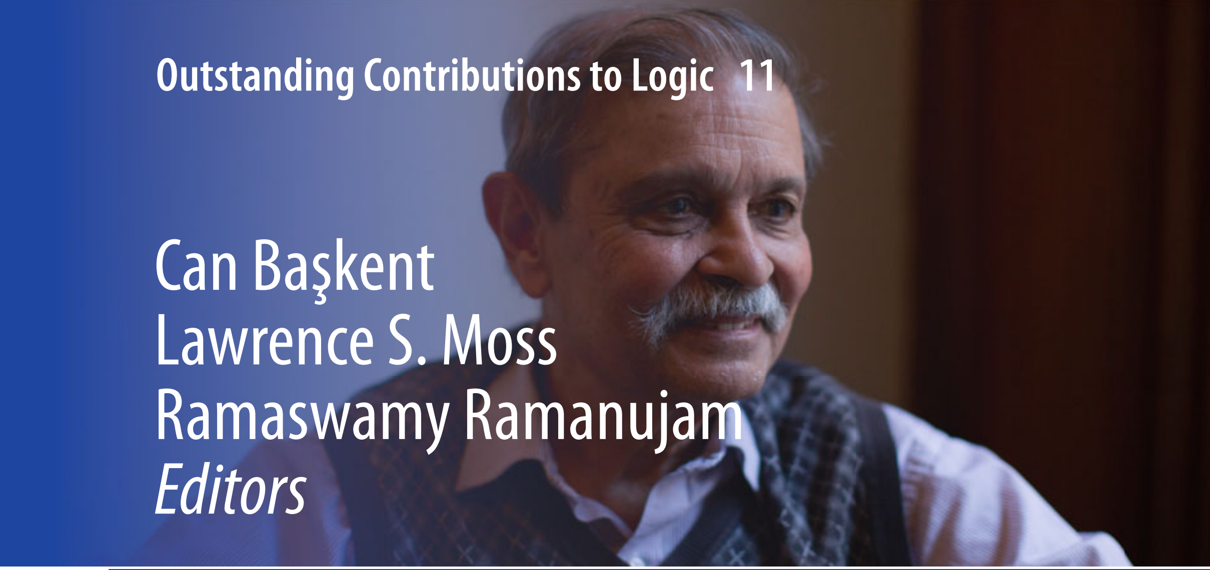
Outstanding Contributions to Logic 11



Can Başkent
Lawrence S. Moss
Ramaswamy Ramanujam
Editors

Rohit Parikh on Logic, Language and Society

Springer

Outstanding Contributions to Logic

Volume 11

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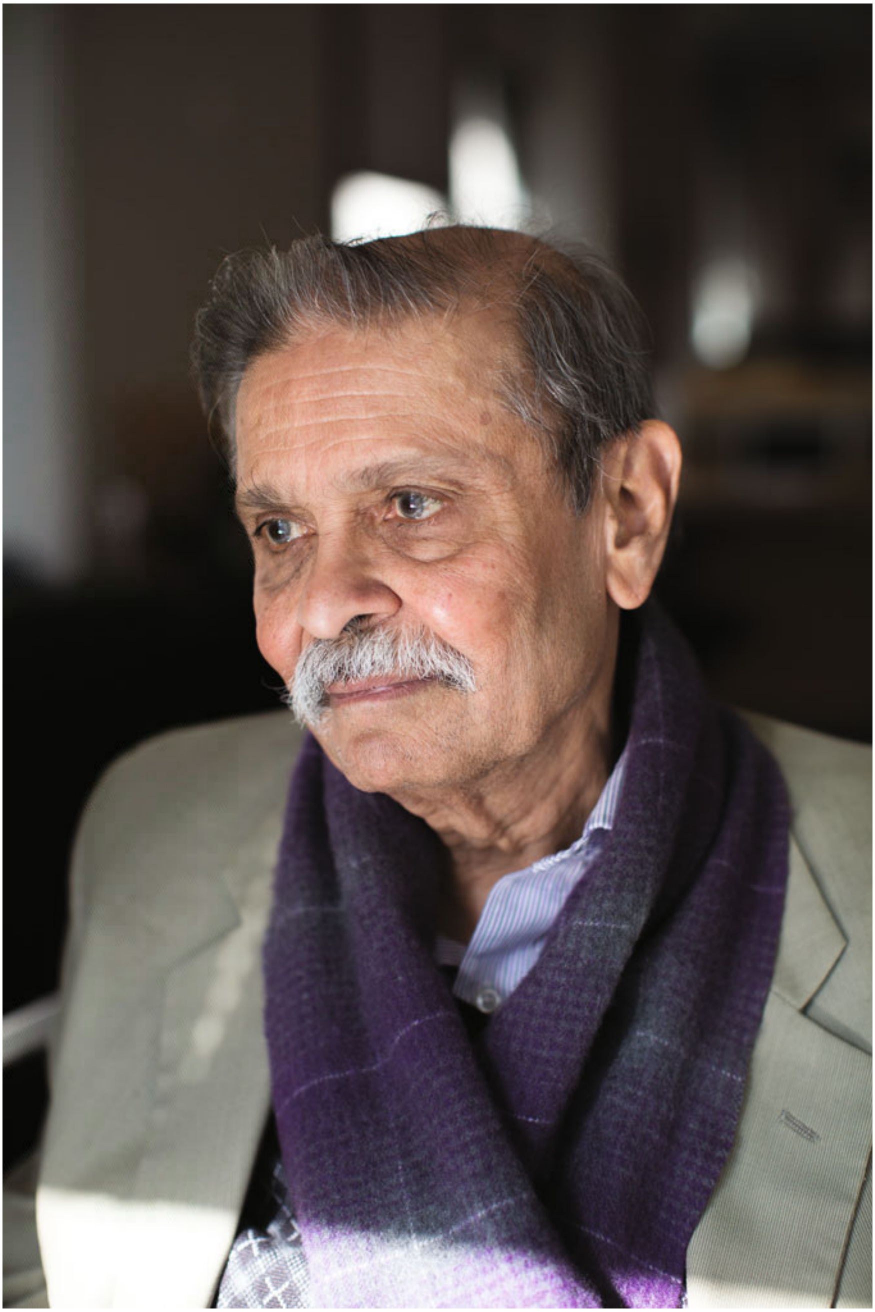
ISSN 2211-2758 ISSN 2211-2766 (electronic)
Outstanding Contributions to Logic
ISBN 978-3-319-47842-5 ISBN 978-3-319-47843-2 (eBook)
DOI 10.1007/978-3-319-47843-2

Library of Congress Control Number: 2016954021

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland



Cover and in-page photographs of Rohit Parikh: Lauren Fleishman

Editors' Preface

This volume is part of Springer's book series *Outstanding Contributions to Logic*. Without doubt the contributions made by Rohit Jivanlal Parikh to logic have been deep and outstanding. Rohit is a leader in the realm of ideas, offering concepts and definitions that enrich the field and lead to new research directions.

Rohit has contributed to a variety of areas in logic, computer science and game theory: in mathematical logic they have been in recursive function theory, proof theory and non-standard analysis; in computer science, in the areas of modal, temporal and dynamic logics of programs and semantics of programs, as well as logics of knowledge; in artificial intelligence in the area of belief revision; in game theory in the formal analysis of social procedures; in all this there is a strong undercurrent of philosophy as well. With such a wide-ranging set of contributions and ever-expanding intellectual interests, we have no possibility of summarizing his work or even take the rider for an extensive tour in Parikh-land. What we do is more in the spirit of the tour brochure, listing attractions 'not to be missed'.

Finally, this volume attempts to underline the academic trajectory of a brilliant scholar whose work opened up various new avenues in research. We will briefly discuss the milestones of Parikh's scholarly work, hoping to give a sense of how he has developed his recent research program *social software* and the formal and philosophical toolkit that lies behind it. We believe that such an impressive trajectory can only help motivate the young researchers and scholars.

