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Chang-Ock Lee · Xiao-Chuan Cai
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Domain Decomposition Methods in Science and Engineering XXIII

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Domain Decomposition Methods in Science and Engineering XXIII

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Preface of DD23 Book of Proceedings

The proceedings of the 23rd International Conference on Domain Decomposition Methods contain developments up to 2015 in various aspects of domain decomposition methods bringing together mathematicians, computational scientists, and engineers who are working on numerical analysis, scientific computing, and computational science with industrial applications. The conference was held on Jeju Island, Korea, July 6–10, 2015.

Background of the Conference Series

The International Conference on Domain Decomposition Methods has been held in 14 countries throughout Asia, Europe, and North America beginning in Paris in 1987. Held annually for the first 14 meetings, it has been spaced out since DD15 at roughly 18-month intervals. A complete list of the past meetings appears below. The 23rd International Conference on Domain Decomposition Methods was the first one held in Korea, and it took place on the beautiful Jeju Island.

The main technical content of the DD conference series has always been mathematical, but the principal motivation was and is to make efficient use of distributed memory computers for complex applications arising in science and engineering. As we approach the dawn of exascale computing, where we will command 10^{18} floating-point operations per second, clearly efficient and mathematically well-founded methods for the solution of large-scale systems become more and more important—as does their sound realization in the framework of modern HPC architectures. In fact, the massive parallelism, which makes exascale computing possible, requires the development of new solution methods, which are capable of efficiently exploiting this large number of cores as the connected hierarchies for memory access. Ongoing developments such as parallelization in time asynchronous iterative methods or nonlinear domain decomposition methods show that this massive parallelism does not only demand for new solution and discretization methods but also allows to foster the development of new approaches.

The progress obtained in domain decomposition techniques during the last decades has led to a broadening of the conference program in terms of methods and applications. Multiphysics, nonlinear problems, and space-time decomposition methods are more prominent these days than they have been previously. Domain decomposition has always been an active and vivid field, and this conference series is representing well the highly active and fast advancing scientific community behind it. This is also due to the fact that there is basically no alternative to domain decomposition methods as a general approach for massively parallel simulations at a large scale. Thus, with growing scale and growing hardware capabilities, also the methods can—and have to—improve.

However, even if domain decomposition methods are motivated historically by the need for efficient simulation tools for large-scale applications, there are also many interesting aspects of domain decomposition, which are not necessarily motivated by the need for massive parallelism. Examples are the choice of transmission conditions between subdomains, new coupling strategies, or the principal handling of interface conditions in problem classes such as fluid-structure interaction or contact problems in elasticity.

While research in domain decomposition methods is presented at numerous venues, the International Conference on Domain Decomposition Methods is the only regularly occurring international forum dedicated to interdisciplinary technical interactions between theoreticians and practitioners working in the development, analysis, software implementation, and application of domain decomposition methods.

The list of previous Domain Decomposition Conferences is the following:

1. Paris, France, January 7–9, 1987
2. Los Angeles, USA, January 14–16, 1988
3. Houston, USA, March 20–22, 1989
4. Moscow, USSR, May 21–25, 1990
5. Norfolk, USA, May 6–8, 1991
6. Como, Italy, June 15–19, 1992
7. University Park, Pennsylvania, USA, October 27–30, 1993
8. Beijing, China, May 16–19, 1995
9. Ullensvang, Norway, June 3–8, 1996
10. Boulder, USA, August 10–14, 1997
11. Greenwich, UK, July 20–24, 1998
12. Chiba, Japan, October 25–29, 1999
13. Lyon, France, October 9–12, 2000
14. Cocoyoc, Mexico, January 6–11, 2002
15. Berlin, Germany, July 21–25, 2003
16. New York, USA, January 12–15, 2005
17. St. Wolfgang-Strobl, Austria, July 3–7, 2006
18. Jerusalem, Israel, January 12–17, 2008
19. Zhangjiajie, China, August 17–22, 2009
20. San Diego, California, USA, February 7–11, 2011

21. Rennes, France, June 25–29, 2012
22. Lugano, Switzerland, September 16–20, 2013
23. Jeju Island, Korea, July 6–10, 2015

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About the 23rd Conference

The 23rd International Conference on Domain Decomposition Methods had 108 participants from over 22 countries. It was the first one to be held in Korea.

As in previous meetings, DD23 featured a well-balanced mixture of established and new topics, such as space-time domain decomposition methods, isogeometric analysis, exploitation of modern HPC architectures, optimal control and inverse problems, and electromagnetic problems. From the conference program, it is evident that the growing capabilities in terms of theory and available hardware allow for increasingly complex nonlinear and multiscale simulations, confirming the huge potential and flexibility of the domain decomposition idea. The conference, which was organized over an entire week, featured presentations of three different types: The conference contained:

- Eleven invited presentations, fostering also younger scientists and their scientific development, selected by the International Scientific Committee
- A poster session, which also gave rise to intense discussions with the mostly younger presenting scientists
- Nine minisymposia, arranged around a special topic
- Seven sessions of contributed talks

The present proceedings volume contains a selection of 42 papers, split into 8 plenary papers, 21 minisymposium papers, and 13 contributed papers and posters.

Sponsoring Organizations

- KAIST Mathematics Research Station
- National Institute for Mathematical Sciences
- The Korean Federation of Science and Technology Societies
- KISTI Supercomputing Center
- A3 Foresight Program
- NVIDIA
- Jeju Convention & Visitors Bureau

The organizing committee would like to thank the sponsors for the financial support.

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Research Activity in Domain Decomposition According to DD23 and Its Proceedings

The conference and the proceedings contain three parts: the plenary presentations, the minisymposium presentation, and the contributed talks and posters.

