Applied and Numerical Harmonic Analysis

$$\hat{f}(\gamma) = \int f(x) e^{-2\pi i x \gamma} dx$$

Sagun Chanillo, Bruno Franchi
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Harmonic Analysis, Partial Differential Equations and Applications

In Honor of Richard L. Wheeden

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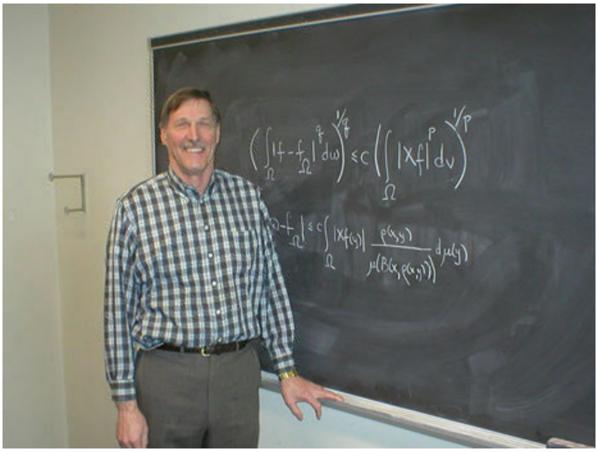
ISSN 2296-5009 ISSN 2296-5017 (electronic)
Applied and Numerical Harmonic Analysis
ISBN 978-3-319-52741-3 ISBN 978-3-319-52742-0 (eBook)
DOI 10.1007/978-3-319-52742-0

Library of Congress Control Number: 2017933468

Mathematics Subject Classification (2010): 35R03, 35J70, 42B20, 42B37, 42B25

Printed on acid-free paper

This book is published under the trade name Birkhäuser, www.birkhauser-science.com
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland



Richard L. Wheeden

Preface

It is a pleasure to bring out this volume of contributed papers on the occasion of the retirement of Richard Wheeden. Dick Wheeden as he is known to his many friends and collaborators spent almost all his professional life at Rutgers University since 1967, other than sabbatical periods at the Institute for Advanced Study, Princeton, Purdue University, and the University of Buenos Aires, Argentina. He has made many original contributions to Potential Theory, Harmonic Analysis, and Partial Differential equations. Many of his papers have profoundly influenced these fields and have had long lasting effects, stimulating research and shedding light. In addition many colleagues and especially young people have benefitted from the generosity of his spirit, where he has shared mathematical insight and provided encouragement. We hope this volume showcases some of the research directions Dick Wheeden was instrumental in pioneering.

1 Potential Theory and Weighted Norm Inequalities for Singular Integrals

Dick Wheeden's work in Analysis can be broken into two periods. The first period consists of his work in Potential Theory, the theory of singular integrals with a deep emphasis on weighted norm inequalities, and a second period from the late 1980s where he and his collaborators successfully applied weighted norm inequalities to the study of degenerate elliptic equations, subelliptic operators, and Monge-Ampère equations.

Wheeden obtained his Ph.D. in 1965 from the University of Chicago under the supervision of Antoni Zygmund. One very productive outcome of this association with Zygmund is the beautiful graduate textbook *Measure and Integral* [36].

Wheeden's thesis dealt with hypersingular integrals. These are singular integrals of the form

$$Tf(x) = \int_{\mathbb{R}^n} (f(x+y) - f(x)) \frac{\Omega(y)}{|y|^{n+\alpha}} dy, \qquad 0 < \alpha < 2,$$

where $\Omega(y)$ is homogeneous of degree zero, integrable on $\mathbb{S}^{n-1}$ and satisfies

$$\int_{\mathbb{S}^{n-1}} y_i \Omega(y)\, d\sigma(y) = 0, \qquad 1 \le i \le n.$$

Since the singularity of the kernel $\frac{\Omega(y)}{|y|^{n+\alpha}}$ is more than that of a standard Calderón-Zygmund kernel, one needs some smoothness on f to ensure boundedness. A typical result found in [34] is

$$\|Tf\|_{L^p(\mathbb{R}^n)} \le C \|f\|_{W^{\alpha,p}(\mathbb{R}^n)}, \qquad 1 < p < \infty,$$

where $W^{\alpha,p}(\mathbb{R}^n)$ is the fractional Sobolev space of order α. These results are developed further in [35].

Another important result that Wheeden obtained at Chicago and in his early time at Rutgers was with Richard Hunt. This work may be viewed as a deep generalization of a classic theorem of Fatou which states that nonnegative harmonic functions in the unit disk in the complex plane have nontangential limits a.e. on the boundary, that is on the unit circle. The theorem of Fatou was generalized to higher dimensions and other domains by Calderón and Carleson. The works [17, 18] extend the Fatou theorem to Lipschitz domains, where now one is dealing with harmonic measure on the boundary. The main result is

Theorem 1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let $\omega^{P_0}(Q)$, $Q \in \partial\Omega$, denote harmonic measure with respect to a fixed point $P_0 \in \Omega$. Then any nonnegative harmonic function $u(P)$ in Ω has nontangential limits a.e. with respect to harmonic measure ω^{P_0} on $\partial\Omega$.*

The proof relies on constructing clever barriers and in particular on a penetrating analysis using Harnack's inequality on the kernel function $K(P,Q)$, $P \in \Omega$, $Q \in \partial\Omega$, which is the Radon-Nikodym derivative

$$K(P,Q) = \frac{d\omega^P(Q)}{d\omega^{P_0}(Q)}.$$

1.1 Singular Integrals and Weighted Inequalities

In 1967, Wheeden moved to Rutgers University and began a long and fruitful collaboration with his colleague B. Muckenhoupt. Two examples of many seminal results proved by Muckenhoupt and Wheeden are the theorems on weighted norm inequalities for the Hilbert transform and the fractional integral operator. To state these results we recall a definition.