As our mentor, colleague, co-author, and always as our teacher, Rohit taught us a lot. This volume is a simple but honest thank you to him from all of us.

A Brisk Tour of Parikh-Land

Parikh's (1966a) paper is a classic: it introduces the notion of *semi-linearity* and proves that commutative maps of context-free languages are semi-linear sets. The paper is a revised version of a 1961 MIT report, and it was published at the invitation of Donald Knuth. Today Parikh maps and Parikh vectors are such common usage in formal language theory that many papers use lower case as in 'parikh maps'. The paper also establishes the existence of inherently ambiguous context-free languages, which again led to a slew of results informal language theory.

Parikh (1966b) is an early paper of Parikh on *well-orderings*, a theme that would recur in his work over the next couple of decades. This paper shows that there are linear orders which are pseudo-well-ordered in that they have no recursive descending sequences, but exponentiating them yields orders that have primitive recursive descending sequences. A decade later, the paper (Parikh and de Jongh 1977) sets up a very interesting connection between hierarchies defined by closure operations and well-partial-orderings. A decade later, Parikh (1986) makes connections between this topic and two very different ones: formulas in logics of knowledge and automata theory. This paper shows precisely what levels of knowledge are achievable by a collection of communicating processes.

Parikh (1971) may well have launched the area of *bounded arithmetic*. 0 is a *small* number and whenever n is small, surely $n + 1$ is small. But a number like 10^{20} is of course not small. The paper then suggests that the length of proof establishing smallness of a number should be relevant (rather than unconstrained induction), and we can have an arithmetic that is conservative for proofs of low complexity. The paper also shows that there are formulae whose proofs are long, but the proof that they are provable could be short. Today, the study of systems of bounded arithmetic and their connection to complexity theory is a fertile area of research.

Parikh (1982, 1994) on *vagueness* are what we might call a typical Parikh phenomenon, addressing a problem of intense philosophical interest and offering conceptual guidance to those addressing it mathematically. Even more than the results, what the paper offers is clarity on concepts, and has influenced a generation of research on approximate and inexact reasoning.

The paper (Parikh 1978) and its companion (Kozen and Parikh 1981) provide an essential element that is now founded in the toolkit of every graduate student in logics of computation today. Following Kozen and Parikh, one proves a completeness theorem for the propositional dynamic logic of regular programs and combines this with a small model property leading to a decision procedure. This is perhaps the most authoritative handling of Kleene iteration in logic yet. Parikh went on to made extensive mathematical contributions to the study of dynamic and process logics.

Parikh (1983) is another landmark, a brilliant logicisation of two-person games that offered technical questions that remain open to this day. If logic can be given its meaning entirely in game theoretic terms, it is reasonable to ask what reasoning about existence of strategies in such games may mean. Game logics are a fertile area of study today.

Parikh (1984) connects logics of knowledge and non-monotonic reasoning in essential ways that would influence thinking in research on dynamic epistemic logics almost two decades later. In a series of papers over the next decade, Parikh has enriched the theory of knowledge in its interaction with communication and action, exploring intrinsic logical questions.

Moss and Parikh (1992) opens up another area, the study of topology via logic. The use of epistemic logic in this context enriches and expands our understanding of topological spaces and this study flourishes to date.

Parikh (1995) formally launches Parikh's idea of *social software*, one that has kept him creatively occupied for the last two decades, enriching interaction between logic and game theory. Social software is the analysis of social procedures by exploring their logical underpinnings. For instance, notions like social obligation, why politicians lie during election campaigns, can be studied logically.

Parikh (1999) is another landmark, this time offering a new technique, that of language splitting, in the extensively studied area of belief revision. The paper shows how we can incorporate a formal notion of relevance, which allows one's information to be split uniquely into a number of disjoint subject areas. This idea has been subsequently been extended fruitfully by Makinson and others, leading to many new directions of research.

Through all this, Parikh's contributions to fundamental logic (e.g. Parikh and Väänänen 2005) and mathematics (Parikh and Nathanson 2007) continue. His recent contributions to philosophical thought (e.g. Parikh 2013; Ginés and Parikh 2015) in the arena of logic, games, language and computation, raise a number of conceptual issues and offer new approaches which will guide research for a long time to come.

This Volume

The contributors to this volume honor Rohit Parikh and his *oeuvre* in many ways. This is not a collection of articles on one theme, or even directly connected to specific works of Parikh, but inspired and influenced by Parikh in some way, adding structures to Parikh-land, enriching it. Our goal in what follows is to say a few brief words about each of the contributions.

Juliet Floyd illustrates the transition in Parikh's interest from formal languages to natural languages, and how Parikh approached Wittgenstein's philosophy of language. In fact, the article describes, as Floyd put it why "Wittgenstein owes Parikh a big 'Thank You'."

Prashant Parikh notes that "Rohit Parikh may have been the first person to study how communication with vague expressions can be fruitful even when the speaker and addressee interpret them differently". Prashant's paper continues this line of work, introducing models from cognitive psychology to analyze vague communication.

Robert van Rooij's paper employs non-classical logic in an analysis of a well-known epistemic paradox, Fitch's Paradox. The topic is attractive for both philosophers and epistemic logicians alike, and van Rooij's contribution hopefully will initiate more interaction between these relatively separate research communities.

If we wish to take seriously Parkih's idea of social software, *money* would seem to be a central social and computational resource. Jan van Eijck and Philip Elsas orchestrate a thought-provoking Socratic dialogue on the function of money, its organizing role in society and its underlying logical principles.

Dominik Klein and Eric Pacuit focus on another important aspect of social software, and one of Parikh's recent interests: voting and political campaigns. Klein and Pacuit discuss a qualitative analysis of voters' changing opinions during a political campaign. Such analyses will be crucial to advance Parikh's program of analysis of social procedures.

Can Başkent discusses the role of classical logic in the social sofware enterprise, and offers an interesting extension of social software to paraconsistent logic, arguing that non-classical logics provide the theme with a broader domain of applications. The paper follows the footsteps of Parikh (2002) and takes a logical pluralist stand on the subject.

Epistemic logic is a central arena of Parikh's work, and much of his work is on delineating notions of knowledge, syntactically and semantically. Knowing a proposition is distinct from the sentential knowledge of the proposition. While economists in general prefer to work directly with propositions at the model level, logicians prefer to work with compositional syntax. This distinction runs through Parikh's work (Parikh 2005) and the paper by Joseph Halpern addresses the issue. He gives strong arguments why syntax can help make finer distinctions and describe notions in a model-independent manner.

Another contribution to this volume on knowledge is Johan van Benthem's exploration of epistemology. The paper is foundational, a far-reaching exploration of a number of themes weaving epistemology, dynamic logic, information, modality, and action. Many of these themes run through Parikh's oeuvre as well. (It should be noted here that Parikh was an early enthusiast for Jan Plaza's work, and that work itself is a major forerunner of dynamic epistemic logic.)

A central lesson of Parikh's work on knowledge is that communication and knowledge ought to be studied together. In their paper on *Gossip protocols*, Maduka Attamah, Hans van Ditmarsch, Davide Grossi and Wiebe van der Hoek consider agents who exchange private information about others over a telephone network. They study how many message exchanges are needed for a particular state of knowledge to be achieved.

Analyzing puzzles through logic and games is an endearing component of Parikh's style, and Sandu and Velica's paper in this volume pays tribute to it. Hintikka and Sandu (1989) introduced independence-friendly (IF) logic in order to express patterns of dependencies among quantifiers which go beyond what can be expressed in first-order logic. The paper by Gabriel Sandu and Silviu Velica offers a formulation of the Monty Hall puzzle in IF logic via a game-based modeling of the problem. In the process, they endow IF logic with a probabilistic semantics.