Plenary Presentations

The plenary presentations of the conference have been dealing with established topics in domain decomposition as well as with new approaches:

- Global convergence rates of some multilevel methods for variational and quasi-variational inequalities, Lori Badea (Institute of Mathematics of the Romanian Academy, Romania)

- Robust solution strategies for fluid-structure interaction problems with applications, Yuri Bazilevs (University of California, San Diego, USA)
- BDDC algorithms for discontinuous Petrov-Galerkin methods, Clark Dohrmann (Sandia National Laboratories, USA)
- Schwarz methods for the time-parallel solution of parabolic control problems, Felix Kwok (Hong Kong Baptist University, Hong Kong)
- Computational science activities in Korea, Jysoo Lee (KISTI, Korea)
- Recent advances in robust coarse space construction, Frédéric Nataf (Université Paris 6, France)
- Domain decomposition preconditioners for isogeometric discretizations, Luca F. Pavarino (University of Milano, Italy)
- Development of nonlinear structural analysis using co-rotational finite elements with improved domain decomposition method, SangJoon Shin (Seoul National University, Korea)
- Adaptive coarse spaces and multiple search directions: tools for robust domain decomposition algorithms, Nicole Spillane (Universidad de Chile, Chile)
- Element-based algebraic coarse spaces with applications, Panayot Vassilevski (Lawrence Livermore National Laboratory, USA)
- Preconditioning for nonsymmetry and time dependence, Andrew Wathen (University of Oxford, United Kingdom)

Minisymposia

There are nine minisymposia organized within DD23:

1. Space-time domain decomposition methods (Ulrich Langer, Olaf Steinbach)
   The space-time discretization of transient partial differential equations by using general space-time finite and boundary elements in the space-time computational domain allows for an almost optimal, adaptive space-time resolution of wave fronts and moving geometries. The global solution of the resulting systems of algebraic equations can easily be done in parallel, but requires appropriate preconditioning techniques by means of multilevel and domain decomposition methods. This minisymposium presents recent results on general space-time discretizations and parallel solution strategies.
2. Domain decomposition with adaptive coarse spaces in finite element and isogeometric applications (Durkbin Cho, Luca F. Pavarino, Olof B. Widlund)
   The aim of the minisymposium is to bring together researchers in both fields of finite elements and isogeometric analysis (IGA) to discuss the latest research developments in domain decomposition methods with adaptive coarse spaces. While coarse spaces are essential for the design of scalable algorithms, they can become quite expensive for problems with a large number of subdomains, or very irregular coefficients/domains, or for IGA discretizations where the high irregularity of the NURBS basis functions yields large interface and coarse

problems. This minisymposium will focus on recently proposed novel adaptive coarse spaces, generalized eigenproblems, and primal constraints selection.

3. Domain decomposition and high-performance computing (Santiago Badia, Jakub Šístek, Kab Seok Kang)

   The next generation of supercomputers, able to reach 1 exaflop/s, is expected to reach billions of cores. The success of domain decomposition for large-scale scientific computing will be strongly related to the ability to efficiently exploit extreme core counts. This MS is mainly oriented to novel algorithmic and implementation strategies that will boost the scalability of domain decomposition methods and their application for large-scale problems. Since large-scale computing is demanded by the most complex applications, generally multiscale, multiphysics, nonlinear, and/or transient in nature, tailored algorithms for these types of applications will be particularly relevant.
4. Domain decomposition methods and parallel computing for optimal control and inverse problems (Huibin Chang, Xue-Cheng Tai, Jun Zou)

   This minisymposium will bring together active experts working on domain decomposition methods and parallel computing for large-scale ill-posed problems from image processing, optimal control, and inverse problems to discuss and exchange the latest developments in these areas.
5. Efficient solvers for electromagnetic problems (Victorita Dolean, Zhen Peng)

   In this minisymposium we explore domain decomposition-type solvers for electromagnetic wave propagation problems. These problems are very challenging (especially in time-harmonic regime where the problem is indefinite in nature and most of the iterative solvers will fail) . The mini-symposium will discuss different areas of recent progress as parallel domain decomposition libraries, sweeping preconditioners, iterative methods based on multi-trace formulations, or new results on optimized Schwarz methods.
6. Domain decomposition methods for multiscale PDEs (Eric Chung, Hyea Hyun Kim)

   It is well known that classical ways to construct coarse spaces are not robust and give large condition numbers depending on the heterogeneities and contrasts of the coefficients. Recently, there are increasing interests in constructing domain decomposition methods with enriched coarse spaces or adaptive coarse spaces. The purpose of this minisymposium is to bring together researchers in the area of domain decomposition methods for PDEs with highly oscillatory coefficients and provide a forum for them to present the latest findings.
7. Birthday minisymposium Ralf Kornhuber (60th Birthday) (Rolf Krause, Martin Gander)

   This MS will bring together talks which are related to the scientific work of Ralf Kornhuber. This includes fast numerical methods for variational inequalities, multigrid methods, numerical methods for phase field equations, and biomechanics.
8. Recent approaches to nonlinear doman decomposition methods (Axel Klawonn, Oliver Rheinbach)

For a few decades already, Newton-Krylov algorithms with suitable preconditioners such as domain decomposition (DD) or multigrid (MG) methods (Newton-Krylov-DD or Newton-Krylov-MG) have been the workhorse for the parallel solution of nonlinear implicit problems. The standard Newton-Krylov approaches are based on a global linearization and the efficient parallel solution of the resulting linear (tangent) systems in each linearization step ("first linearize, then decompose"). Increasing local computational work and reducing communication are key ingredient for the efficient use of future exascale machines. In Newton-Krylov-DD/Newton-Krylov-MG methods, these aspects can be mainly treated at the level of the solution of the linear systems by the preconditioned Krylov methods. Computational work can be localized, and communication can be reduced by a complete reordering of operations: the nonlinear problem is first decomposed and then linearized, leading to nonlinear domain decomposition methods. An early approach in this direction is the ASPIN (additive Schwarz preconditioned inexact Newton) method by Cai and Keyes. Recently, there has been work on nonlinear FETI-DP and BDDC methods by Klawonn, Lanser, and Rheinbach. In this minisymposium, recent approaches to nonlinear domain decomposition methods will be presented.

9. Tutorial for domain decomposition on heterogenous HPC (Junard Lee)

   At this minisymposium, we will have a tutorial session. We will cover heterogeneous HPC architecture, CUDA programming language, Open ACC directives, and how to implement these technologies to accelerate PDE solvers specially domain decomposition method.