Definition 1 Let $1 < p < \infty$, and let $w \in L^1_{loc}(\mathbb{R}^n)$ be a positive function on $\mathbb{R}^n$. Then $w \in A_p$ if and only if for all cubes Q,

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}\right)^{p-1} < \infty.$$

The A_p condition had already appeared in Muckenhoupt's pioneering work on the Hardy-Littlewood maximal function [51]. But now Wheeden along with Hunt and Muckenhoupt [19] carried it further. They considered the prototypical one-dimensional singular integral, the Hilbert transform,

$$Hf(x) = p.v. \int_{-\infty}^{\infty} \frac{f(y)}{x-y}dy,$$

and established the following trailblazing theorem.

Theorem 2 *A nonnegative $w \in L^1_{loc}(\mathbb{R})$ satisfies the L^p weighted norm inequality for the Hilbert transform,*

$$\left(\int_{\mathbb{R}} |Hf|^p w\right)^{\frac{1}{p}} \leq C_p \left(\int_{\mathbb{R}} |f|^p w\right)^{\frac{1}{p}},$$

if and only if $w \in A_p$.

Their key difficulty in establishing this result was to prove it when $p = 2$. Then one can adapt the Calderón-Zygmund scheme for singular integrals and finish with an interpolation. The case $p = 2$ had been studied earlier by Helson and Szegö [47] using a completely different function theoretic approach, where they obtained the equivalence of the weighted norm inequality with a subtle decomposition of the weight involving the conjugate function. Theorem 2 finally characterized these two equivalent properties in terms of a remarkably simple and checkable criterion, the A_p condition. Theorem 2 was the forerunner to a deluge of results by Wheeden in the decades since, to multiplier operators by Kurtz and Wheeden [20], to the Lusin square function by Gundy and Wheeden [16] (preceded by Segovia and Wheeden [33]), and the Littlewood-Paley g^*_λ function by Muckenhoupt and Wheeden [24], to name just a few. With Muckenhoupt, Wheeden also initiated a study of the two weight theory for the Hardy-Littlewood maximal operator and Hilbert transform [25] and with Chanillo a study of the two weight theory for the square function [6]. That is one now seeks conditions on nonnegative functions v, w so that one has

$$\left(\int_{\mathbb{R}^n} |Tf|^p v\right)^{\frac{1}{p}} \leq C_p \left(\int_{\mathbb{R}^n} |f|^p w\right)^{\frac{1}{p}},$$

where T could be a singular integral operator, a square function, or the Hardy-Littlewood maximal operator. The papers [25] and [6] stimulated much research in a search for an appropriate two weight theory for singular integrals. In the early 1990s Wheeden returned to this question and undertook a study of two weight problems for the fractional integral. These results are described later in this preface.

The later "one weight" results mentioned above relied on the so-called good-λ inequalities [37], [40], a beautiful stratagem with which Wheeden was wholly won over. We cite two instances of results proved by Wheeden, where good-λ inequalities play a key step in the proofs. The first example is joint work with Chanillo [1] where he investigated a complete theory of differentiation based on the Marcinkiewicz integral

$$Mf(x) = \left(\int_{\mathbb{R}^n} \frac{|f(x+t) + f(x-t) - 2f(x)|^2}{|t|^{n+2}} dt \right)^{\frac{1}{2}}.$$

This work viewed Mf as a rough square function and the aim was to treat it in the spirit of the work of Burkholder and Gundy [37] for the Lusin square function and establish control via a good-λ inequality and maximal functions.

The second work with Muckenhoupt, destined to play a major role in Wheeden's interest in degenerate elliptic PDE in the late 1980s onward, was the paper [23] on fractional integral operators I_α. Define for $0 < \alpha < n$,

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

Theorem 3 *Let v be a positive function on $\mathbb{R}^n$. Then for $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{p} + \frac{1}{p'} = 1$, the weighted norm inequality for I_α,*

$$\left(\int_{\mathbb{R}^n} |(I_\alpha f)\, v|^q \right)^{\frac{1}{q}} \leq C_p \left(\int_{\mathbb{R}^n} |f v|^p \right)^{\frac{1}{p}},$$

holds if and only if for all cubes Q,

$$\sup_Q \left(\frac{1}{|Q|} \int_Q v^q \right)^{\frac{1}{q}} \left(\frac{1}{|Q|} \int_Q v^{-p'} \right)^{\frac{1}{p'}} < \infty.$$

The corresponding inequality for the fractional maximal operator

$$M_\alpha f(x) = \sup_{Q:\, x \in Q} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)|\, dy,$$

which is dominated by the fractional integral $I_\alpha |f|(x) = \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy$, can be established by a variety of techniques, and from this, the inequality for the larger (at least when $f \geq 0$) fractional integral I_α can be obtained as a consequence of the *good-λ inequality*:

$$\begin{aligned}&|\{x \in \mathbb{R}^n : M_\alpha f(x) \leq \beta\lambda \text{ and } |I_\alpha f(x)| > 2\lambda\}|_{v^q}\\ &\leq C\beta\, |\{x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda\}|_{v^q},\end{aligned}$$

valid for nonnegative functions f, and for all $\lambda > 0$ and $0 < \beta < 1$ and where we have used the notation:

$$|E|_{v^q} = \int_E v^q(x)dx.$$

This striking inequality says, loosely speaking, that the conditional probability of doubling the size of $I_\alpha f$, given a fixed lower threshold, is small unless the maximal function $M_\alpha f$ exceeds a smaller threshold—in other words, $I_\alpha f$ cannot increase by much at a given location unless $M_\alpha f$ is already large there.

