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Fractional Elliptic Problems with Critical Growth in the Whole of $\mathbb{R}^n$

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APPUNTI

LECTURE NOTES

Serena Dipierro
School of Mathematics and Statistics
University of Melbourne
813 Swanston Street
Parkville VIC 3010, Australia

María Medina
Departamento de Matemáticas
Universidad Autónoma de Madrid 28049, Madrid, Spain
and
Facultad de Matemáticas
Pontificia Universidad Católica de Chile
Avenida Vicuña Mackenna 4860
Santiago, Chile

Enrico Valdinoci
School of Mathematics and Statistics
University of Melbourne
813 Swanston Street
Parkville VIC 3010, Australia
and
Università degli studi di Milano
Via Saldini 50, 20133 Milano, Italia
and
Istituto di Matematica Applicata e Tecnologie Informatiche
Consiglio Nazionale delle Ricerche
Via Ferrata 1, 27100 Pavia, Italia

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Preface

This is a research monograph devoted to the analysis of a nonlocal equation in the whole of the Euclidean space. In studying this equation, we will introduce all the necessary material in the most self-contained way as possible, giving precise reference to the literature when necessary.

In further detail, we study here the following nonlinear and nonlocal elliptic equation in $\mathbb{R}^n$

$$(-\Delta)^s u = \varepsilon\, h\, u^q + u^p \ \text{ in } \mathbb{R}^n,$$

where $s \in (0, 1)$, $n > 2s$, $\varepsilon > 0$ is a small parameter, $p = \frac{n+2s}{n-2s}$, $q \in (0, 1)$, and $h \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. The problem has a variational structure, and this allows us to find a positive solution by looking at critical points of a suitable energy functional. In particular, in this monograph, we find a local minimum and a different solution of this functional (this second solution is found by a contradiction argument which uses a mountain pass technique, so the solution is not necessarily proven to be of mountain pass type).

One of the crucial ingredient in the proof is the use of a suitable Concentration-Compactness principle.

Some difficulties arise from the nonlocal structure of the problem and from the fact that we deal with an equation in the whole of $\mathbb{R}^n$ (and this causes lack of compactness of some embeddings). We overcome these difficulties by looking at an equivalent extended problem.

This monograph is organized as follows.

Chapter 1 gives an elementary introduction to the techniques involved, providing also some motivations for nonlocal equations and auxiliary remarks on critical point theory.

Chapter 2 gives a detailed description of the class of problems under consideration (including the main equation studied in this monograph) and provides further motivations.

Chapter 3 introduces the analytic setting necessary for the study of nonlocal and nonlinear equations (this part is of rather general interest, since the functional analytic setting is common in different problems in this area).

The research oriented part of the monograph is mainly concentrated in Chapters 4, 5 and 6 (as a matter of fact, Chapter 5 may also be of general interest, since it deals with a regularity theory for a general class of equations).

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Chapter 1
Introduction

This research monograph deals with a nonlocal problem with critical nonlinearities. The techniques used are variational and they rely on classical variational methods, such as the mountain pass theorem and the Concentration-Compactness Principle (suitably adapted, in order to fit with the nonlocal structure of the problem under consideration). The subsequent sections will give a brief introduction to the fractional Laplacian and to the variational methods exploited.

Of course, a comprehensive introduction goes far beyond the scopes of a research monograph, but we will try to let the interested reader get acquainted with the problem under consideration and with the methods used in a rather self-contained form, by keeping the discussion at the simplest possible level (but trying to avoid oversimplifications). The expert reader may well skip this initial overview and go directly to Chapter 2.

1.1. The fractional Laplacian

The operator dealt with in this paper is the so-called fractional Laplacian.

For a "nice" function u (for instance, if u lies in the Schwartz Class of smooth and rapidly decreasing functions), the s power of the Laplacian, for $s \in (0, 1)$, can be easily defined in the Fourier frequency space. Namely, by taking the Fourier transform

$$\hat{u}(\xi) = \mathcal{F}u(\xi) = \int_{\mathbb{R}^n} u(x)\, e^{-2\pi i x\cdot\xi}\, dx,$$

and by looking at the Fourier Inversion Formula

$$u(x) = \mathcal{F}^{-1}\hat{u}(x) = \int_{\mathbb{R}^n} \hat{u}(\xi)\, e^{2\pi i x\cdot\xi}\, d\xi,$$

one notices that the derivative (say, in the kth coordinate direction) in the original variables corresponds to the multiplication by $2\pi i\xi_k$ in the frequency variables, that is

$$\partial_k u(x) = \int_{\mathbb{R}^n} 2\pi i \xi_k \, \hat{u}(\xi) \, e^{2\pi i x\cdot\xi} \, d\xi = \mathcal{F}^{-1}\big(2\pi i \xi_k \, \hat{u}\big).$$