Contributed Presentations and Posters

The contributed talks have been distributed over seven different sessions:

1. Domain Decomposition Methods for Applications
2. Optimized Schwarz Methods
3. Fast Solvers for Nonlinear and Unsteady Problems
4. Domain Decomposition Methods with Lagrange Multipliers
5. Efficient Methods and Solvers for Applications
6. Multiphysics Problems
7. Coarse Space Selection Strategies

The proceedings part with poster presentations is also a real treasure trove for new ideas in domain decomposition methods.

Acknowledgements In closing, we would like to thank all the participants gathered on Jeju Island for their contributions to the scientific success of this conference. Moreover, it is our pleasure to express our sincere thanks to everybody who has supported this conference on the administrative side. This includes the chairs of the conference sessions, the volunteers from KAIST and Jeju

National University helping on the practical and technical issues, and last but not least the KSIAM staff who has provided invaluable support.

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Global Convergence Rates of Some Multilevel Methods for Variational and Quasi-Variational Inequalities

Lori Badea

1 Introduction

The first multilevel method for variational inequalities has been proposed in Mandel (1984a) for complementarity problems. An upper bound of the asymptotic convergence rate of this method is derived in Mandel (1984b). The method has been studied later in Kornhuber (1994) in two variants, standard monotone multigrid method and truncated monotone multigrid method. These methods have been extended to variational inequalities of the second kind in Kornhuber (1996, 2002). Also, versions of this method have been applied to Signorini's problem in elasticity in Kornhuber and Krause (2001). In Badea (2003, 2006) global convergence rates of some projected multilevel relaxation methods of multiplicative type are given. Also, a global convergence rate was derived in Badea (2008) for a two-level additive method. Two-level methods for variational inequalities of the second kind and for some quasi variational inequalities have been analyzed in Badea and Krause (2012). In Badea (2014), it was theoretically justified the global convergence rate of the standard monotone multigrid methods and, in Badea (2015), this result has been extended to the hybrid algorithms, where the type of the iterations on the levels is different from the type of the iterations over the levels. Finally, a multigrid method for inequalities containing a term given by a Lipschitz operator is analyzed in Badea (2016). Evidently, the above list of citations is not exhaustive and, for further information, we can see the review article (Gräser and Kornhuber, 2009).

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This is a review paper regarding the convergence rate of some multilevel methods for variational inequalities and also, for more complicated problems such as variational inequalities of the second kind, quasi-variational inequalities and inequalities with a term containing a Lipschitz operator. The methods are first introduced as some subspace correction algorithms in a reflexive Banach space and, under some assumptions, general convergence results (error estimations, included) are given. In the finite element spaces, we prove that these assumptions are satisfied and that the introduced algorithms are in fact one-, two-, multilevel or multigrid methods. The constants in the error estimations are explicitly written in functions of the overlapping and mesh parameters for the one- and two-level methods and in function of the number of levels for the multigrid methods.

In this paper, we denote by V a reflexive Banach space and $K \subset V$ is a non empty closed convex subset. Also, $F : K \to \mathbf{R}$ is a Gâteaux differentiable functional and we assume that there exist two real numbers $p, q > 1$ such that for any $M > 0$ there exist $\alpha_M,\ \beta_M > 0$ for which

$$\begin{aligned}&\alpha_M ||v-u||^p \leq < F'(v) - F'(u), v-u > \\ &\text{and } ||F'(v) - F'(u)||_{V'} \leq \beta_M ||v-u||^{q-1},\end{aligned}$$

for any $u, v \in K$, $||u||, ||v|| \leq M$. In view of these properties, we can prove that F is a convex functional and $1 < q \leq 2 \leq p$.

2 One- and Two-Level Methods

In this section we introduce one- and two-level methods of multiplicative type, first as a general subspace correction algorithm. Details concerning the proof of its global convergence can be found in Badea (2003). The one- and two-level methods are derived from this algorithm by the introduction of the finite element spaces and details are given in Badea (2006). Similar results can be proved for the additive variant of the methods [see Badea (2008)].

We consider the variational inequality

$$u \in K : \ < F'(u), v-u > \geq 0, \text{ for any } v \in K, \tag{1}$$

and if K is not bounded, we suppose that F is coercive, i.e. $F(v) \to \infty$ as $||v|| \to \infty$. Then, problem (1) has an unique solution. Let $V_1, \cdots, V_m$ be some closed subspaces of V for which we make the following.

Assumption 1 *There exists a constant* $C_0 > 0$ *such that for any* $w, v \in K$ *and* $w_i \in V_i$ *with* $w + \sum_{j=1}^{i} w_j \in K$, $i = 1, \cdots, m$, *there exist* $v_i \in V_i$, $i = 1, \cdots, m$, *satisfying*

$$w + \sum_{j=1}^{i-1} w_j + v_i \in K,\ v - w = \sum_{i=1}^{m} v_i,\ \sum_{i=1}^{m} ||v_i||^p \le C_0^p \left(||v - w||^p + \sum_{i=1}^{m} ||w_i||^p \right).$$

For linear problems, the last condition has a more simple form and is named the stability condition of the space decomposition. To solve problem (1), we introduce the following subspace correction algorithm.

Algorithm 1 *We start the algorithm with an arbitrary* $u^0 \in K$. *At iteration* $n+1$, *having* $u^n \in K$, $n \ge 0$, *we sequentially compute for* $i = 1, \cdots, m$,

$$w_i^{n+1} \in V_i,\ u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K : \langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1} \rangle \ge 0,$$

for any $v_i \in V_i$, $u^{n+\frac{i-1}{m}} + v_i \in K$, *and then we update* $u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}$.

The following result proves the global convergence of this algorithm [see Theorem 2 in Badea (2003)].