The fractional integral operator $I_\alpha f$ plays a major role in the proofs of Sobolev inequalities and localized versions of Sobolev inequalities called Poincaré inequalities. These inequalities, together with the energy inequalities of Cacciopoli, are used to derive via an iteration scheme due to Moser, a fundamental inequality for elliptic PDE, called the Harnack inequality. The Harnack inequality can be then used to obtain regularity in Hölder classes of weak solutions of second order elliptic PDE. Thus Wheeden was now led in a second period to the study of degenerate elliptic PDE and the particular problem of regularity of weak solutions to degenerate elliptic PDE. One of the earliest Poincaré-Sobolev inequalities he obtained was a natural outcome of earlier work for the Peano maximal function [2] and is contained in his paper with Chanillo [3]. To state the main theorem in [3], we need some notation. We consider v, w locally integrable positive functions on $\mathbb{R}^n$. Fix a ball B. We now consider balls $B_r(x_0) \subset B$, centered at x_0 with radius $r > 0$. We assume that v is doubling, i.e.,

$$\int_{B_{2r}(x_0)} v \leq C \int_{B_r(x_0)} v,$$

and we also assume the balance condition (which turns out to be necessary)

$$\frac{r}{h}\left(\frac{\int_{B_r(x_0)} v}{\int_B v}\right)^{\frac{1}{q}} \leq C\left(\frac{\int_{B_r(x_0)} w}{\int_B v}\right)^{\frac{1}{p}}, \tag{1}$$

$$\text{where } h = |B|^{\frac{1}{n}}.$$

For $f \in C^1(B)$, we set $f_{\text{avg}} = \frac{1}{|B|}\int_B f$.

Theorem 4 *Let $1 < p < q < \infty$. Assume that $w \in A_p$, that v is doubling, and that the balance condition (1) holds. Then*

(1) *For $f \in C_0^1(B)$ we have the Sobolev inequality,*

$$\left(\frac{1}{\int_B v}\int_B |f|^q\, v\right)^{\frac{1}{q}} \leq Ch\left(\frac{1}{\int_B w}\int_B |\nabla f|^p\, w\right)^{\frac{1}{p}}.$$

(2) *For $f \in C^1(B)$ we have the Poincaré inequality,*

$$\left(\frac{1}{\int_B v}\int_B \left|f - f_{\text{avg}}\right|^q v\right)^{\frac{1}{q}} \leq Ch\left(\frac{1}{\int_B w}\int_B |\nabla f|^p\, w\right)^{\frac{1}{p}}.$$

The results in [3] when combined with energy estimates like Cacciopolli's inequality and an appropriate Moser iteration scheme lead to Harnack inequalities [5] and estimates for Green's function for elliptic operators in divergence form [7].

2 Degenerate Elliptic Equations, Subelliptic Operators, and Monge-Ampére Equations

In the early 1990s, Wheeden's interests turned to the study of Sobolev-Poincaré inequalities in the setting of metric spaces, focusing in particular on Carnot-Carathéodory metrics generated by a family of vector fields and on the associated degenerate elliptic equations. Let $X := \{X_1, \ldots, X_m\}$ be a family of Lipschitz continuous vector fields in an open set $\Omega \subset \mathbb{R}^n$, $m \leq n$. We can associate with X a metric in Ω—the Carnot-Carathéodory (CC) metric $d_c = d_c(x, y)$ or the control metric—by taking the minimum time we need to go from a point x to a point y along piecewise integral curves of $\pm X_1, \ldots, \pm X_m$ (if such curves exist). The generating vector fields of the Lie algebra of connected and simply connected, stratified nilpotent Lie groups, also called Carnot groups, as well as vector fields of the form $\lambda_1\partial_1, \ldots, \lambda_n\partial_n$, where the λ_j's are Lipschitz continuous nonnegative functions, provide basic examples of vector fields for which the CC distance is always finite. The latter vector fields are said to be of Grushin type in [10].

A (p, q)-Sobolev-Poincaré inequality in this setting is an estimate of the form

$$\left(\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} |g - g_B|^q\, dx\right)^{\frac{1}{q}} \leq C\, r\left(\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)}\left(\sum_j |X_j g|^2\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}} \tag{2}$$

for all metric balls $B(x_0, r) := \{x ; d_c(x_0, x) < r\}$ and for all smooth functions g with average g_B on $B(x_0, r)$. Moreover, Lebesgue measure on both sides of (2) can be replaced by two different measures which may arise from weight functions. This case gives rise to what is called a weighted (or two-weight) Sobolev-Poincaré inequality. The weight functions are chosen to satisfy conditions akin to [3] and [5].

