavalry officers –, we may conclude that János was a young man of great physical strength.

Fate willed that when he was transferred to Arad in 1826, his superior became Johann Wolter Eckwehr, who had been his professor of mathematics at the Academy of Military Engineers. János had corresponded with Eckwehr before. That year, he handed over his manuscript in German to his former professor, in which he summarized his investigations in non-Euclidean geometry. Regrettably, this manuscript has been lost.

In 1831, János was transferred to Lemberg, and in 1832, Olmütz became the last station of his military career. On his way to Lemberg, he visited his father in Marosvásárhely.

In Arad, János had recurrent fever. Presumably, he caught malaria because there was a marshland around the town. Later, he suffered from cholera, too; his health had deteriorated significantly. This was aggravated by the fact that on his way from Lemberg to Olmütz, his coach turned over and he suffered a serious head injury. He had already neglected his job because he was uninterested in the routine drafting he was required to do. Instead, he tried to use all his spare time to work on solution of mathematical problems. He tried to apply for a 3-year leave from service to pursue his mathematical research. In 1832, his application was forwarded to Archduke János, who rejected it. Finally, in 1833, he was discharged with a pension as a second-class captain. A further reason for his discharge was that on his way from Lemberg to Olmütz he had an argument with customs officers at the border because he refused to open his trunk. The officers then reported him to the authorities.

Coming back to the year of 1831, the most important event was the publication of the Appendix, as a preprint, in Latin. Volume 1 of Tentamen, written also in Latin and bound together with János' Appendix, was published in 1832. Volume 2 was published in 1833. It is an important fact that the date of the imprimatur of Tentamen is October 12, 1829.

Farkas Bolyai sent Gauss a copy of the Appendix, almost immediately after its first publication in April, 1831, asking him for his opinion. That copy had been lost, but the accompanying letter, dated June 20, 1831, exists (see Bolyai Letters, 1975, 168-173). Farkas sent another copy on January 16, 1832. Gauss' reply of March 6, 1832, is widely known. In one of the most devastating parts of the letter he writes:

'Regarding your son's work: If I began by saying that I am unable to praise this work, you would certainly be surprised for a moment. To praise it would be to praise myself. Indeed, the whole contents of the work, the path taken by your son, the results to which he is led, coincide almost exactly with my own ideas I have been developing for thirty to thirty-five years'.

Then he continues: he, too, intended to write all down eventually, so that at least it would not finally perish with him. In other letters, Gauss denotes an earlier date when he had already been concerned with non-Euclidean geometry. But in his letter to Gerling, he recognizes that his ideas in 1798 were far from the maturity found in the work of János Bolyai. Gauss also praised János Bolyai and his work but the praises could not alleviate the pain the first letter had caused the young Titan.

János Bolyai moved to his father's house in Marosvásárhely in 1833, but a year later he went to Domáld, where he lived until 1846. From 1834, he cohabited with Rozália Kibédi Orbán. Legal marriage was out of the question because they were unable to raise the money for a deposit called 'caution money' that was required as János was an army officer. They had two children: Dénes (1837-1913) and Amália (1840-1893). Amália had no children but Dénes had several from his three marriages. Among several other descendants, János Bolyai the great-grandchild of our János, lives in Edelény, Hungary.

The year of 1837 brought a significant event in the lives of both Bolyais. The Jablonowski Scientific Society of Leipzig announced a competition for the foundation of the theory of imaginary numbers (the original text of the competition is rather long and it appears strange by modern standards). The Bolyais learned about it not long before the deadline in November 1837, but both of them submitted competition papers. Beside them Ferenc Kerekes, professor of the College of Debrecen (1784-1850) took part in the competition. The Bolyais did not win but Kerekes was granted half the prize. This work of János Bolyai, known as Responsio, is based on principles similar to those of Hamilton's, who founded the theory of complex numbers. Although János Bolyai submitted his paper in 1837, his theory was complete in 1831, which is earlier than Hamilton's submission of his own paper to the Academy of Dublin.

János Bolyai had several other new mathematical results which have been discussed in a recently published book by Elemér Kiss (1999).

In 1846, János Bolyai moved to Marosvásárhely with his family because his father was discontent with János's management of the estate of Domáld and leased the estate to a tenant.

The year of 1848 gave a surprise to János. He read Lobachevskii's work, published in 1840 in German, whose content coincided with that of the Appendix to a large extent. First, he was suspicious that he had been a victim of theft but later he commented on the work. Paul Stäckel and József Kürschák published the comments in 1902.

During the war of independence in 1849, when caution money was not required, he married Rozália Orbán. But after the war the Army did not recognize the legality of the marriage.

In 1852, János Bolyai moved away from his family, leaving the house to his wife and giving a considerable amount of money for the support of the children. However, he continued looking after them. He was in bad health and taken care of by a servant, Júlia Szőts.

In 1857, with his half brother Gergely, who ran the estate at Bolya, he sold Domáld for 1600 Rhenish Ft.

On January 27, 1860, Júlia Szőts wrote a letter to Gergely, asking him to come urgently because János was unwell. Having signed the letter, she looked at her master and continued: 'While I was writing this letter, he died, thus, there is nothing to be said: the Captain is gone'.

In addition to the obligatory military escort, there were three civilian persons present at the funeral. Below the formal records in the registry of the Calvinist Church the following notes were added: 'He was a famous mathematician of great mind. He was first even among the first. It is a pity that his talent was buried unused' (Kiss, 1999).

However, as we will show later, people much more competent than the aforesaid registrar, were unable to assess the greatness of János Bolyai's personality and work at the time.

No pictures of János Bolyai exist. There was a picture that showed him in uniform, however, at one occasion, Bolyai cut it to pieces with his sword in rage. Recently, the opinion that one of the reliefs on the top of the façade of the Palace of Culture at Marosvásárhely portrays him has gained acceptance. Out of the six reliefs in total in five have been proved to represent the individuals, whose names are inscribed on them. Below the sixth one the name of János Bolyai is inscribed and it is just next to that of Farkas Bolyai's relief. There is further proof, namely, the testimonies of those who had known János Bolyai personally at the time the Palace was built. Moreover, there is a striking similarity between the relief and the portrait of György Klapka. It is known that János Bolyai resembled György Klapka, a general of the Hungarian revolutionary army of 1848-49. Kinga Széchenyi made a plaquette of János Bolyai for the anniversary of 2002 based on this relief (see Figure 1).

Figure 1. János Bolyai. The plaquette made by Kinga Széchenyi.

4. Forerunners of the non-Euclidean geometry

The word 'geometry' derives from Old Greek 'geometrein' that meant land surveying. Originally, geometry was the collection of simple rules that had been obtained through experimentation, data-collection, and intuition. The Egyptians, Babylonians, and Chinese were also aware of such rules but the Greeks were the first who deduced geometric propositions from known or evident ones. Among them Thales of Miletos should primarily be mentioned as the main inventor of the deductive proof in geometry and mathematics. He established the first logical geometry.