Theorem 1 *On the above conditions on the spaces and the functional* F, *if Assumption 1 holds, then there exists an* $M > 0$ *such that* $||u^n|| \le M$, *for any* $n \ge 0$, *and we have the following error estimations:*

(i) if $p = q = 2$ *we have* $||u^n - u||^2 \le \frac{2}{\alpha_M} \left(\frac{\tilde{C}_1}{\tilde{C}_1 + 1} \right)^n \left[F(u^0) - F(u) \right]$.

(ii) if $p > q$ *we have* $||u - u^n||^p \le \frac{p}{\alpha_M} \frac{F(u^0) - F(u)}{\left[1 + n\tilde{C}_2 \left(F(u^0) - F(u) \right)^{\frac{p-q}{q-1}} \right]^{\frac{q-1}{p-q}}}$,

where

$$\tilde{C}_1 = \beta_M (\tfrac{p}{\alpha_M})^{\frac{q}{p}} m^{2-\frac{q}{p}} \Big[(1 + 2C_0) \left(F(u^0) - F(u) \right)^{\frac{p-q}{p(p-1)}} + \left(\beta_M (\tfrac{p}{\alpha_M})^{\frac{q}{p}} m^{2-\frac{q}{p}} \right)^{\frac{1}{p-1}} C_0^{\frac{p}{p-1}} / \eta^{\frac{1}{p-1}} \Big] / (1 - \eta) \text{ and}$$

$$\tilde{C}_2 = \frac{p - q}{(p-1) \left(F(u^0) - F(u) \right)^{\frac{p-q}{q-1}} + (q-1) \hat{C}^{\frac{p-1}{q-1}}}.$$

The value of η *in the expression of* $\tilde{C}_1$ *can be arbitrary in* $(0, 1)$, *but we can also chose a* $\eta_0 \in (0, 1)$ *such that* $\tilde{C}_1(\eta_0) \le \tilde{C}_1(\eta)$ *for any* $\eta \in (0, 1)$.

One-level methods are obtained from Algorithm 1 by using the finite element spaces. To this end, we consider a simplicial regular mesh partition $\mathcal{T}_h$ of mesh size h over $\Omega \subset \mathbf{R}^d$. Also, let $\Omega = \cup_{i=1}^m \Omega_i$ be a domain decomposition of Ω, the overlapping parameter being δ, and we assume that $\mathcal{T}_h$ supplies a mesh partition for each subdomain Ω_i, $i = 1, \ldots, m$. In Ω, we use the linear finite element space V_h whose functions vanish on the boundary of Ω and, for each $i = 1, \ldots, m$, we consider the linear finite element space $V_h^i \subset V_h$ whose functions vanish outside Ω_i. Spaces V_h and V_h^i, $i = 1, \ldots, m$, are considered as subspaces of $W^{1,\sigma}$, $1 \le \sigma \le \infty$, and let $K_h \subset V_h$ be a convex set satisfying.

Property 1 If $v, w \in K_h$, and if $\theta \in C^0(\bar{\Omega})$, $\theta|_\tau \in C^1(\tau)$ for any $\tau \in \mathcal{T}_h$, and $0 \le \theta \le 1$, then $L_h(\theta v + (1-\theta)w) \in K_h$, where L_h is the P_1-Lagrangian interpolation.

We see that the convex sets of obstacle type satisfy this property, and we have (see Proposition 3.1 in Badea (2006) for the proof)

Proposition 1 *Assumption 1 holds for the linear finite element spaces, $V = V_h$ and $V_i = V_h^i$, $i = 1, \ldots, m$, and for any convex set $K = K_h \subset V_h$ having Property 1. The constant C_0 in Assumption 1 can be written as $C_0 = C(m+1)(1 + \frac{m-1}{\delta})$, where C is independent of the mesh parameter and the domain decomposition.*

In the case of the two-level methods, we consider two regular simplicial mesh partitions $\mathcal{T}_h$ and $\mathcal{T}_H$ on $\Omega \subset \mathbf{R}^d$, $\mathcal{T}_h$ being a refinement of $\mathcal{T}_H$. Besides the finite element spaces V_h, V_h^i, $i = 1, \ldots, m$ and the convex set K_h, defined for the one-level methods, we introduce the linear finite element space V_H^0 corresponding to the H-level, whose functions vanish on the boundary of Ω. The two-level method is obtained from the general subspace correction Algorithm 1 for $V = V_h$, $K = K_h$, and the subspaces $V_0 = V_H^0$, $V_1 = V_h^1$, $V_2 = V_h^2, \ldots, V_m = V_h^m$. Also, these spaces are considered as subspaces of $W^{1,\sigma}$, $1 \le \sigma \le \infty$, and we have the following (see Proposition 4.1 in Badea (2006) for the proof)

Proposition 2 *Assumption 1 is satisfied for the linear finite element spaces $V = V_h$ and $V_0 = V_H^0$, $V_i = V_h^i$, $i = 1, \ldots, m$, and any convex set $K = K_h$ having Property 1. The constant C_0 can be taken of the form $C_0 = Cm\left(1 + (m-1)\frac{H}{\delta}\right) C_{d,\sigma}(H, h)$, where C is independent of the mesh and domain decomposition parameters, and*

$$C_{d,\sigma}(H,h) = \begin{cases} 1 & \text{if } d = \sigma = 1 \text{ or } 1 \le d < \sigma \le \infty \\ \left(\ln \frac{H}{h} + 1\right)^{\frac{d-1}{d}} & \text{if } 1 < d = \sigma < \infty \\ \left(\frac{H}{h}\right)^{\frac{d-\sigma}{\sigma}} & \text{if } 1 \le \sigma < d < \infty. \end{cases}$$

Some numerical results have been given in Badea (2009) to compare the convergence of the one-level and two-level methods. They concern the two-obstacle problem of a nonlinear elastic membrane,

$$u \in [a, b] : \int_\Omega |\nabla u|^{\sigma-2} \nabla u \nabla (v - u) \ge 0, \text{ for any } v \in [a, b] \tag{2}$$

where $\Omega \subset \mathbf{R}^2$, $K = [a, b]$, $a \leq b$, $a, b \in W_0^{1,\sigma}(\Omega)$, $1 < \sigma < \infty$. These numerical experiments have confirmed the previous theoretical results.