Accordingly, the operator $(-\Delta) = -\sum_{k=1}^{n} \partial_k^2$ corresponds to the multiplication by $(2\pi|\xi|)^2$ in the frequency variables, that is

$$-\Delta u(x) = \int_{\mathbb{R}^n} (2\pi|\xi|)^2 \, \hat{u}(\xi) \, e^{2\pi i x\cdot\xi} \, d\xi = \mathcal{F}^{-1}\big((2\pi|\xi|)^2 \, \hat{u}\big).$$

With this respect, it is not too surprising to define the power s of the operator $(-\Delta)$ as the multiplication by $(2\pi|\xi|)^{2s}$ in the frequency variables, that is

$$(-\Delta)^s u(x) := \mathcal{F}^{-1}\big((2\pi|\xi|)^{2s} \, \hat{u}\big). \tag{1.1.1}$$

Another possible approach to the fractional Laplacian comes from the theory of semigroups and fractional calculus. Namely, for any $\lambda > 0$, using the substitution $\tau = \lambda t$ and an integration by parts, one sees that

$$\int_0^{+\infty} t^{-s-1}(e^{-\lambda t} - 1)\, dt = \Gamma(-s)\, \lambda^s,$$

where Γ is the Euler's Gamma-function. Once again, not too surprising, one can define the fractional power of the Laplacian by formally replacing the positive real number λ with the positive operator $-\Delta$ in the above formula, that is

$$(-\Delta)^s := \frac{1}{\Gamma(-s)} \int_0^{+\infty} t^{-s-1}(e^{\Delta t} - 1)\, dt,$$

which reads as

$$(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^{+\infty} t^{-s-1}(e^{\Delta t} u(x) - u(x))\, dt. \tag{1.1.2}$$

Here above, the function $U(x,t) = e^{\Delta t} u(x)$ is the solution of the heat equation $\partial_t U = \Delta U$ with initial datum $U|_{t=0} = u$.

The equivalence between the two definitions in (1.1.1) and (1.1.2) can be proved by suitable elementary calculations, see *e.g.* [14].

The two definitions in (1.1.1) and (1.1.2) are both useful for many properties and they give different useful pieces of information. Nevertheless, in this monograph, we will take another definition, which is equivalent to the ones in (1.1.1) and (1.1.2) (at least for nice functions), but which is more flexible for our purposes. Namely, we set

$$\begin{aligned}(-\Delta)^s u(x) &:= c_{n,s}\, PV \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy \\ &:= c_{n,s} \lim_{r\to 0} \int_{\mathbb{R}^n \setminus B_r(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy,\end{aligned} \tag{1.1.3}$$

where

$$c_{n,s} := \frac{2^{2s}\, s\, \Gamma\left(\frac{n}{2}+s\right)}{\pi^{n/2}\,\Gamma(1-s)}.$$

See for instance [14] for the equivalence of (1.1.3) with (1.1.1) and (1.1.2).

Roughly speaking, our preference (at least for what concerns this monograph) for the definition in (1.1.3) lies in the following features. First of all, the definition in (1.1.3) is more unpleasant, but geometrically more intuitive (and often somehow more treatable) than the ones in (1.1.1) and (1.1.2), since it describes an incremental quotient (of differential order $2s$) weighted in the whole of $\mathbb{R}^n$. As a consequence, one may obtain a "rough" idea on how $(-\Delta)^s$ looks like by considering the oscillations of the original function u, suitably weighted.

Conversely, the definitions in (1.1.1) and (1.1.2) are perhaps shorter and more evocative, but they require some "hidden calculations" since they involve either the Fourier transform or the heat flow of the function u, rather than the function u itself.

Moreover, the definition in (1.1.3) has straightforward probabilistic interpretations (see *e.g.* [14] and references therein) and can be directly generalized to other singular integrodifferential kernels (of course, in many cases, even when dealing in principle with the definition in (1.1.3), the other equivalent definitions do provide additional results).