Wheeden in 1994, in collaboration with Franchi and Gutiérrez [10], proved a two-weight Sobolev-Poincaré inequality for a class of Grushin type vector fields that best illustrates this circle of ideas.

In the Poincaré-Sobolev inequality that follows, the weight function u is assumed doubling. The vector fields are given by, $\nabla_\lambda g(z) = \left(\nabla_x g(z), \lambda(x)\nabla_y g(z)\right)$ for $z = (x, y) \in \mathbb{R}^{n+m}$. $\lambda(x)$ is assumed continuous. The continuity of $\lambda(x)$ allows the notion of a metric $d_c(\cdot,\cdot)$ which is naturally associated with the vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \lambda(x)\frac{\partial}{\partial y_1}, \ldots, \lambda(x)\frac{\partial}{\partial y_n}$ by means of subunit curves to be defined [42]. To this metric one may now associate balls $B(z_0, r)$ which are balls in $\mathbb{R}^{n+m}$ with center z_0 and radius r defined by $B(z_0, r) := \{z ; d_c(z_0, z) < r\}$. With some further stipulation on the weight v that will be stated later, one has the Poincaré-Sobolev inequality displayed below:

$$\left(\frac{1}{|B(z_0, r)|_u}\int_{B(z_0,r)} |g(z) - g_B|^q\, u(z)\, dz\right)^{\frac{1}{q}}$$

$$\leq Cr\left(\frac{1}{|B(z_0, r)|_v}\int_{B(z_0,r)} |\nabla_\lambda g(z)|^p\, v(z)\, dz\right)^{\frac{1}{p}}$$

Further assumptions on the coefficients of the vector fields are $\lambda(x)$ lies in some Reverse Hölder class, i.e., $\lambda \in RH_\infty$ and $\lambda^n \in$ *strong* A_∞, in the sense of David and Semmes [41] suitably adapted to the Carnot-Caratheodory metric situation. The key assumption on the weight v above is that there exists $w \in$ *strong* A_∞ for which we have

$$vw^{\frac{1}{N}-1} \in A_p\left(w^{1-\frac{1}{N}}dz\right),\ N = n + m.$$

The following "balance condition" which is by now well known to be necessary [3] is also assumed:

$$\frac{r(B)}{r(B_0)}\left(\frac{u(B)}{u(B_0)}\right)^{\frac{1}{q}} \leq C\left(\frac{v(B)}{v(B_0)}\right)^{\frac{1}{p}}, \quad B \subset (1+\varepsilon)B_0.$$

A typical example of a function λ is $\lambda(x) = |x|^\alpha$ for $\alpha > 0$. The results allow weights v that vanish to high order and include new classes of weight functions even in the case $\lambda(x) \equiv 1$. Other important examples arise from weight functions v that are Jacobians of Quasiconformal maps.

The paper [10] contains two remarkable technical results: first of all, it is proved that metric balls for the CC distance satisfy the so-called *Boman condition*, that by now has been proved to be equivalent to several other geometric conditions. This allows one to apply the *Boman chain technique* (as studied by Chua and Bojarski) to suitable metric spaces equipped with doubling measures. This makes it possible to obtain (2) from a "weak" type Sobolev-Poincaré inequality, where in the right-hand side of (2) we replace the ball $B(\bar{x}, r)$ by an "homothetic" ball $B(\bar{x}, \tau r)$, $\tau > 1$. The second important idea in this paper consists in the clever use of a technique inspired by Long and Nie [50], and that will become more or less standard in the future. To illustrate this idea, consider a fractional integral I and let $u \to |Xu|$ be a *local* operator, where $|Xu|$ denotes the norm of the Euclidean gradient or of some generalized gradient $(X_1 u, \ldots, X_m u)$. This technique makes it possible to obtain strong type inequalities from weak type inequalities of the form:

$$|\{x \in B := B(\bar{x}, r)\,;\, |u(x) - u_B| > \lambda\}| \le C \left(\frac{\|I(|Xu|)\|_{L^1(\tau B)}}{\lambda} \right)^{\frac{1}{q}}, \qquad \tau > 1,$$

In particular one obtains $(1, q)$-Sobolev-Poincaré inequalities in situations where one has no recourse to the Marcinkiewicz interpolation theorem. This is achieved by slicing the graph of $u(x) - u_B$ in strips $[2^{-k+1}, 2^{-k}]$. The local character of $|Xu|$ yields that $|Xu|$ vanishes on constants, so that it is possible to reconstruct $|Xu|$ from these slices.

This technique enabled Wheeden in [9] (in collaboration with Franchi and Gallot) and in [11] (in collaboration with Franchi and Lu) to prove Sobolev type inequalities and Sobolev-Poincaré type inequalities on Carnot groups in the *geometric case* $p = 1$, starting from a subrepresentation formula of a compactly supported function (or of a function of zero average on a ball) which expressed the function in terms of a suitable fractional integral of its generalized gradient. In particular, this argument yields forms of Sobolev inequalities which are related to isoperimetric inequalities on Carnot groups.