3 Multilevel and Multigrid Methods

Details concerning the results in this section can be found in Badea (2014, 2015). As in the case of the one- and two-level methods, we consider problem (1). Let V_j, $j = 1, \ldots, J$, be closed subspaces of $V = V_J$ which will be associated with the level discretizations, and V_{ji}, $i = 1, \ldots, I_j$, be closed subspaces of V_j which will be associated with the domain decompositions on the levels. We consider $K \subset V$ a non empty closed convex subset and write $I = \max_{j=J,\ldots,1} I_j$.

To get sharper error estimations in the case of the multigrid method, we consider some constants $0 < \beta_{jk} \leq 1$, $\beta_{jk} = \beta_{kj}$, $j, \ k = J, \ldots, 1$, for which $\langle F'(v + v_{ji}) - F'(v), v_{kl}\rangle \leq \beta_M \beta_{jk} ||v_{ji}||^{q-1}||v_{kl}||$, for any $v \in V$, $v_{ji} \in V_{ji}$, $v_{kl} \in V_{kl}$ with $||v||$, $||v + v_{ji}||$, $||v_{kl}|| \leq M$, $i = 1, \ldots, I_j$ and $l = 1, \ldots, I_l$. Also, we fix a constant $\frac{p}{p-q+1} \leq \sigma \leq p$ and assume that there exists a constant C_1 such that $|| \sum_{j=1}^{J} \sum_{i=1}^{I_j} w_{ji}|| \leq C_1 (\sum_{j=1}^{J} \sum_{i=1}^{I_j} ||w_{ji}||^{\sigma})^{\frac{1}{\sigma}}$, for any $w_{ji} \in V_{ji}$, $j = J, \ldots, 1$, $i = 1, \ldots, I_j$. Evidently, in general, we can take $\beta_{jk} = 1$, $j, \ k = J, \ldots, 1$ and $C_1 = (IJ)^{\frac{\sigma-1}{\sigma}}$. In the multigrid methods, the convex sets where we look for the corrections are iteratively constructed from a level to another during the iterations in function of the current approximation. In this general background we make the following.

Assumption 2 *For a given $w \in K$, we recursively introduce the level convex sets $\mathcal{K}_j$, $j = J, \ J-1, \ \ldots, 1$, satisfying*

- *at level J: we assume that $0 \in \mathcal{K}_J, \ \mathcal{K}_J \subset \{v_J \in V_J : w + v_J \in K\}$ and consider a $w_J \in \mathcal{K}_J$,*
- *at a level $J-1 \geq j \geq 1$: we assume that $0 \in \mathcal{K}_j, \ \mathcal{K}_j \subset \{v_j \in V_j : w + w_J + \ldots + w_{j+1} + v_j \in K\}$ and consider a $w_j \in \mathcal{K}_j$.*

Also, we make a similar assumption with that in the case of the -one and two-level methods,

Assumption 3 *There exists two constants $C_2, \ C_3 > 0$ such that for any $w \in K$, $w_{ji} \in V_{ji}$, $w_{j1} + \ldots + w_{ji} \in \mathcal{K}_j$, $j = J, \ldots, 1$, $i = 1, \ldots, I_j$, and $u \in K$, there exist $u_{ji} \in V_{ji}$, $j = J, \ldots, 1$, $i = 1, \ldots, I_j$, which satisfy*

$$u_{j1} \in \mathcal{K}_j \text{ and } w_{j1} + \ldots + w_{ji-1} + u_{ji} \in \mathcal{K}_j, \ i = 2, \ldots, I_j, \ j = J, \ldots, 1,$$
$$u - w = \sum_{j=1}^{J} \sum_{i=1}^{I_j} u_{ji}, \ \sum_{j=1}^{J} \sum_{i=1}^{I_j} ||u_{ji}||^{\sigma} \leq C_2^{\sigma} ||u - w||^{\sigma} + C_3^{\sigma} \sum_{j=1}^{J} \sum_{i=1}^{I_j} ||w_{ji}||^{\sigma}$$

The convex sets $\mathcal{K}_j$, $j = J, \dots, 1$, *are constructed as in Assumption 2 with the above* w *and* $w_j = \sum_{i=1}^{I_j} w_{ji}$, $j = J, \dots, 1$.

The general subspace correction algorithm corresponding to the multigrid method is written as [see Algorithm 2.2 in Badea (2014) or Algorithm 1.1 in Badea (2015)],

Algorithm 2 *We start with an arbitrary* $u^0 \in K$. *At iteration* $n+1$ *we have* $u^n \in K$, $n \geq 0$, *and successively perform:*

- *at level J: as in Assumption 2, with* $w = u^n$, *we construct* $\mathcal{K}_J$.

  Then, we write $w_J^n = 0$, *and, for* $i = 1, \dots, I_J$, *we successively calculate* $w_{Ji}^{n+1} \in V_{Ji}$, $w_J^{n+\frac{i-1}{I_J}} + w_{Ji}^{n+1} \in \mathcal{K}_J$,

$$\langle F'(u^n + w_J^{n+\frac{i-1}{I_J}} + w_{Ji}^{n+1}), v_{Ji} - w_{Ji}^{n+1} \rangle \geq 0$$

for any $v_{Ji} \in V_{Ji}$, $w_J^{n+\frac{i-1}{I_J}} + v_{Ji} \in \mathcal{K}_J$, *and write* $w_J^{n+\frac{i}{I_J}} = w_J^{n+\frac{i-1}{I_J}} + w_{Ji}^{n+1}$.

- *at a level* $J-1 \geq j \geq 1$*: as in Assumption 2, we construct* $\mathcal{K}_j$ *with* $w = u^n$ *and* $w_J = w_J^{n+1}, \dots, w_{j+1} = w_{j+1}^{n+1}$.