In addition, by taking the definition in (1.1.3), we do not need u to be necessarily in the Schwartz Class, but we can look at weak, distributional solutions, in a similar way to the theory of classical Sobolev spaces. We refer for instance to [25] for a basic discussion on the fractional Sobolev spaces and to [44] for the main functional analytic setting needed in the study of variational problems.

To complete this short introduction to the fractional Laplacian, we briefly describe a simple probabilistic motivation arising from game theory on a traced space (here, we keep the discussion at a simple, and even heuristic level, see for instance [10, 42] and the references therein for further details). The following discussion describes the fractional Laplacian occurring as a consequence of a classical random process in one additional dimension (see Figure 1.1.1).

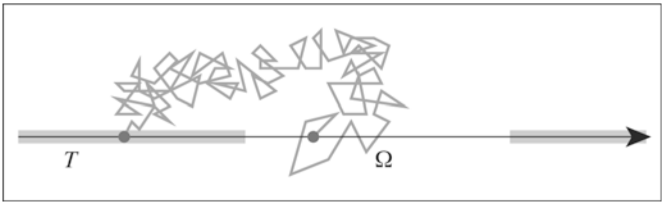

Figure 1.1.1. A Brownian motion on $\mathbb{R}^{n+1}$ and the payoff on $\mathbb{R}^n \times \{0\}$.

We consider a bounded and smooth domain $\Omega \subset \mathbb{R}^n$ and a nice (and, for simplicity, rapidly decaying) payoff function $f : \mathbb{R}^n \setminus \Omega \to [0, 1]$. We immerse this problem into $\mathbb{R}^{n+1}$, by defining $\Omega_* := \Omega \times \{0\}$.

The game goes as follows: we start at some point of Ω_* and we move randomly in $\mathbb{R}^{n+1}$ by following a Brownian motion, till we hit $(\mathbb{R}^n \setminus \Omega) \times \{0\}$ at some point p: in this case we receive a payoff of $f(p)$ livres.

For any $x \in \mathbb{R}^n$, we denote by $u(x)$ the expected value of the payoff when we start at the point $(x, 0) \in \mathbb{R}^{n+1}$ (that is, roughly speaking, how much we expect to win if we start the game from the point $(x, 0) \in \Omega \times \{0\}$). We will show that u is solution of the fractional equation

$$\begin{cases} (-\Delta)^{1/2} u = 0 & \text{in } \Omega, \\ u = f & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.1.4}$$

Notice that the condition $u = f$ in $\mathbb{R}^n \setminus \Omega$ is obvious from the construction (if we start directly at a place where a payoff is given, we get that). So the real issue is to understand the equation satisfied by u.

For this scope, for any $(x, y) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$, we denote by $U(x, y)$ the expected value of the payoff when we start at the point (x, y). We observe that $U(x, 0) = u(x)$. Also, we define $T := (\mathbb{R}^n \setminus \Omega) \times \{0\}$ (T is our "target" domain) and we claim that U is harmonic in $\mathbb{R}^{n+1} \setminus T$, *i.e.*

$$\Delta U = 0 \text{ in } \mathbb{R}^{n+1} \setminus T. \tag{1.1.5}$$

To prove this, we argue as follows. Fix $P \in \mathbb{R}^{n+1} \setminus T$ and a small ball of radius $r > 0$ around it, such that $B_r(P) \subset \mathbb{R}^{n+1} \setminus T$. Then, the expected value that we receive starting the game from P should be the average of the expected value that we receive starting the game from another point $Q \in \partial B_r(P)$ times the probability of drifting from Q to P. Since the Brownian motion is rotationally invariant, all the points on the sphere have the same probability of drifting towards P, and this gives that

$$U(P) = \fint_{\partial B_r(P)} U(Q)\, d\mathcal{H}^n(Q).$$

That is, U satisfies the mean value property of harmonic functions, and this establishes (1.1.5).

Furthermore, since the problem is symmetric with respect to the $(n+1)$th variable we also have that $U(x, y) = U(x, -y)$ for any $x \in \Omega$ and so

$$\partial_y U(x, 0) = 0 \text{ for any } x \in \Omega. \tag{1.1.6}$$

Now we take Fourier transforms in the variable $x \in \mathbb{R}^n$, for a fixed $y > 0$. From (1.1.5), we know that $\Delta U(x, y) = 0$ for any $x \in \mathbb{R}^n$ and $y > 0$, therefore

$$-(2\pi|\xi|)^2 \hat{U}(\xi, y) + \partial_{yy}\hat{U}(\xi, y) = 0,$$

for any $\xi \in \mathbb{R}^n$ and $y > 0$. This is an ordinary differential equation in $y > 0$, which can be explicitly solved: we find that