Amy Greenwald, Jiacui Li and Eric Sodomka establish a formally appealing connection between a specific presentation of games (called *extensive normal form games*) and a process of randomized decisions. Uniting two different approaches to decision making provides us with a broader understanding of game theoretical decision processes.

Similarly, Jouko Väänänen discusses an iteration of a logical framework which arises from a situation of limited information. The solution is given within the context of dependence logic with team semantics.

Melvin Fitting's paper in this volume is a contribution to an important issue in modal logic, the relation between *intensions* and *extensions*. His paper proposes an appealing formal treatment of *predicate abstraction*. This paper not only discusses examples, it presents a formal proof system which is sure to be of independent interest.

Konstantinos Georgatos' paper on *epistemic conditionals* addresses belief revision, an important topic in artificial intelligence and related areas of philosophical logic. Yet another connection to Parikh's work comes in this paper's use of the logic of *subset spaces*, a topic founded by Parikh and Moss.

Dexter Kozen's contribution to this volume is a contribution to *Kleene algebra*, an area pioneered by Kozen himself. Kleene algebras may be thought of as algebraic structures, which generalize and illuminate the algebra of regular expressions. Thus, it harks back to Parikh's interest in formal language theory and dynamic logic.

Vaughan Pratt's paper shares with Kozen's a decidedly algebraic flavor. In fact, Pratt's paper takes up several topics relating category theory and syllogistic logic. Although at first this seems an unlikely match, Pratt shows how algebraic perspectives can illuminate the technical sides of logical systems which we thought we knew well.

Noson Yanofsky discusses a deeply foundational issue in theoretical computer science: algorithms. Parikh's work on the logic of programs is carried to a more abstract level. The relation between programs and algorithms is discussed using a novel category theoretic approach.

All Aboard

We hope to have presented you with a brochure-view of Parikh-land and then given an "introductory video" on the sights and sounds that you will experience when reading the book. We now invite you to board the bus. The drivers are grateful to all the authors for their contributions, and to the Series Editor Sven Ove Hansson for giving us the opportunity for this wonderful ride.

Bath, UK — Can Başkent
Bloomington, USA — Lawrence S. Moss
Chennai, India — Ramaswamy Ramanujam

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Contents

Chapter 1
Parikh and Wittgenstein

Juliet Floyd

Abstract A survey of Parikh's philosophical appropriations of Wittgensteinian themes, placed into historical context against the backdrop of Turing's famous paper, "On computable numbers, with an application to the *Entscheidungsproblem*" (Turing in Proc Lond Math Soc 2(42): 230–265, 1936/1937) and its connections with Wittgenstein and the foundations of mathematics. Characterizing Parikh's contributions to the interaction between logic and philosophy at its foundations, we argue that his work gives the lie to recent presentations of Wittgenstein's so-called metaphilosophy (e.g., Horwich in Wittgenstein's metaphilosophy. Oxford University Press, Oxford, 2012) as a kind of "dead end" quietism. From early work on the idea of a feasibility in arithmetic (Parikh in J Symb Log 36(3):494–508, 1971) and vagueness (Parikh in Logic, language and method. Reidel, Boston, pp 241–261, 1983) to his more recent program in social software (Parikh in Advances in modal logic, vol 2. CSLI Publications, Stanford, pp 381–400, 2001a), Parikh's work encompasses and touches upon many foundational issues in epistemology, philosophy of logic, philosophy of language, and value theory. But it expresses a unified philosophical point of view. In his most recent work, questions about public and private languages, opportunity spaces, strategic voting, non-monotonic inference and knowledge in literature provide a remarkable series of suggestions about how to present issues of fundamental importance in theoretical computer science as serious *philosophical* issues.

Keywords Social software · Wittgenstein · Turing · Parikh · Common knowledge

1.1 Introduction

The influence of Wittgenstein's later philosophy on Rohit Parikh's work has been thoroughgoing, quite distinctive, and partly explains the creativity and breadth of

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C. Başkent et al. (eds.), *Rohit Parikh on Logic, Language and Society*,
Outstanding Contributions to Logic 11, DOI 10.1007/978-3-319-47843-2_1

Parikh's contributions. This essay surveys the long and brilliant career of his appropriations of Wittgensteinian themes, using this to show why it is that his biggest notion, that of *social software*, should be attended to by mathematicians, philosophers, and computer scientists.

Of course there are many Wittgensteins, just as there are many Peirces, many Kants, many Platos. The best advice here is to think *with* a philosopher, rather than primarily *about* that philosopher; to make as good sense of as much of the thought as you can, given its context; and to see how the best parts might be projected forward in your own thinking. This is what Parikh has done, aided and abetted by four twentieth century philosophers from whom he has learned a great deal: W.V. Quine, Burton Dreben, Nelson Goodman, and Hilary Putnam.[1] For Parikh, in a very distinctive way, Wittgenstein isn't the subject matter, he's the way through.

So in what follows I shall not debate interpretations, so much as try to say why I think Parikh's Wittgenstein is worthwhile. The important point is that he is putting Wittgensteinian insights to *use*. If philosophy lives and dies by its applications, then that is a very important thing. In fact I believe Wittgenstein owes Parikh a big "Thank You", no matter if the biographical Wittgenstein would have balked at the appropriation and development of his ideas under the name "Wittgenstein". The real, historical Wittgenstein, obsessed with controlling his out-of-control vanity, worried that all his writing and lecturing might do would be to sow the seeds of a certain jargon, and he feared and hated the humiliation of anyone writing about *him*.[2] But he did still dare to hope that someone else would draw out and apply his thinking on the foundations of mathematics.[3] That hope has not been in vain.

From Parikh's earliest work on the idea of "feasibility" in arithmetic (Parikh 1971), through his work on vagueness (Parikh 1983, 1994, 1996b) and on social software (Parikh 2001a, 2002), he has touched on many foundational issues in epistemology, philosophy of logic and mathematics, philosophy of language, and value theory. But his work embodies and develops a unified philosophical point of view, particularly about what "foundations" really are (and are not). Although in a broad sense the point of view he has developed belongs in the "finitist" tradition, it offers a surprising number of undogmatic series of new twists, reaching horizontally, and not just vertically, and thereby going to the heart of what we *mean* by a "foundation". In general, Parikh has suggested that we learn how to replace the sort of familiar objections to finitism, and to Wittgenstein, namely,

> But isn't the statement, concerning a particular system of logistic, that "*there is no contradiction with fewer than* 10^{100} *lines*" true or false? This is not just a question of our "form of life"![4]

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[1] Strictly speaking, Parikh studied officially only with Quine and Dreben, for he left Harvard in 1961 (cf. Parikh 1996a, p. 89). We might also add to the list of philosophers who have influenced him Peirce and Ramsey. If we did, that might well explain the attraction Parikh's work has had for the present author, who was similarly influenced and educated, at least in part, two generations later on.

[2] Monk and Wittgenstein (1990), Chap. 24.

[3] Monk and Wittgenstein (1990), Chap. 22.

[4] Putnam (1979), p. 511, n. 4.

with questions such as the following—and here I distill the subject matter of many of Parikh's papers, without quoting precisely from him:

> Can we rigorously formalize aspects of our "form of life"—
> including the idea of a "feasible" formalization? In what sense of "works" is it that we may hope to find a philosophy of mathematics (and of knowledge) that *does* "work"? What is knowledge really like?

Parikh thus serves as a counterexample to Paul Horwich's recent claim in *The New York Times* that Wittgenstein's thought today is ignored by "all a small and ignored clique of hard-core supporters" (Horwich 2013).[5] In fact, given the interdisciplinary breadth of Parikh's influence, and his lack of dogmatism, Parikh shows that Wittgenstein left us something much more important than a *philosophie du jour* or a method of exposing nonsense or a great name to add to a pantheon; rather, he left us open problems and a *way of thinking* that is constructive, pointing a way forward.[6] Like much philosophy, Parikh's lies in a long and developing, if ever-contested tradition. He has specified a vision of a certain kind. Even if the vision is different from others, and may not even *look* to some like a vision, it is one.