Then, we write $w_j^n = 0$, *and for* $i = 1, \dots, I_j$, *we successively calculate* $w_{ji}^{n+1} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1} \in \mathcal{K}_j$,

$$\langle F'(u^n + \sum_{k=j+1}^{J} w_k^{n+1} + w_j^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}), v_{ji} - w_{ji}^{n+1} \rangle \geq 0$$

for any $v_{ji} \in V_{ji}$, $w_j^{n+\frac{i-1}{I_j}} + v_{ji} \in \mathcal{K}_j$, *and write* $w_j^{n+\frac{i}{I_j}} = w_J^{n+\frac{i-1}{I_j}} + w_{ji}^{n+1}$.

- *we write* $u^{n+1} = u^n + \sum_{j=1}^{J} w_j^{n+1}$.

Convergence of this algorithm is given by [see Theorem 1.1 in Badea (2015)]

Theorem 2 *Under the above conditions on the spaces and the functional F, if Assumptions 2 and 3 hold, then there exists an* $M > 0$ *such that* $||u^n|| \leq M$, *for any* $n \geq 0$, *and we have the following error estimations:*

(i) *if* $p = q = 2$ *we have* $||u^n - u||^2 \leq \frac{2}{\alpha_M}(\frac{\tilde{C}_1}{\tilde{C}_1+1})^n[F(u^0) - F(u)]$,

(ii) *if* $p > q$ *we have* $||u - u^n||^p \leq \frac{p}{\alpha_M} \frac{F(u^0)-F(u)}{[1+n\tilde{C}_2(F(u^0)-F(u))^{\frac{p-q}{q-1}}]^{\frac{q-1}{p-q}}}$,

where

$$\tilde{C}_1 = \frac{1}{C_2\varepsilon}\left[\frac{C_2}{\varepsilon} + 1 + C_1C_2 + C_3\right],$$

$$\tilde{C}_2 = \frac{p-q}{(p-1)(F(u^0)-F(u))^{\frac{p-q}{q-1}} + (q-1)\tilde{C}_3^{\frac{p-1}{q-1}}} \text{ with}$$

$$\tilde{C}_3 = \frac{\frac{\alpha_M}{p}}{C_2\varepsilon}\left[\frac{C_2}{\varepsilon^{\frac{1}{p-1}}(\frac{\alpha_M}{p})^{\frac{q-1}{p-1}}} + \frac{(1+C_1C_2+C_3)(IJ)^{\frac{p-\sigma}{p\sigma}}}{(\frac{\alpha_M}{p})^{\frac{q}{p}}}(F(u^0)-F(u))^{\frac{p-q}{p(p-1)}}\right]$$

$$\varepsilon = \frac{\alpha_M}{p}\frac{1}{2C_2\beta_M I^{\frac{\sigma-1}{\sigma}+\frac{p-q+1}{p}} J^{\frac{\sigma-1}{\sigma}-\frac{q-1}{p}}(\max_{k=1,\cdots,J}\sum_{j=1}^{J}\beta_{kj})}.$$

To get the multilevel method corresponding to Algorithm 2, we consider a family of regular meshes $\mathcal{T}_{h_j}$ of mesh sizes $h_j, j = 1, \ldots, J$, over the domain $\Omega \subset \mathbf{R}^d$ and assume that $\mathcal{T}_{h_{j+1}}$ is a refinement of $\mathcal{T}_{h_j}$. Let, at each level $j = 1, \ldots, J$, $\{\Omega_j^i\}_{1\le i\le I_j}$ be an overlapping decomposition of Ω, of overlapping size δ_j. We also assume that, for $1 \le i \le I_j$, the mesh partition $\mathcal{T}_{h_j}$ of Ω supplies a mesh partition for each Ω_j^i, $\mathrm{diam}(\Omega_{j+1}^i) \le Ch_j$ and $I_1 = 1$.

We introduce the linear finite element spaces, $V_{h_j} = \{v \in C(\bar{\Omega}_j) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_{h_j},\ v = 0 \text{ on } \partial\Omega_j\}, j = 1, \ldots, J$, corresponding to the level meshes, and $V_{h_j}^i = \{v \in V_{h_j} : v = 0 \text{ in } \Omega_j\backslash\Omega_j^i\}$, $i = 1, \ldots, I_j$, associated with the level decompositions. Spaces V_{h_j} $j = 1, \ldots, J-1$, will be considered as subspaces of $W^{1,\sigma}$, $1 \le \sigma \le \infty$.

The multilevel and multigrid methods will be obtained from Algorithm 2 for a two sided obstacle problem (1), i.e. the convex set is of the form $K = \{v \in V_{h_J} : \varphi \le v \le \psi\}$, with $\varphi,\ \psi \in V_{h_J}$, $\varphi \le \psi$. Concerning the construction of the level convex sets, we have [Proposition 3.1 in Badea (2014)]

Proposition 3 *Assumption 2 holds for the convex sets $\mathcal{K}_j$, $j = J, \ldots, 1$, defined as,*

- *for $w \in K$, at the level J, we take $\varphi_J = \varphi - w$, $\psi_J = \psi - w$, $\mathcal{K}_J = [\varphi_J, \psi_J]$, and consider an $w_J \in \mathcal{K}_J$,*
- *at a level $j = J-1, \ldots, 1$, we define $\varphi_j = I_{h_j}(\varphi_{j+1} - w_{j+1})$, $\psi_j = I_{h_j}(\psi_{j+1} - w_{j+1})$, $\mathcal{K}_j = [\varphi_j, \psi_j]$, and consider an $w_j \in \mathcal{K}_j$, $I_{h_j} : V_{h_{j+1}} \to V_{h_j}$, $j = 1, \ldots, J-1$, being some nonlinear interpolation operators between two consecutive levels.*

Also, our second assumption holds [see Proposition 2 in Badea (2015)],

Proposition 4 *Assumption 3 holds for the convex sets $\mathcal{K}_j$, $j = J, \ldots, 1$, defined in Proposition 3. The constants C_2 and C_3 are written as*

$$C_2 = CI^{\frac{\sigma+1}{\sigma}}(I+1)^{\frac{\sigma-1}{\sigma}}(J-1)^{\frac{\sigma-1}{\sigma}}[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma}]^{\frac{1}{\sigma}}$$
$$C_3 = CI^2(I+1)^{\frac{\sigma-1}{\sigma}}(J-1)^{\frac{\sigma-1}{\sigma}}[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma}]^{\frac{1}{\sigma}}$$

We proved that Assumptions 2 and 3 hold, and have explicitly written constants C_2 and C_3 in function of the mesh and overlapping parameters. We can then conclude from Theorem 2 that Algorithm 2 is globally convergent. Convergence rates given in Theorem 2 depend on the functional F, the maximum number of the subdomains on each level, I, and the number of levels J. Since the number of subdomains on levels can be associated with the number of colors needed to mark the subdomains such that the subdomains with the same color do not intersect with each other, we can conclude that the convergence rate essentially depends on the number of levels J.