More generally, on a metric space $(\mathcal{S}, \rho, m)$ endowed with a doubling measure m, we say that a (p, q)-Sobolev-Poincaré inequality holds ($1 \le p \le q \le \infty$) if for any Lipschitz continuous function u there exists $g \in L^p_{\mathrm{loc}}(\mathcal{S})$ such that

$$\left(\frac{1}{|B(\bar{x}, r)|} \int_{B(\bar{x},r)} \left| u - \frac{1}{|B(\bar{x}, r)|} \int_{B(\bar{x},r)} u\, dm \right|^q dm \right)^{\frac{1}{q}} \le C\, r \left(\frac{1}{|B(\bar{x}, r)|} \int_{B(\bar{x},r)} |g|^p\, dm \right)^{\frac{1}{p}p}, \tag{3}$$

where g depends on u but is independent of $B(\bar{x}, r)$ (notice again we could look for similar inequalities where we replace the measure m by two measures μ, ν). We recall that the metric space $(\mathcal{S}, \rho, m)$ endowed with a measure m is said to be locally doubling, if for the measure m there exists $A > 0$ such that the measure m satisfies the doubling condition $m(B(x, 2r)) \le A m(x, r)$ for all $x \in \mathcal{S}$ and $r \le r_0$. That is the doubling condition holds for all balls with small enough radii. The central point in the proof in [12] (see also [8]) consists in establishing the equivalence between (p,q)-Sobolev-Poincaré inequalities in metric measure spaces and subrepresentation formulae. The following result is typical:

Theorem 5 *Let $(\mathcal{S},\rho,m)$ be a complete metric space endowed with a locally doubling measure m and satisfying the segment property (i.e., for each pair of points $x,y\in\mathcal{S}$, there exists a continuous curve γ connecting x and y such that $\rho(\gamma(t),\gamma(s))=|t-s|$.) Let μ, ν be locally doubling measures on $(\mathcal{S},\rho,m)$. Let $B_0=B(x_0,r_0)$ be a ball, let $\tau>1$ be a fixed constant, and let $f,g\in L^1(\tau B_0)$ be given functions. Assume there exists $C>0$ such that, for all balls $B\subseteq\tau B_0$,*

$$\frac{1}{\nu(B)}\int_B|f-f_{B,\nu}|\,d\nu\le C\frac{r(B)}{\mu(B)}\int_B|g|\,d\mu, \tag{4}$$

where $f_{B,\nu}=\frac{1}{\nu(B)}\int_B f d\nu$. If there is a constant $\theta(r_0)>0$ such that for all balls B, $\tilde B$ with $\tilde B\subseteq B\subseteq\tau B_0$,

$$\frac{\mu(B)}{\mu(\tilde B)}\ge\theta(r_0)\,\frac{r(B)}{r(\tilde B)},$$

then for $(d\nu)$-a.e. $x\in B_0$,

$$|f(x)-f_{B_0,\nu}|\le C\int_{\tau B_0}|g(y)|\frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))}\,d\mu(y). \tag{5}$$

We notice that, by Fubini-Tonelli Theorem, clearly (5) implies (4).

The proof of the above result relies on the construction of a suitable chain of balls with controlled overlaps, starting from a ball B and shrinking around a point $x\in B$. Repeated use of the Poincaré inequality (4) yields the subrepresentation formula (5) for any Lebesgue point of u.

Applying results on L^p-L^q continuity for fractional integrals, like for example in [3, 29], from (5) one obtains (two-weight) Sobolev-Poincaré inequalities. In [22] Lu and Wheeden were able to get rid of the constant $\tau>1$ in the subrepresentation formula (5).

A $(p_0,1)$-Poincaré-inequality with $p_0\ge1$ yielding a (p,q)-Sobolev-Poincaré inequalities (and possibly with weights) is referred to as the *self-improving property* of the Poincaré inequality. This notion had been introduced by Saloff-Coste in [53] in the Riemannian or sub-Riemannian setting. The result in [53] states that the $(p_0,1)$-Poincaré inequality plus the doubling property of the measure yields (p,q)-Sobolev-Poincaré inequalities. The arguments of [12] can be carried out only in the case $p_0=1$, basically since $a(B)$ defined by

$$a(B)=r(B)\left(\frac{1}{|B|}\int_B|g|^{p_0}\,dx\right)^{1/p_0}\qquad(g\text{ and }p_0\text{ fixed})$$

is not easy to sum, even over a class of disjoint balls B if $p_0>1$.

In particular, in [15], this difficulty was overcome by considering a sum operator $T(x)$ which is formed by summing $a(B)$ over an appropriate chain of balls associated with a point x:

$$T(x) = \sum_{B \text{ in a chain for } x} a(B).$$

In case $p_0 = 1$, the sum operator becomes an integral operator. The L^p to L^q mapping properties of the sum operator can be derived in much the same way as those for fractional integral operators, and these norm estimates for T lead to correspondingly more general Poincaré estimates. These results by Wheeden and collaborators for the weighted self-improving property of the Poincaré inequality on general metric spaces may be found in [13–15].

In [21], the authors proved a counterpart of the equivalence between subrepresentation formulae and Sobolev-Poincaré inequality for higher order differential operators. These results are counterparts of earlier results for the gradient derived in [12]. In the higher order case, on the left-hand side of the Poincaré inequality, instead of subtracting a constant given by the average of the function, one subtracts appropriate polynomials, related to the Taylor polynomial on Euclidean spaces, and related Folland-Stein polynomials [43] for the situation on stratified groups.