In his most recent work, Parikh is surveying and analyzing questions about public and private languages, the limits of formalization, opportunity spaces, elections and social spaces of coordination, knowledge's effects on obligations, truthfulness and sincerity, and knowledge in literature—thereby drawing together game theory, logic, and computer science into a remarkable series of (rigorously presented) bouquets. Thereby he often shows, quite critically, how certain ideas that people think go together, ideas that appear to give us an ultimate foundation (say, of *homo economicus*, for example, or our notions of *truth* and *belief* and *meaning*) do not, but are better seen as "preliminary" analyses (Parikh 2007, Answer 3), embedded in collections of other structures and systems, in which the characterization of a total collection is not the point, so much as the study of interactions among them in ordinary life, which includes the slippages, bendings, and breakings that go on.

Parikh's mathematical facility—as well as his conviviality and generosity as a teacher and colleague—have led to a constant stream of interesting and fruitful putting-togethers and rearrangements of heretofore disparate areas of logic and the theory of "rationality": proof theory and bounded arithmetic (Parikh 1971, 1998), temporal logic and social levels (Parikh 2001a, 2003), Bayesian probability theory and defeasible inference (Arló-Costa and Parikh 2005), epistemic and dynamic epistemic logic (Parikh 2006, 2008), modal, deontic, and finite information logics (Parikh and Väänänen 2005; Parikh 2006; Pacuit et al. 2006; Baskent 2012), belief revision

[5] In a recent YouTube debate with Timothy Williamson http://www.youtube.com/\penalty\exhyphenpenaltywatch?v=IpOmFTRcwUM), Horwich appealed to problems about logical omniscience to draw out criticisms of taking quantified modal logic without several grains of salt. It is unfortunate that he didn't take the next step, which would have been to point out the positive results and work that is going on as a result of those criticisms, rooted in Wittgenstein himself. However, this would have contradicted his own reading of Wittgenstein as having an essentially negative "metaphilosophy".

[6] Cf. Collins (2013).

theory, relevance and topology (Chopra and Parikh 2000; Chopra et al. 2001; Parikh 2001b; Dabrowski et al. 1996), electoral and political theory (Pacuit and Rohit 2005; Dean and Parikh 2011), and even literature and life (Parikh 2009b, forthcoming). Juxtaposing these traditions of research with one another by asking philosophical questions yields interesting accounts of tensions and presuppositions among them.

Yet Parikh's papers, while mathematically creative and rigorous, also provide a remarkable series of suggestions about how to present issues of fundamental importance in computer science *as* serious philosophical and mathematical issues: perhaps most important, *ways of thinking* that may be accepted, changed, confronted, or turned, allowing us to rethink and perhaps change our lives.

It is from this, above all, that emerges the characteristic Wittgensteinian touch within Parikh's work: he is able to see philosophical problems where others see nothing but "rags and dust" (*Philosophical Investigations* (Wittgenstein 2009) (hereafter "PI") §52). There are only a few of his papers in which Parikh takes Wittgenstein up as an explicit topic (cf. Parikh 1995; Parikh and Renero, forthcoming), and even in these he is not doing straight exposition, but making a series of points: drawing connections, working out analogies, testing the limits of other analogies.

Why are the analogies so important? Because analogies always lie at the basis of any rigorization, and one can fail to see something as philosophical when it truly is so. One can also mistake a problem in one's analogical thinking for a metaphysical literalism, eclipsing the root of the matter. So the frequent and informed allusions and quotations Parikh often adduces to Wittgenstein's remarks should be taken seriously, as part of what he is doing, and not merely decorative or literary asides. They make a series of points that economists, philosophers, game theorists and theoretical computer scientists should listen to.

First of all, as Parikh has himself pointed out, logic in the broad sense of critical reflection on rationality and discourse is an activity, not a particular standpoint or theory, and in this sense belongs to many traditions.[7] (As Quine and Wittgenstein might have said, each sounding a rather different note, logic in the broadest sense is part and parcel of having any view at all, it is not just one optional point of view.) Yet secondly, and more specifically, Parikh's work stands in close proximity to a certain quite particular philosophical tradition of reflection on foundations, stemming not only from the classical works of Gödel and Turing in the 1930s, but also from the earliest reception of Wittgenstein's later writings on mathematics. I shall be emphasizing this stage-setting in what follows, in order to make the case that there is something philosophical at stake in every paper Parikh writes.

Wittgenstein's reception began in public space, as a piece of "social software", in 1956, when the first edition of *Remarks on the Foundations of Mathematics* (Wittgenstein 1978, hereafter "RFM") first appeared. This book was heavily and posthumously reconstructed by Wittgenstein's literary executors from manuscripts never intended for publication, aided and abetted in their editorial task by Kreisel, an important interlocutor of Wittgenstein's (and Parikh's). (Afterwards Kreisel reviewed the book, calling it "the surprisingly insignificant product of a sparkling mind", perhaps to cover

[7] See the discussions of Indian sagas in Parikh's (2009b).

himself.[8]) Dummett (1959) saw numerous mistakes and errors in RFM, especially in the remarks on Gödel.[9] (These remarks, suffering from a variety of weaknesses, had been eliminated by Wittgenstein himself from all drafts of the PI that appeared after 1937.)

Bernays's somewhat less critical review of RFM (Bernays 1959), though it still accused Wittgenstein of embracing nihilism and irrationality, did note that one might see emerging from the scattered remarks a point of view, which he called "anthropological". Hao Wang, partly on the basis of conversations with Gödel, wrote several papers on this perspective, and associated it with his own interest in automated theorem proving.[10] Nevertheless, at least initially, Wang was proud of never mentioning Wittgenstein at all (cf. e.g., Wang 1961): the *brouhaha* of enthusiasm for the famed philosopher during this early period of the 1950 s was oppressive, and like Hilary Putnam and John Rawls, Wang resisted labels, certainly not wanting to become known as a "Wittgensteinian" (cf. my Floyd 2011). Parikh, entering the philosophical scene later on, has felt it important, instead, to develop Wittgenstein's ideas quite explicitly, and this under the rubric of "common sense" (Parikh 1971, p. 494).

Since then, Dummett and others developed the point of view of "strict finitism" (Dummett 1975; Wright 1980, 1993), and Kripke wrote his famed book on Wittgenstein's rule-following remarks (Kripke 1982). These made somewhat more kosher to the wider philosophical public at least some of Wittgenstein's writings on mathematics. But outside of these writers, there has not been a more serious appropriator of this reading of Wittgenstein's ideas about logic, foundations, and mathematics than Parikh. Actually Parikh has done far more than any of these authors to spread and develop the "anthropologism" associated with Wittgenstein. For one thing, he is far more sympathetic to the literal usefulness of Wittgenstein's ways of thinking. For another, he has absorbed the anti-dogmatic side of Wittgenstein more deeply. For a third, his facility as a proof theorist developed very early, in a very focussed way, building on Gödel's pioneering work on speed up theorems, where tradeoffs between how we represent the length of a proof, a "step", or a symbol are placed front and center as part of mathematics (Parikh 1986). For a fourth, he was immersed at the early stages in developing proof and complexity theory as part and parcel of theoretical computer science—a turn anticipated by Wang, but only dimly.

Wang (1996) took the interaction between Gödel and Wittgenstein to have been one of the most important in twentieth century philosophy, but he sided with Gödel and Bernays in regarding "anthropologism" as a limited and partial, though possibly entertainable, point of view. He also attempted, following Gödel, to develop a notion of "intuition" of concepts. Parikh, by contrast, has no use for "intuition" in any

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[8]See Parikh's reminiscence of Kreisel in his (Parikh 1996a).