$$\hat{U}(\xi, y) = \alpha(\xi)\, e^{2\pi|\xi|y} + \beta(\xi)\, e^{-2\pi|\xi|y},$$

for suitable functions α and β. As a matter of fact, since

$$\lim_{y \to +\infty} e^{2\pi|\xi|y} = +\infty,$$

to keep $\hat{U}$ bounded we have that $\alpha(\xi) = 0$ for any $\xi \in \mathbb{R}^n$. This gives that

$$\hat{U}(\xi, y) = \beta(\xi)\, e^{-2\pi|\xi|y}.$$

We now observe that

$$\hat{u}(\xi) = \hat{U}(\xi, 0) = \beta(\xi),$$

therefore

$$\hat{U}(\xi, y) = \hat{u}(\xi)\, e^{-2\pi|\xi|y}$$

and so

$$\mathcal{F}(\partial_y U)(\xi, y) = \partial_y \hat{U}(\xi, y) = -2\pi|\xi|\, \hat{u}(\xi)\, e^{-2\pi|\xi|y}.$$

In particular, $\mathcal{F}(\partial_y U)(\xi, 0) = -2\pi|\xi|\, \hat{u}(\xi)$. Hence we exploit (1.1.6) (and we also recall (1.1.1)): in this way, we obtain that, for any $x \in \Omega$,

$$0 = \partial_y U(x, 0) = -\mathcal{F}^{-1}\Big(2\pi\, |\xi|\, \hat{u}(\xi)\Big)(x) = -(-\Delta)^{1/2} u(x),$$

which proves (1.1.4).

1.2. The mountain pass theorem

Many of the problems in mathematical analysis deal with the construction of suitable solutions. The word "construction" is often intended in a "weak" sense, not only because the solutions found are taken in a "distributional" sense, but also because the proof of the existence of the solution is often somehow not constructive (though some qualitative or quantitative properties of the solutions may be often additionally found).

In some cases, the problem taken into account presents a variational structure, namely the desired solutions may be found as critical points of a functional (this functional is often called "energy" in the literature, though it is in many cases related more to a "Lagrangian action" from the physical point of view).

When the problem has a variational structure, it can be attacked by all the methods which aim to prove that a functional indeed possesses a critical point. Some of these methods arise as the "natural" generalizations from basic Calculus to advanced Functional Analysis: for instance, by a variation of the classical Weierstraß Theorem, one can find solutions corresponding to local (or sometimes global) minima of the functional.

In many circumstances, these minimal solutions do not exhaust the complexity of the problem itself. For instance, the minimal solutions happen in many cases to be "trivial" (for example, corresponding to the zero solution). Or, in any case, solutions different from the minimal ones may exist, and they may indeed have interesting properties. For example, the fact that they come from a "higher energy level" may allow them to show additional oscillations, or having "directions" along which the energy is not minimized may produce some intriguing forms of instabilities.

Detecting non-minimal solutions is of course, in principle, harder than finding minimal ones, since the direct methods leading to the Weierstraß Theorem (basically reducing to compactness and some sort of continuity) are in general not enough.

As a matter of fact, these methods need to be implemented with the aid of additional "topological" methods, mostly inspired by Morse Theory (see [41]). Roughly speaking, these methods rely on the idea that critical points prevent the energy graph to be continuously deformed by following lines of steepest descent (*i.e.* gradient flows).

One of the most important devices to detect critical points of non-minimal type is the so called mountain pass theorem. This result can be pictorially depicted by thinking that the energy functional is simply the elevation $\mathcal{E}$ of a given point on the Earth. The basic assumption of the mountain pass theorem is that there are (at least) two low spots in the landscape, for instance, the origin, which (up to translations) is supposed to lie at the sea level (say, $\mathcal{E}(0) = 0$) and a far-away place p which also lies at the sea level, or even below (say, $\mathcal{E}(p) \leqslant 0$).

The origin is also supposed to be surrounded by points of higher elevation (namely, there exist $r, a > 0$ such that $\mathcal{E}(u) \geqslant a$ if $|u| = r$). Under this assumption, any path joining the origin with p is supposed to "climb up" some mountains (*i.e.*, it has to go up, at least at level $a > 0$, and then reach again the sea level in order to reach p).