In the general framework of multilevel methods we take $C_1 = CJ^{\frac{\sigma-1}{\sigma}} \max_{k=1,\cdots,J} \sum_{j=1}^{J} \beta_{kj} = J$ and, as functions depending only of J, we have

$$C_2 = C(J-1)^{\frac{\sigma-1}{\sigma}} S_{d,\sigma}(J) \text{ and } C_3 = C(J-1)^{\frac{\sigma-1}{\sigma}} S_{d,\sigma}(J) \text{ where}$$

$$S_{d,\sigma}(J) = \left[\sum_{j=2}^{J} C_{d,\sigma}(h_{j-1}, h_J)^{\sigma}\right]^{\frac{1}{\sigma}} = \begin{cases} (J-1)^{\frac{1}{\sigma}} & \text{if } d = \sigma = 1 \\ & \text{or } 1 \le d < \sigma < \infty \\ CJ & \text{if } 1 < d = \sigma < \infty \\ C^J & \text{if } 1 \le \sigma < d < \infty. \end{cases}$$

In the above multilevel methods a mesh is the refinement of that one on the previous level, but the domain decompositions are almost independent from one level to another. We obtain similar multigrid methods by decomposing the domain by the supports of the nodal basis functions of each level. Consequently, the subspaces $V_{h_j}^i$, $i = 1, \ldots, I_j$, are one-dimensional spaces generated by the nodal basis functions associated with the nodes of $\mathcal{T}_{h_j}$, $j = J, \ldots, 1$. In the case of the multigrid methods, we can take $C_1 = C$ and $\max_{k=1,\cdots,J} \sum_{j=1}^{J} \beta_{kj} = C$. Now we can write the convergence rate of the multigrid method corresponding to Algorithm 2 in function of the number of levels J for a given particular problem. In Badea (2014), the convergence rate of the multigrid method for the example in (2) has been written.

Remark 1 (See also Badea (2014))

1. The above results referred to problems in $W^{1,\sigma}$ with Dirichlet boundary conditions, but they also hold for Neumann or mixed boundary conditions.
2. Similar convergence results can be obtained for problems in $(W^{1,\sigma})^d$.
3. The analysis and the estimations of the global convergence rate which are given above refers to two sided obstacle problems which arise from the minimization of functionals defined on $W^{1,\sigma}$, $1 < \sigma < \infty$.

4. We can compare the convergence rates we have obtained with similar ones in the literature in the case of H^1 ($p = q = 2$) and $d = 2$. In this case, we get that the global convergence rate of Algorithm 2 is $1 - \frac{1}{1+CJ^3}$. The same estimate, of $1 - \frac{1}{1+CJ^3}$, is obtained by R. Kornhuber for the asymptotic convergence rate of the standard monotone multigrid methods for the complementarity problems.

Algorithm 2 is of multiplicative type over the levels as well as on each level, i.e. the current correction is found in function of all corrections on both the previous levels and the current level. We can also imagine hybrid algorithms where the type of the iteration over the levels is different from the type of the iteration on the levels. This idea can be also found in Smith et al. (1996). In Badea (2015), such hybrid algorithms (multiplicative over the levels—additive on levels, additive over the levels—multiplicative on levels and additive over the levels as well as on levels) have been introduced and analyzed in a similar manner with that of Algorithm 2. The following remark contains some conclusions withdrawn in Badea (2015) concerning the convergence rate (expressed only in function of J) of these hybrid algorithms for problem (2).

Remark 2

1. Regardless of the iteration type on levels, algorithms having the same type of iterations over the levels have the same convergence rate, provided that additive iterations on levels are parallelized.
2. The algorithms which are of multiplicative type over the levels converge better, by a factor of between $1/J$ and 1 (depending on σ), than their additive similar variants.

4 One- and Two-Level Methods for Variational Inequalities of the Second Kind and Quasi-Variational Inequalities

The results in this section are detailed in Badea and Krause (2012) where one- and two-level methods have been introduced and analyzed for the second kind and quasi-variational inequalities. In the case of the variational inequalities of the second kind, let $\varphi : K \to \mathbf{R}$ be a convex, lower semicontinuous, not differentiable functional and, if K is not bounded, we assume that $F + \varphi$ is coercive, i.e. $F(v) + \varphi(v) \to \infty$, as $\|v\| \to \infty,\ v \in K$. We consider the variational of the second kind

$$u \in K : \langle F'(u), v - u\rangle + \varphi(v) - \varphi(u) \geq 0, \text{ for any } v \in K \tag{3}$$

which, in view of the properties of F and φ, has a unique solution. An example of such a problem is given by the contact problems with Tresca friction. To solve problem (3), we introduce

Algorithm 3 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \geq 0$, we compute sequentially for $i = 1, \cdots, m$, the local corrections $w_i^{n+1} \in V_i$, $u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K$ as the solution of the variational inequality*

$$\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle + \varphi(u^{n+\frac{i-1}{m}} + v_i) - \varphi(u^{n+\frac{i-1}{m}} + w_i^{n+1}) \geq 0,$$

for any $v_i \in V_i$, $u^{n+\frac{i-1}{m}} + v_i \in K$, and then we update $u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}$.