[9]For a survey of reactions and a response, see Floyd (2001).

[10]Wang (1996), p. 214; Wang (1961). As we are now beginning to learn (Floyd and Kanamori 2016), Gödel was thoroughly immersed in Russell's philosophy in 1942–1943, and thoroughly involved in combatting what he took, rightly, to have been the constructivistic effects of Wittgenstein on the second edition of *Principia*. Gödel himself wrote of an "anthropological" sense of truth in his notebooks, and one may conjecture that although the later Wittgenstein's writings were not known to him then, he discussed this idea with Bernays.

classical philosophical sense, but works with common sense examples in a strategic way, developing "anthropologism" to see just how far it can go. He has shown that it can go very far. Not of course by contradicting the classical results of Gödel, without which Parikh's earliest foundational work in proof theory would have been impossible. But working against this whole approach when it is conceived of as the *only* foundational way. Parikh is a kind of conceptual and ethical pluralist in his very bones, and in this sense a true student, both of Quine and Putnam, if not also the Buddha. He is also a realist, in Cora Diamond's a-metaphysical sense of someone who is *realistic* about what logic is and can do (Diamond 1991).

Gödel's completeness theorem gave us the non-standard models of arithmetic Parikh would later exploit as engines of semantic development. It also laid into neat conceptual space the classical consequence relation, on which so many later results (Gödel's incompleteness theorems included) turned. Classical first order logic tells how logic would be, if things were simple.[11] Parikh urges us to develop logic in the face of the fact that things are not so simple. Better and more deeply put, Parikh's work emphasizes that what counts as "simplicity" is relative to who and where we are, and whom we are with, and how we incur obligations and draw consequences differently in the face of particular acts of speaking. This is explicitly a theme, not only in Plato, but also in the later Wittgenstein.[12]

Tarski's analysis of truth for formalized languages not only served the development of model and set theory, it also allowed philosophers to skirt, through the schematic approach to structures, head-on confrontation with the distinction between the infinite and the finite in their syntax for treating the quantifiers. For Tarski's schematic, metatheoretic analysis of truth in formalized languages is neutral with respect to the size of the domain, as well it should be in connection with the formal aspect of truth in general, as it serves us in the development of a general notion of definability.

But, as Parikh shows, analogous versions of the old problems keep on rearing their heads throughout logic. In a sense, even to claim that

> From two integers k, l one passes immediately to k^l; this process leads in a few steps to numbers which are far larger than any occurring in experience, e.g., $67^{(257^{729})}$.

is in the end "an application of the general method of analogy, consisting in extending to inaccessible numbers the relations which we can concretely verify for accessible numbers" as Bernays had said (Parikh 1971, p. 494). Parikh showed (Parikh 1971) that the analogy could be cashed in differently, that the intuitive notions of "feasible" and "reasonable length" are rigorizable, and the complexity of proofs in that sense may be made mathematically rigorous. As Buss has said, this work (Parikh 1971, 1973) was "seminal and influential and led to large research areas which are still active and

[11] I owe this way of putting the matter to Colin McClarty, in conversation.

[12] Cf. PI §§48ff, which allude to Plato's *Theaetetus* discussion of whether knowledge can be reduced to perception. For an explicit connection with Turing, see Parikh and Renero (forthcoming). As Floyd and Kanamori (2016) and Floyd (forthcoming) show, Gödel himself discussed the *Theaetetus* in his Max Phil notebooks, 1942–1943, following Russell (1940, pp. 116ff).

fruitful 25 years later" (Buss 1999, p. 43).[13] It also showed, philosophically speaking, that the *conventional* answer to the question

> Does the Bernays number $67^{(257^{729})}$ actually belong to every set containing 0 and closed under the successor function?

—viz., "Yes"—may still be accepted. Nevertheless,

> …we have seen that there is a very large element of phantasy in conventional mathematics which one may accept if one finds it pleasant, but which one could equally sensibly (perhaps more sensibly) reject (Parikh 1971, p. 507).

Going further, into the foundations of logic itself, we are not—individually or collectively—logically omniscient, either about the logical consequences of our thoughts or one another's points of view. We can of course study how things would be if we were, and develop notions of *proposition* using either Frege's structured thought approach, or the Hintikka-Lewis-Stalnaker idea of propositions as sets of possible worlds. But, like Wittgenstein, Parikh's suggestion is to complicate the logic so as to keep it, and the notion of "correctness", applicable, and show how conceptually relative are classical idealizations.[14] This sheds light on what knowledge and logic and truth are, and not merely on how they might be, if things were simple.

1.2 Surveyability (*Übersichtlichkeit*)

In looking at Wittgenstein's RFM, Parikh was struck by how many questions bearing on fundamentals of theoretical computer science were already being explored in Wittgenstein's writings from the 1930s (Parikh 1995, p. 92):

> I would like to make the case that people in AI who are actually unaware of much of what Wittgenstein says on this issue are in fact actually carrying out some of what might have been his program, namely that if you look at various activities that are going on right now in Artificial Intelligence and Logics oriented towards it, then you'll find that very many of these developed theories can be seen as expansions of relatively off-hand remarks that Wittgenstein makes.
>
> A rather interesting historical fact is that when the RFM first came out in the early 1950s, Complexity Theory as we know it now did not exist. The book was criticized very sharply by Kreisel (1958). It was criticized also by Dummett (1959), though somewhat more mildly. Now, whether it was because of this or simply because the time was not right, RFM was not taken as seriously as the PI was.
>
> If knowing Complexity Theory one goes back and reads the RFM, then what are impressive are the things that he says which could not make sense in the 50's but which make perfect sense now.

[13] For example, Sazonov (1995).

[14] Parikh (1971) acknowledges earlier work on the notion of "feasible" by Esenine (1970), but states (p. 494, n. 1) that Parikh "preferred the more conservative approach of using 'standard' methods for [his] technical proofs".

There are some historical reasons for this, and they confirm Parikh's sense of what is going on. They are important to appreciate philosophically. For the history shows that Parikh's questions were always part and parcel of the classical tradition in logic and foundations, as well as the foundations of computer science and artificial intelligence, even before they emerged as separate departments in universities, or separate ventures, and even though philosophers tended to carve them off from what was called "philosophy" of "foundations of mathematics" as the twentieth century progressed. The issues may have been forgotten or overlooked while logicians worked out the fundamental classical notions of *truth*, *definability*, and *logical consequence*. But they were always there, entangled with that classical project.

The notion of *Übersichtlichkeit* as a developed philosophical idea in connection with mathematical proof derives from Wittgenstein's writings of 1939–1940, and, significantly, precisely those manuscripts that were written in the wake of his conversations with Alan Turing in 1937 and 1939 (cf. Floyd 2001, 2012, forthcoming). Turing personally sent off only five offprints of his famous paper "On Computable Numbers, with an application to the *Entscheidungsproblem*" (Turing 1936/1937) in the first round, and one was to Wittgenstein, who was then trying to put together the PI.[15] Wittgenstein's interest in the machine as *symbolizing its own action*—famously explored in PI §§193–194— probably derives, in part, from his pondering the relation of Turing's results to Gödel's. What is clear is that after discussions with Turing the notion of *Übersichtlichkeit* springs to the fore of his thinking about the nature of logic and proof in mathematics, and is brought to bear in his criticisms of the idea that *Principia Mathematica* can serve as a "foundation" of arithmetic in logic. Wittgenstein also became interested in the context-bound status of the distinction between the notions of *proof*, *calculation*, and *experiment* after discussions with Turing.[16]

In RFM Wittgenstein appears to offer a version of the Poincaré objection to logicism: because formal *Principia* proofs and terms are not always surveyable, we need to use mathematical induction in setting out the very formulae of the system, thereby helping ourselves to arithmetic and counting already at the outset, in trying to "take in" a formal proof.[17] In working through a proof, for example, we may need to index and count the variables, draw lines in the margins connecting parts with one another, and so on. But if these pieces of "software" are necessary for us to take in the proof, then they are no less working parts of the proof than the formalism, and then

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[15]Turing to his mother February 11, 1947, from Princeton [AMT/K/1/54, at the King's College archives (http://www.turingarchive.org/viewer/?id=414\&title=54)].