Thus, each of the path joining 0 to p will have a highest point. If one needs to travel in "real life" from 0 to p, then (s)he would like to minimize the value of this highest point, to make the effort as small as possible. This corresponds, in mathematical jargon, to the search of the value

$$c := \inf_{\Gamma} \sup_{t\in[0,1]} \mathcal{E}(g(t)), \tag{1.2.1}$$

where Γ is the collection of all possible path g such that $g(0) = 0$ and $g(1) = p$.

Roughly speaking, one should expect c to be a critical value of saddle type, since the "minimal path" has a maximum in the direction "transversal to the range of mountains", but has a minimum with respect to the tangential directions, since the competing paths reach a higher altitude.

A possible picture of the structure of this mountain pass is depicted in Figure 1.2.2. On the other hand, to make the argument really work, one needs a compactness condition, in order to avoid that the critical point "drifts to infinity".

We stress that this loss of compactness for critical points is not necessarily due to the fact that one works in complicate functional spaces, and indeed simple examples can be given even in Calculus curses, see for instance the following example taken from Exercise 5.42 in [21]. One can consider the function of two real variables

$$f(x, y) = (e^x + e^{-x^2})y^2(2 - y^2) - e^{-x^2} + 1.$$

By construction $f(0, 0) = 0$,

$$\begin{aligned} \partial_x f &= (e^x - 2xe^{-x^2})y^2(2 - y^2) + 2xe^{-x^2}, \\ \partial_y f &= 2(e^x + e^{-x^2})y(2 - y^2) - 2(e^x + e^{-x^2})y^3 \\ \text{and} \qquad D^2 f(0, 0) &= \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}. \end{aligned}$$

As a consequence, the origin is a nondegenerate local minimum for f. In addition, $f(0, \sqrt{2}) = 0$, so the geometry of the mountain pass is satisfied.

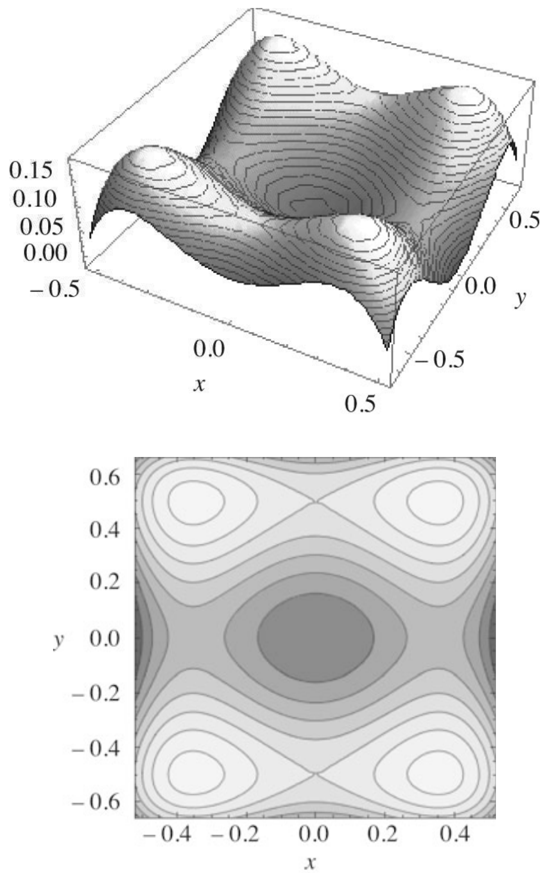

Figure 1.2.2. The function $z = x^2 + y^2 - 4x^4 - 2y^4$ (3D plot and level sets).

Nevertheless, the function f does not have any other critical points except the origin. Indeed, a critical point should satisfy

$$(e^x - 2xe^{-x^2})y^2(2 - y^2) + 2xe^{-x^2} = 0 \tag{1.2.2}$$

$$\text{and} \quad 2(e^x + e^{-x^2})y(2 - y^2) - 2(e^x + e^{-x^2})y^3 = 0. \tag{1.2.3}$$

If $y = 0$, then we deduce from (1.2.2) that also $x = 0$, which gives the origin. So we can suppose that $y \neq 0$ and write (1.2.3) as

$$2(e^x + e^{-x^2})(2 - y^2) - 2(e^x + e^{-x^2})y^2 = 0,$$

which, after a further simplification gives $(2 - y^2) - y^2 = 0$, and therefore $y = \pm 1$.

By inserting this into (1.2.2), we obtain that

$$0 = (e^x - 2xe^{-x^2}) + 2xe^{-x^2} = e^x,$$