2.1 Two Weight Norm Inequalities for Fractional Integrals

Beginning in 1992, Wheeden returned to the study of the two weight inequality for fractional integrals,

$$\left(\int_{\mathbb{R}^n} (I_\alpha f)^q \, w\right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} f^p v^p\right)^{\frac{1}{p}}, \quad f \geq 0,$$

and showed with Sawyer [29] that for $1 < p < q < \infty$ and $0 < \alpha < n$, this inequality could be characterized by a simple two weight analogue of Muckenhoupt's condition:

$$A^\alpha_{p,q} \equiv \sup_Q |Q|^{1-\frac{\alpha}{n}} \left(\int_Q s_Q^q w\right)^{\frac{1}{q}} \left(\int_Q s_Q^{p'} v^{-p'}\right)^{\frac{1}{p'}} < \infty,$$

where $s_Q(x) \equiv |Q|^{\frac{\alpha}{n}-1} + |x - x_Q|^{\alpha-n}$ is a "tailed" version of the scaled indicator $|Q|^{\frac{\alpha}{n}-1} \mathbf{1}_Q(x)$. This work built on the weak type work of Kokilashvili and Gabidzashvili. Unfortunately, this simple solution fails when $p = q$, but there it was shown that a "bumped-up" version of $A^{\alpha}_{p,q}$ suffices for the fractional integral inequality: there is $r > 1$ such that

$$\sup_{Q} |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{|Q|} \int_{Q} w^{r} \right)^{\frac{1}{qr}} \left(\frac{1}{|Q|} \int_{Q} v^{-rp'} \right)^{\frac{1}{p'r}} < \infty.$$

A forerunner to the situation when $p = q$ is the paper by Chanillo and Wheeden [4], where weighted fractional integral inequalities when $p = q$ are derived and then applied to obtain Weyl type eigenvalue estimates for the Schrödinger operator with appropriate potential.

The results on weighted norm inequalities for two weights for fractional integrals and other similar results were then extended in [29] to spaces of homogeneous type. Along the way two discoveries were made which we list:

- The failure of the Besicovitch covering lemma for the Heisenberg group equipped with the usual left invariant metric and where all balls are chosen using this metric (also obtained independently by Koranyi and Reimann).
- A construction of a dyadic grid for spaces of homogeneous type (a variant was also obtained independently and a bit earlier by M. Christ [38], and a precursor of this by G. David even earlier).

2.2 Fefferman-Phong and Hörmander Regularity

The 2006 Memoir of Wheeden with Sawyer [30] is concerned with regularity of solutions to *rough subelliptic* equations. Previously, regularity had been reasonably well understood in two cases:

1. when the equation is subelliptic, and the coefficients are restricted to being smooth,
2. when the equation is elliptic, and the coefficients are quite rough.

In the subelliptic case, there were two main types of result. First, there was the algebraic commutator criterion of Hörmander for sums of squares of smooth vector fields [48]. These operators had a special "sum of squares" form for the second order terms, but no additional restriction on the smooth first order term. Second, there was the geometric "control ball" criterion of Fefferman and Phong that applies to operators with general smooth subelliptic second order terms, but the operators were restricted to be self-adjoint. They obtained the following analogue of the Fefferman-Phong theorem for rough coefficients, namely a quadratic form $\mathcal{Q}(x,\xi) = \xi' Q(x)\,\xi$ is *subelliptic* (which means that in a quantitative sense we leave unspecified, all weak solutions u to the equation $\nabla' Q(x)\,\nabla u = \phi$ are Hölder continuous, i.e., $u \in$

C^α for $\alpha > 0$), if the control balls $K(x,r)$ relative to $\mathcal{Q}$ satisfy

1. $|K(x,2r)| \le C|K(x,r)|$ (doubling),
2. $D(x,r) \subset K(x, Cr^\varepsilon)$ (containment),
3. $\left\{\frac{1}{|K|}\int_K |w|^{2\sigma}\right\}^{\frac{1}{2\sigma}} \le Cr(K)\left\{\frac{1}{|K|}\int_K \left(\|\nabla w\|_{\mathcal{Q}}^2 + |w|^2\right)\right\}^{\frac{1}{2}}$,
4. $\left\{\frac{1}{|K|}\int_K |w-w_K|^2\right\}^{\frac{1}{2}} \le Cr(K)\left\{\frac{1}{|K^*|}\int_{K^*} \|\nabla w\|_{\mathcal{Q}}^2\right\}^{\frac{1}{2}}$,

where $w \in W_0^{1,2}(K)$ in the Sobolev inequality and $w \in W^{1,2}(K^*)$ in the Poincaré inequality, where K^* is the double of K.

The containment condition 2. is necessary. The Sobolev inequality 3. is necessary for a related notion of subellipticity for the homogeneous Dirichlet problem for $L = \nabla' Q(x)\nabla$: for all balls B there exists a weak solution u satisfying

$$\begin{cases} Lu = f & in\ B \\ u\ = 0 & on\ \partial B \end{cases} \text{ and } \sup_B |u| \lesssim \left(\int_B |f|^{\frac{q}{2}}\right)^{\frac{2}{q}}.$$

The Poincare inequality 4. is necessary for a related notion of hypoellipticity for the homogeneous Neumann problem for $\mathbf{n}_Q = \mathbf{n}' Q(x)\nabla$: for all balls B there exists a weak solution u satisfying

$$\begin{cases} Lu\ = f & in\ B \\ \mathbf{n}_Q u = 0 & on\ \partial B \end{cases} \text{ and } \|u\|_{L^2(B)} \lesssim r(B)^2 \|f\|_{L^2(B)}.$$

The doubling condition 1. is not needed and has been replaced more recently with the theory of nondoubling measures pioneered by Nazarov, Treil, and Volberg.