[16]Contrary to what many have assumed, Wittgenstein did not read Turing's famous paper on the "Turing Test" for intelligence (Turing 1950), at least until the very end of his life, when there is no evidence of its having influenced him. This is shown by his letter to Malcolm of January 12, 1950, where he tells Malcolm he has not read it, but states, "I imagine it is no leg pull" (Malcolm and Wittgenstein 2001, pp. 129–130, also in Wittgenstein 2005). Wittgenstein did discuss the question, "Can a machine think?" as early as 1926 with Schlick. This was because Russell asked the question already in his (Russell 1921) Lecture XIII, "On Truth and Falsehood", arguing that while there is no difficulty in attributing to a machine "correctness" of response, the issue of truth is more complex.

[17]Parikh (1971, p. 502) already interprets the objection in terms of what is called there an "anthropomorphic system" of bounded arithmetic, in which the notion of "feasibility" is rigorized.

there is no true "reduction" of arithmetic to logic. Instead we are using mathematics to make things "surveyable" or *übersichtlich*, just as Hilbert would have said we must.

The difficulty with this objection, of course, is not to have it bring in irrelevancies of psychology into discussions of the foundations of logic and mathematics.[18] Frege and Russell aimed to extrude psychology altogether from their respective conceptions of justification, and Wittgenstein followed them here. Justifications do not reduce to strikings, experiences, acts of acquaintance, or seemings: we cannot experience in that way the "because" (PI §176). Nevertheless, if mathematics itself is necessary for us to take logic in, then it is arguably a matter of logic, and not merely psychology, that "a proof must be surveyable" (*übersichtlich*, *übersehbar*, *überblickbar*) (RFM III §§1,21–22,39,55; IV §41).

The word "*Übersicht*" is difficult to translate; some readers like the term "perspicuous" or "open to view", or "surview", but this can imply that one has achieved a kind of mountain-top angle on things, and is able to take in all details in a glance, and understand the place of all of them within a larger whole. This is not Wittgenstein's meaning, and as has been argued by Mühlhölzer (2006), a better translation would be "surveyable". When Wittgenstein writes that one of his methods in philosophy is to present us with *übersichtliche Darstellungen*, or "surveyable representations" of grammar, he means that he is providing snapshots of portions of human linguistic activity designed to let us work through them usefully, to survey them in that sense. What matters is, first of all, that the proof can be easily reproduced or copied, "in the manner of a picture": a proof must be communicable, easily reidentifiable, recognizable, as *this* proof and not another, it must not be an experiment each time it is ventured, but rather it must serve as a kind of *calculation* in the sense that it entangles us with what *has* to come out, if the procedure is correctly followed (cf. RFM I §§1,22,39,55). Thus what also matters, second, is that a proof is a proof without something (some thing) behind it: it shows *itself* as a proof, no one general foundational theory of it being needed.

Let us generalize this idea of *Übersichtlichkeit*. The saying "I only know my way about like a rat in a maze" voices an idea Parikh has elaborated for many years: the rat may well know the right cues about when to turn left or right, but lack an *overview* or *map* of the situation—anyway, a map that she can use to see how to read other maps, using maps as members of a category. Deferring a fuller discussion of animal consciousness as such to §4 below, let us consider ourselves, as human beings with our widely shared capabilities. As a human with faculties and experience enough, I can easily memorize one or two station changes on the Boston MBTA transportation line. But holding in mind the whole, so as to make creative or novel choices, requires a model or structure that I am capable of projecting in a variety of ways. Fortunately, officials provide posters of the surview in all stations, and make it available on the web, and the map is comprehensive enough to do the job. In this regard, more than its *accuracy*, we should emphasize the map's *usefulness* in being the sort of representation that can be taken in just a few seconds, followed and discussed by

[18]On this see Goldfarb (1985) and Stenlund (1996).

most of us (including those who may not speak very fluent English), easily posted on the wall of a train car, and used. In a sense, its *truthfulness* is its loyalty, loyalty being the originally root meaning of the word "true", at least in English.

Parikh substitutes "truthfulness" for "truth" in many of his writings, and this is no accident. He is suggesting that truth is a matter of degree, insofar, at least, as we know and act on it. Peirce, for one, seems to have thought of a true belief, not necessarily as an ideal limit point, i.e., a fixed ideal belief that scientists would agree about in the ultimate long run, but, rather, a tenacious and loyal one that will not let us down.[19] Generally speaking, the MBTA map does not let us down and we expect it not to. It works, and in this sense serves as an *Übersicht*, i.e., as a guide to action and communication, not merely a piece of information. Or, perhaps better, insofar as it codifies things to be known, it is *sensitive* to understandings and perceived and expected needs among us.[20] Could this "usefulness" break down? Of course. New lines may be built, and it becomes outdated; the map could be systematically misprojected by a group, Martians could land who couldn't understand it. But it is not the job of a piece of social software to take into account all possibilities. Nor even to justify us sufficient unto the day we act. Instead, the map opens up a space of possible actions: it is analogous to an operating system, within which we make choices.

Parikh is not suggesting that we reduce "knowing that" to "knowing how" in any facile way. More deeply, he is after what the knowing of "knowing that" really amounts to, if it is logically structured. The MBTA map is a representation, surely, a piece of information, perhaps a paradigmatic one. We may not wish to honor it with the title of "depiction", if by depiction we mean something like a portrait, *in* which we may see, not only the Mona Lisa, but Lisa herself. Or perhaps it is a "depiction" in a very malleable sense of a "symbol", but then it may not be considered a very good one, except in a school of graphic design.

The important point here is that there is a broader point of view, both on a depictive portrait and on the MBTA map: both are also, as Parikh has called them since his originating works (Parikh 2001a, 2002), pieces of *social software*. The map of the MBTA helps us bear in mind more easily a great deal of action-points, as individuals. The portrait of Lisa tells us something, not only about her, but about the artist, the time and manner in which she lived, and her person. That is why it can serve as a touchstone for each of us, gazing upon it: it is a gesture of Leonardo's—which is

[19]Read this way, as Cheryl Misak does (Misak 2000), Peirce does not fall into the errors attributed to him by Quine at the end of Chap. 1 of *Word and Object*, where Quine complains about the unclarity of "ideal truth in the long run" as a *definition* of truth (Quine 1960, §6). Parikh's work with San Ginés (manuscript) explores the relevance of a pragmatic approach to belief, augmented by a 3-dimensional approach analyzing the "success" of a belief into its relativity to the (first person) thinker, the situation of a co-participant (second person) as their joint activities unfold over time, and the (third person) unfolding of circumstances themselves over time.

[20]On context sensitivity and understandings, see Travis (2005, 2006, 2008).

why it matters whether he painted two Mona Lisa portraits, and not just one.[21] The social aspect opens up for us humans the possibility of all kinds of collective action: communication, teaching, learning, and, generally, the enlargement of spaces of opportunities for knowing, believing, and acting—things we do, after all, in concert *with* one another, not merely *to* one another.