To prove the convergence of the algorithm, we introduce a technical assumption,

$$\sum_{i=1}^{m}[\varphi(w + \sum_{j=1}^{i-1} w_j + v_i) - \varphi(w + \sum_{j=1}^{i-1} w_j + w_i)] \leq \varphi(v) - \varphi(w + \sum_{i=1}^{m} w_i)$$

for $v, w \in K$, and $v_i, w_i \in V_i$, $i = 1, \ldots, m$, in Assumption 1. In general, φ has not such a property and to show that this assumption holds when the finite element spaces are used, we have to take a numerical approximation of φ. The convergence of Algorithm 3 is proved by the following

Theorem 3 *Under the above assumptions on V, F and φ, let u be the solution of the problem and u^n, $n \geq 0$, be its approximations obtained from Algorithm 3. If Assumption 1 holds, then there exists $M > 0$ such that such that $\|u^{n+\frac{i}{m}}\| \leq M$, $n \geq 0, 1 \leq i \leq m$, and we have the following error estimations:*

(i) $\|u^n - u\|^2 \leq \frac{p}{\alpha_M}\left(\frac{\tilde{C}_1}{\tilde{C}_1+1}\right)^n \left[F(u^0) + \varphi(u^0) - F(u) - \varphi(u)\right]$ *if* $p = q = 2$,

(ii) $\|u - u^n\|^p \leq \frac{p}{\alpha_M}\frac{F(u^0)+\varphi(u^0)-F(u)-\varphi(u)}{\left[1+n\tilde{C}_2\left(F(u^0)+\varphi(u^0)-F(u)-\varphi(u)\right)^{\frac{p-q}{q-1}}\right]^{\frac{q-1}{p-q}}}$ *if* $p > q$,

where

$$\tilde{C}_1 = \beta_M(1+2C_0)m^{2-\frac{q}{p}}(\frac{p}{\alpha_M})^{\frac{q}{p}}\left(F(u^0) - F(u) + \varphi(u^0) - \varphi(u)\right)^{\frac{p-q}{p(p-1)}} + \beta_M C_0 m^{\frac{p-q+1}{p}} \frac{1}{\varepsilon^{\frac{1}{p-1}}}(\frac{p}{\alpha_M})^{\frac{q-1}{p-1}} \text{ with } \varepsilon = \alpha_M / \left(p\beta_M C_0 m^{\frac{p-q+1}{p}}\right),$$

$$\tilde{C}_2 = \frac{p-q}{(p-1)(F(u^0) + \varphi(u^0) - F(u) - \varphi(u))^{\frac{p-q}{q-1}} + (q-1)C_1^{\frac{p-1}{q-1}}}$$

In the case of the quasivariational inequalities, we consider only the case of $p = q = 2$ and let $\varphi : K \times K \to \mathbf{R}$ be a functional such that, for any $u \in K$, $\varphi(u, \cdot) : K \to \mathbf{R}$ is convex, lower semicontinuous and, if K is not bounded, $F(\cdot) + \varphi(u, \cdot)$ is coercive, i.e. $F(v) + \varphi(u, v) \to \infty$ as $\|v\| \to \infty, v \in K$. We assume that for any $M > 0$ there exists a constant $c_M > 0$ such that

$$|\varphi(v_1, w_2) + \varphi(v_2, w_1) - \varphi(v_1, w_1) - \varphi(v_2, w_2)| \leq c_M||v_1 - v_2||||w_1 - w_2||$$

for any $v_1,\ v_2,\ w_1\ w_2 \in K$, $||v_1||,\ ||v_2||,\ ||w_1||\ ||w_2|| \le M$. If φ has the above property, the quasi-variational inequality

$$u \in K : \langle F'(u), v-u\rangle + \varphi(u,v) - \varphi(u,u) \ge 0, \text{ for any } v \in K$$

has a unique solution. An example of such a problem is given by the contact problems with non-local Coulomb friction. We can write three algorithms depending on the first argument of φ.

Algorithm 4 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \ge 0$, we compute sequentially for $i = 1,\cdots,m$, the local corrections $w_i^{n+1} \in V_i$, $u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K$, satisfying*

$$\begin{aligned}&\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle + \varphi(v_i^{n+1}, u^{n+\frac{i-1}{m}} + v_i)\\ &-\varphi(v_i^{n+1}, u^{n+\frac{i-1}{m}} + w_i^{n+1}) \ge 0,\end{aligned}$$

for any $v_i \in V_i$, $u^{n+\frac{i-1}{m}} + v_i \in K$, and then we update $u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}$.

Above, the first argument v_i^{n+1} of φ can be taken either $u^{n+\frac{i-1}{m}} + w_i^{n+1}$ or $u^{n+\frac{i-1}{m}}$ or even u^n. As we shall see in the next convergence theorem, the three variants of the algorithm are convergent. Similarly with the case of the inequalities of the second kind, we introduce the technical assumption

$$\sum_{i=1}^{m}[\varphi(u, w + \sum_{j=1}^{i-1} w_j + v_i) - \varphi(u, w + \sum_{j=1}^{i} w_j)] \le \varphi(u,v) - \varphi(u, w + \sum_{i=1}^{m} w_i)$$

for any $u \in K$ and for $v, w \in K$ and $v_i, w_i \in V_i$, $u^{n+\frac{i-1}{m}} + v_i \in K$, $i = 1,\ldots,m$, in Assumption 1. Also, in the finite element spaces, φ of the continuous problem is numerically approximated in order to get the above assumption satisfied. Convergence of the three algorithms is proved by

Theorem 4 *Under the above assumptions on V, F and φ, let u be the solution of the problem and u^n, $n \ge 0$, be its approximations obtained from one of the variants of Algorithm 4. If Assumption 1 holds, and if $\frac{\alpha_M}{2} \ge mc_M + \sqrt{2m(25C_0+8)\beta_M c_M}$, for any $M > 0$, then there exists an $M > 0$ such that $\|u^{n+\frac{i}{m}}\| \le M$, $n \ge 0, 1 \le i \le m$, and we have the following error estimation*

$$\|u^n - u\|^2 \le \frac{2}{\alpha_M}\left(\frac{\tilde{C}_1}{\tilde{C}_1 + 1}\right)^n \left[F(u^0) + \varphi(u,u^0) - F(u) - \varphi(u,u)\right].$$