They also obtained an analogue of the Hörmander theorem for diagonal vector fields with rough coefficients. As a starting point, they showed that if the vector fields $X_j = a_j(x)\frac{\partial}{\partial x_j}$ were analytic, then the X_j satisfied a "flag condition" if and only if they satisfied the Hörmander commutation condition. They then extended the flag condition to rough vector fields and obtained regularity theorems for solutions to the corresponding sums of squares operators.

2.3 *The Monge-Ampére Equation*

Using the regularity theorems in their 2006 Memoir [30] (see also [28], [31] and [32]), Wheeden with Rios and Sawyer [26, 27] obtained the following geometric result: A C^2 convex function u whose graph has smooth Gaussian curvature $k \approx |x|^2$ is itself smooth *if and only* if the sub-Gaussian curvature k_{n-1} of u is positive in Ω.

The question remains open for $C^{1,1}$ convex solutions today—this much regularity is assured for solutions to the Dirichlet problem with smooth data and nonnegative Gaussian curvature k (Guan et al. [46]), but cannot in general be improved to C^2 by the example of Sibony in which the tops and sides of the unit disk are curled up to form a smooth boundary but with second order discontinuities at the start of the curls.

The proof of the regularity theorem for C^2 solutions draws from a broad spectrum of results—an n-dimensional extension of the partial Legendre transform due to the authors [26], Calabi's identity for $\sum u^{ij}\sigma_{ij}$, the Campanato method of Xu and Zuily [54], the Rothschild-Stein lifting theorem for vector fields [52], Citti's idea (see, e.g., [39]) of approximating vector fields by first order Taylor expansions, and earlier work of the authors in [26] generalizing Guan's subelliptic methods in [44, 45]. The proof of the geometric consequence uses the Morse lemma to obtain the sum of squares representation of k. The necessity of $k_{n-1} > 0$ follows an idea of Iaia [49]: the inequality $k \leq (k_{n-1})^{\frac{n}{n-1}}$ shows that for a smooth convex solution u with $k(x) \approx |x|^2$ we must have $k_{n-1} > 0$ at the origin.

3 Papers by Richard Wheeden Referred to in the Preface

References

1. S. Chanillo, R.L. Wheeden, Distribution function estimates for Marcinkiewicz integrals and differentiability. Duke Math. J. **49**(3), 517–619 (1982)
2. S. Chanillo, R.L. Wheeden, Inequalities for Peano maximal functions and Marcinkiewicz integrals. Duke Math. J. **50**(3), 573–603 (1983)
3. S. Chanillo, R.L. Wheeden, Weighted Poincaré and Sobolev inequalities and estimates for weighted Peano maximal functions. Amer. J. Math. **107**(5), 1191–1226 (1985)
4. S. Chanillo, R.L. Wheeden, L^p-estimates for fractional integrals and Sobolev inequalities with applications to Schrödinger operators. Commun. Partial Differ. Equ. **10**(9), 1077–1116 (1985)
5. S. Chanillo, R.L. Wheeden, Harnack's inequality and mean-value inequalities for solutions of degenerate elliptic equations. Commun. Partial Differ. Equ. **11**(10), 1111–1134 (1986)
6. S. Chanillo, R.L. Wheeden, Some weighted norm inequalities for the area integral. Indiana Univ. Math. J. **36**, 277–294 (1987)
7. S. Chanillo, R.L. Wheeden, Existence and estimates of Green's function for degenerate elliptic equations. Ann. Sc. Norm (Pisa), **15**, 309–340 (1988)
8. B. Franchi, R.L. Wheeden, Some remarks about Poincaré type inequalities and representation formulas in metric spaces of homogeneous type. J. Inequal. Appl. **3**(1), 65–89 (1999)
9. B. Franchi, S. Gallot, R.L. Wheeden, Sobolev and isoperimetric inequalities for degenerate metrics. Math. Ann. **300**(4), 557–571 (1994)
10. B. Franchi, C.E. Gutiérrez, R.L. Wheeden, Weighted Sobolev-Poincaré inequalities for Grushin type operators. Commun. Partial Differ. Equ. **19**(3–4), 523–604 (1994)
11. B. Franchi, G. Lu, R.L. Wheeden, Representation formulas and weighted Poincaré inequalities for Hörmander vector fields. Ann. Inst. Fourier(Grenoble) **45**(2), 577–604 (1995)
12. B. Franchi, G. Lu, R.L. Wheeden, A relationship between Poincaré-type inequalities and representation formulas in spaces of homogeneous type. Int. Math. Res. Not. (1), 1–14 (1996)
13. B. Franchi, C. Pérez, R.L. Wheeden, Self-improving properties of John–Nirenberg and Poincaré inequalities on spaces of homogeneous type. J. Funct. Anal. **153**, 108–146 (1988)