It is in these "social software" respects that the rat's powers to articulate, advise its fellows, generalize, and create new opportunities for all of the above is quite limited in comparison with our own. This is shown everywhere in our daily lives. Shopping for an apple in a grocery store—to choose a canonical example from Wittgenstein's PI §2 that has been discussed in detail by Parikh, emphasizing the importance of parameters and protocols (Parikh 2009a)—or, to take some other examples, recounting a story of why someone in the office is angry, or "liking" on the web, or voting, or calling 911: all of these activities are embedded within and by means of social forms of infrastructure that we mostly do not bring to mind as forms of social software when we act within them. Of course we do not: their function is, after all, to *offload* the problem of actually reaching decisions to something else, just like the MBTA map. That offloading can then itself be represented as a piece of social software, and that representation in turn again regarded as part of what we do, hence part of social software as well.

The idea of taking this notion of *Übersichtlichkeit*, or "surveyability", in at the foundational level is this. The notion is nearly as comprehensive as that of *computability* à la Turing, if not more so, since we don't calculate with all of our concepts, but only with special ones, whereas concepts as such are meant for sharing and use in a social setting in which we act, communicate, articulate, express, influence, grow and jointly engage with one another.

Parikh has recently asked, somewhat tongue in cheek, whether there might not be a Church's Thesis for social algorithms (Parikh, forthcoming). That would place what Turing accomplished into a new light—without, I suppose, contradicting his analysis. The point would be, as Parikh says (Parikh 1995, pp. 89–90), a fourfold, Wittgensteinian one:

- **Mathematics as an Applied Science:** The truth of Mathematics is that it fits into our lives.
- **The Importance of the Social:** Language as also Mathematics and even thinking—certainly we think of the last as a private process—are in fact, according to [Wittgenstein], social activities.
- **Locality or Context Dependence:** We do not have general notions like knowledge, truth, number but context dependent ones.
- **Flexibility:** A formal system does not fully determine our behaviour nor how we use it.

[21] Cf. http://www.pbs.org/wnet/secrets/mona-lisa-mystery-full-episode/1821/ and http://www.opencultre.com/2013/12/first-mona-lisa.html.

1.3 Turing Machines: From Language Games to Social Software

Let us return to Turing, to whom we owe the idea, if not in some respects the fact, of the stored program computer, *via* what Church would call his notion of a "Turing Machine". Wittgenstein and social software—hence philosophy—may be seen, in retrospect, to have been entangled with Turing's model, just as Parikh has been arguing.

Many philosophers since the early 1960s have thought that logical analysis gave us great insight into the individual: states of mind, cognitive processes and their computational modeling of cognitive states, and so on. Computational Functionalism—invented in part by Putnam, but later rejected by him—dominated the approach to cognitive content for a very long time. But "Can Machines Think?" was imagined by Turing in his (Turing 1936/1937, §8) explicitly as nothing more (and nothing less) than a *comparison* between a human computer—an individual—and a machine: as what Wittgenstein would have called a *language-game*, i.e., ultimately, an *analogy*. Turing's proofs do not depend upon their serving as descriptions of what actually goes on in our minds, ultimately, when we calculate, any more than Wittgenstein's imagined language-games are intended to describe everything that happens when we use language. They could not, after all, if he was to resolve the *Entscheidungsproblem*, for no mathematical theorem can turn on a thesis in the philosophy of mind.

Turing signaled this in §1 of his famed (Turing 1936/1937) by stating that "we may *compare* a man in the process of computing a real number to a machine which is only capable of a finite number of conditions" (my italics). A language game is not exactly a *description* of what we do, it is instead a *comparison*, as Wittgenstein says (PI §130):

> Our clear and simple language-games are not preliminary studies for a future regimentation of language, as it were, first approximations, ignoring friction and air resistance. Rather, the language-games stand there as *objects of comparison* which, through similarities and dissimilarities, are meant to throw light on relations of our language.

In other words, the idea of a "language game" is intended by Wittgenstein to speak to questions concerning our very idea of what logic is, rather than specifically to matters of psychology. As Parikh has written (Parikh 2009a), Wittgenstein is using language games to stress the importance of social software, protocols, and partial renditions of games in logic.

Turing begins §1 of his (Turing 1936/1937) this way:

> We have said that the computable [real] numbers are those whose decimals are calculable by finite means. This requires rather more explicit definition. No real attempt will be made to justify the definitions given until we reach §9. For the present I shall only say that the justification lies in the fact that the human memory is necessarily limited.

There has been a lot of discussion in the literature of this remark about human memory. Many have assumed that Turing *must* have been committed to a very particular theory of human mentality, and used this theory of mind to achieve his result.[22] In particular, the thought is that he was reducing consciousness to merely bodily or physical processes, limited in space and time, and construing a human thinker as really nothing more than a machine. It is true that throughout his paper Turing speaks of the finite number of "states of mind" of a human computer, and the ability of a human to take in only a small number of figures "at a glance" (Turing 1936/1937, §9). It sounds as if Turing is making a series of epistemological points, directly continuing the Hilbert finitistic or formalistic tradition, or perhaps broaching a behavioristic theory of cognition and/or perception.[23] This invited the idea of a "language of thought" operating automatically inside the head, as if philosophy of mind and/or our perceptual ability to cognize were central or foundational to his model. This in turn lead to the rather irrelevant criticism that in his modeling of the classical consequence relation by way of an infinite tape Turing was analyzing thought in a way useful only for Martians or cognitive scientists.

However, Turing himself was more careful. In fact this opening remark constitutes no theory of mind at all, in the sense that a traditional metaphysics is intended to offer. The response to Hilbert is a response, but also a reorienting of the whole subject back to *us*, to scrutinize the nature of the conditions involved in resolving the *Entscheidungsproblem*. What Turing does, right at the beginning, is to simply reiterate the *point* of effectiveness in the context of mathematics, and therefore, in human life. He is calling attention to what is "right before our eyes".

Turing could not have proved a *mathematical* result based on any theory of mind, however powerful: mathematical theorems about mathematics cannot be based on theories of minds, but only on mathematics. Instead, in the spirit of Wittgenstein's idea of routines that are "plain to view", I suggest we take Turing to be remarking on what is, in the end, a rather obvious point. He is making the whole idea of an "effective calculation", hence the whole idea of a "formal system", *plain*.

The general idea of a Turing Machine boils down the idea of a "step-by-step" routine to its simplest, most intuitive elements, those derived from what we human beings *do*. The tape is unbounded ("infinite") in length: one can always add on to each routine another, then another, and so on (Turing points out that the linearization into a 2-dimensional tape is only one possibility, used for his purposes). Moreover the "paper", or "tape" contains "squares" to be marked with "symbols", and it does not matter *which* particular symbols they are: algorithms can be communicated by means of a whole variety of methods, languages, diagrams, pictures, and so on. The (human) computer is held to (a) "see" only a bounded collection of symbols at each single step, "at a glance"—shades of *Übersichtlichkeit*—and (b) only to have written down, at any specific point in the process, a finite number of symbols that it can move to locally.[24] As Turing says explicitly (Turing 1936/1937, §9, III),

[22]See Gödel (1972). For criticisms of Gödel's assumptions see Webb (1990), Sieg (2007).

[23]Cf. Shanker (1998).

[24]Wilfried Sieg (2008) has "axiomatized" Turing's model with these "boundedness" and "locality" conditions, showing that nothing in what Turing did refutes Hilbert's approach. One might however say that Turing encompasses that approach as but one among others in mathematics and philosophy.

> It is always possible for the computer to break off from his work, to go away and forget all about it, and later to come back and go on with it. If he does this he must leave a note of instructions (written in some standard form) explaining how the work is to be continued. This note is the counterpart of the "state of mind".

This "counterpart", is a shareable command: part of the general *interface* environment. It forms part of our most ordinary sense of a routine that offloads, to save effort, for human beings—as individuals, and within groups. There is no "there" there, if we ask "*Where* is the interface?"

The most comprehensive perspective we have of what Turing modeled is, then, that of social software. In Turing (1936/1937) Turing showed, step by step, how his machines could carry out any effective calculation that the Gödel and Church systems could carry out— and, by imaginative extension, any one we might dream up, thereby fixing a very widely-applicable parameter for titrating the notion of a "step" in a computation. That this parameter is provably robust, impervious to the vagaries of particular formal languages or local conventions of symbolism, struck Gödel as nothing short of a "miracle", once and for all determining the precise generality of his incompleteness results.[25] Turing, he said, had gotten to "the right perspective" mathematically.[26] Gödel objected, however, to what he took to be a prejudicial assumption of Turing's about our mental lives: that our experiences are discrete or discretizeable.

However, philosophically what needs to be stressed is that a Turing machine, as a construct, is double-faced: from one point of view, it is nothing but just another formal system. But from another point of view, it tells us what a formal system *is*, by showing us what it is *for*. To resolve the question what a formal system (or "effective calculation") is, it did no good to write down another formal system (or perform another computation). It also would have done no good to have furthered a theory of mind, if Turing had done so. Instead, one had to *do* something, to make the question clear.

Turing of course also showed that since any Turing machine's recipe of directions, configurations, and symbols can *itself* be coded by numbers and thereby worked on by another, *there is a universal Turing machine* that can carry out the routine of any and every machine. It is this, as Martin Davis has argued, that leads to our sense of the "ubiquity" of computable processes in our world.[27] There is no diagonal "escape" from the class of effective computations by means of anything effectively computable, for the universal computer can always incorporate any effectively computable process into itself.

As Davis has also emphasized, Turing's model shows us that a distinction that may be—and was—naturally drawn by early computer designers between software,

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(Footnote 24 continued)
Wittgenstein himself makes very few remarks on the axiomatic method, although there is nothing in his approach, *per se*, that refutes its importance for tagging assumptions in deductive reasoning contexts, and in fact the axiomatic method may be said to fit nicely with the view of mathematical sentences as norms or commands (cf. Friederich 2011).

[25] See Gödel (1946), p. 1.

[26] Gödel to Wang, in Wang (1974), p. 85; Copeland (2004), pp. 45, 48, (Copeland 2012), Chap. 2.

[27] Davis, forthcoming.

hardware, and data is not rigid, but rather contextual and shifting.[28] This is the essential insight behind the idea of the stored program computer: a single device, as we all know nowadays, can perform multiple tasks, as we open up different routines, or programs, and it can work on its own program and activities as well, for example joining in the activities with other machines and users.[29]

One might add something else, drawing out a philosophical implication that is also a supposition: the distinction between what *I* and *we* do and what the *machine* does is fluid in life, within the interface. This suggests that what it *is* to perform a calculation (correctly or incorrectly) is itself an occasion-sensitive matter; an agent-relative matter sometimes, a matter of social purpose at other times, a matter of particular collectivities within multi-agent systems at others. Our standards of correctness are norms, like commands, and their standards of fulfillment are brought together with language in ways that involve much plasticity, as well as structure.

In the initial philosophical reception of Turing's ideas, however, the community, the environment, the social setting, and the culture, were shoved offstage. This was a sign of the times: philosophers wanted to bring psychology back into the fold of their thinking. Even Paul Grice, whose notion of "conversational implicature" has been an important stimulus for Parikh's work (Parikh 2002, 2003), remained in thrall to a certain picture of correctness based on the notion of "intention" conceived of as a psychological state. And in this Grice continues to be followed by many philosophers of a naturalistic bent, as well as researchers in Artificial Intelligence who believe that we are Turing Machines all the way down.

In his later essays, however, Turing himself was quite clear that a social context would be required for the development of his ideas. His 1948 technical report "Intelligent Machinery: A Report by A.M. Turing" contains the observation that "an isolated [human] does not develop any intellectual power" (Turing 1948/1992), and he emphasized the need for creativity and intuition, as well as rule-following. In fact, as has been argued (Sterrett 2012, forthcoming), his vision of computing machinery presupposed that frequent communication and contact among human beings would be crucial for developing human cognitive abilities that are integrated with machines.

Only with the development of the web's architecture, the ubiquity of analysis of our communications and the crowd-sourcing of intellectual projects has the obviousness of the point risen to philosophical consciousness clearly. Early on, logic and philosophy were too much in thrall to the idea that, as Russell put it in his William James Lectures of 1940 (Russell 1940) (Chap. XIII), "'correct' cannot be used in defining 'true', since 'correct' is a social concept, but 'true' is not".

For Parikh, however, it was always and only about the *system*, about the interplay between truth and correctness, the individual and society, and never about the individual's state of mind or truth as such independent of the opportunity spaces we inhabit.[30] Somehow—perhaps because of his upbringing in India, perhaps because

[28] See Davis (2000, 2001, forthcoming).

[29] See Copeland (2012).

[30] Lately Parikh has absorbed the importance of Amartya Sen's thinking about justice as a notion rooted in capabilities and opportunities, rather than abstract principles of opportunity.

of study with Quine, Goodman and Putnam and others in the philosophy department at Harvard—Parikh made it a point to pursue culture and society as, so to speak, foundational in the development of logic. He never fell for functionalism, but instead kept Turing's analysis of computability in mind as a designed device, an analogy, as Wittgenstein would have always suggested that it was, and as Turing himself said. Already in his early work on bounded arithmetic (Parikh 1971), Parikh was turning away from the classical consequence relation and placing epistemic limitations on it, working with notions such as "feasibility" in arithmetic and in vagueness (Parikh 1983, 1994). He took Turing, quite correctly, to have come up with something that itself was only an "analogy" in analyzing the notion of "effective calculability".

There are two final historical sides about Wittgenstein and Turing that are relevant.

1. Action and Turing's Argument. The importance to Turing's analysis of what it is that we *do* with rules is very clear when we examine the particular argument by means of which Turing resolved Hilbert's *Entscheidungsproblem* in his (Turing 1936/1937), showing that there can be no general way of effectively determining whether or not a given sentence of a language does or does not follow from another.[31] He did so, *not* by producing a contradiction, as is usual in presentations of the proof nowadays *via* the Halting argument, but instead by showing how, if one assumes that there is a decision procedure of this general kind, one would then be committed to the constructability of a *tautological* machine, one that could be made to do something like turning up a card in a game that says "Do whatever it is that you are doing". This would be an empty command that *cannot* be followed, though it can be defined. There can be no such machine, and so no such general procedure.[32]

The argument is a kind of "diagonal" argument, but one that works by reduction to tautology, rather than to contradiction. It bears an analogy to the truth teller paradox ("This sentence is true"), but with an important difference: the focus on action and correctness, the possibility of being able to carry out or *follow* the command, is clearer. What Turing wrote is that the more straightforward approach, via a general use of negation in a diagonal argument, may "leave the reader with a feeling that 'there must be something wrong"' (§8). That person might have been, e.g., an intuitionist—it may even have been the historical Wittgenstein. Be that as it may, in Turing's situation, he did not want to get into the problem of negation, whether or not the "law of contradiction" is a universal law. So instead he artfully dodges foundational controversy, something Bernays suggested he flag explicitly in the revisions to (Turing 1936/1937) that he got Turing to publish right away.[33] He defines

[31]Or, "what is the same thing" (Turing 1936/1937, §11), how to effectively determine in general whether when one adjoins a new axiom in the series of theories Gödel (1931) showed us how to generate, one ends up with a consistent theory.

[32]The negation of this rule, namely, "Do whatever you are *not* doing" could also be defined; yet in the context of Turing's argument in his (Turing, 1936/1937), this would equally well be a command that could not really be *followed*, for the problem would remain of how to specify what it is that you *are* doing at that particular step, such that you are not to do it.

[33]See Turing (1937), discussed in Floyd (2012). Bernays urged, in a suggested revision to Turing's association of machines with real numbers, that Turing make explicit that a Brouwerian fan construction could be used.