14. B. Franchi, C. Pérez, R.L. Wheeden, Sharp geometric Poincaré inequalities for vector fields and non-doubling measures. Proc. Lond. Math. Soc. **80**(3), 665–689 (2000)
15. B. Franchi, C. Pérez, R.L. Wheeden, A sum operator with applications to self improving properties of Poincaré inequalities in metric spaces. J. Fourier Anal. Appl. **9**(5), 511–540 (2003)
16. R.F. Gundy, R.L. Wheeden, Weighted integral inequalities for the nontangential maximal function, Lusin area integral, and Walsh-Paley series. Stud. Math. **49**, 107–124 (1973/74)
17. R.A. Hunt, R.L. Wheeden, On the boundary values of harmonic functions. Trans. Am. Math. Soc. **132**, 307–322 (1968)
18. R.A. Hunt, R.L. Wheeden, Positive harmonic functions on Lipschitz domains. Trans. Am. Math. Soc. **147**, 507–527 (1970)
19. R. Hunt, B. Muckenhoupt, R. Wheeden, Weighted norm inequalities for the conjugate function and Hilbert transform. Trans. Am. Math. Soc. **176**, 227–251 (1973)
20. D.S. Kurtz, R.L. Wheeden, Results on weighted norm inequalities for multipliers. Trans. Am. Math. Soc. **255**, 343–362 (1979)
21. G. Lu, R.L. Wheeden, High order representation formulas and embedding theorems on stratified groups and generalizations. Stud. Math. **142**, 101–133 (2000)
22. G. Lu, R.L. Wheeden, An optimal representation formula for Carnot-Carathéodory vector fields. Bull. Lond. Math. Soc. **30**(6), 578–584 (1998)
23. B. Muckenhoupt, R.L. Wheeden, Weighted norm inequalities for fractional integrals. Trans. Am. Math. Soc. **192**, 261–274 (1974)
24. B. Muckenhoupt, R.L. Wheeden, Norm inequalities for the Littlewood-Paley function $g_\lambda^\star$. Trans. Am. Math. Soc. **191**, 95–111 (1974)
25. B. Muckenhoupt, R.L. Wheeden, Two weight function norm inequalities for the Hardy-Littlewood maximal function and the Hilbert transform. Stud. Math. **55**(3), 279–294 (1976)
26. C. Rios, E. Sawyer, R.L. Wheeden, A higher dimensional partial Legendre transform, and regularity of degenerate Monge-Ampère equations. Adv. Math. **193**, 373–415 (2005)
27. C. Rios, E. Sawyer, R.L. Wheeden, Regularity of subelliptic Monge-Ampère equations. Adv. Math. **217**(3), 967–1026 (2008)
28. E. Sawyer, R.L. Wheeden, Regularity of degenerate Monge-Ampère and prescribed Gaussian curvature equations in two dimensions. preprint available at http://www.math.mcmaster.ca/~sawyer
29. E. Sawyer, R.L. Wheeden, Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces. Amer. J. Math. **114**, 813–874 (1992)
30. E. Sawyer, R.L. Wheeden, Hölder continuity of weak solutions to subelliptic equations with rough coefficients. Mem. Am. Math. Soc. **180**(847), 10+157 (2006)
31. E. Sawyer, R.L. Wheeden, Regularity of degenerate Monge-Ampère and prescribed Gaussian curvature equations in two dimensions. Potential Anal. **24**, 267–301 (2006)
32. E. Sawyer, R.L. Wheeden, A priori estimates for quasilinear equations related to the Monge-Ampère equation in two dimensions. J. Anal. Math. **97**, 257–316 (2005)
33. C. Segovia, R.L. Wheeden, On weighted norm inequalities for the Lusin area integral. Trans. Am. Math. Soc. **176**, 103–123 (1973)
34. R.L. Wheeden, On hypersingular integrals and Lebesgue spaces of differentiable functions. Trans. Am. Math. Soc. **134**, 421–435 (1968)
35. R.L. Wheeden, On hypersingular integrals and Lebesgue spaces of differentiable functions. II. Trans. Am. Math. Soc. **139**, 37–53 (1969)
36. R.L. Wheeden, A. Zygmund, *Measure and Integral: An Introduction to Real Analysis*, 2nd edn. (CRC Press/Chapman and Hall, Boca Raton, 2015)

4 Other Papers Referred to in the Preface

References

37. D.L. Burkholder, R.F. Gundy, Distribution function inequalities for the area integral. Stud. Math. **44**, 527–544 (1972)
38. M. Christ, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral. Colloq. Math. **61**, 601–628 (1990)
39. G. Citti, E. Lanconelli, A. Montanari, Smoothness of Lipschitz continuous graphs with nonvanishing Levi curvature. Acta Math. **188**, 87–128 (2002)
40. R.R. Coifman, C.L. Fefferman, Weighted norm inequalities for maximal functions and singular integrals. Stud. Math. **51**, 241–250 (1974)
41. G. David, S. Semmes, Strong A_∞ weights, Sobolev inequalities and quasiconformal mappings, in *Analysis and Partial Differential Equations*. Lecture Notes in Pure and Applied Mathematics, vol. 122 (Dekker, New York, 1990) pp. 101–111
42. C. Fefferman, D.H. Phong, Subelliptic eigenvalue problems, in *Conference in Honor of A. Zygmund*, Wadsworth Math. Series (1981)
43. G.B. Folland, E.M. Stein, Hardy spaces on homogeneous groups. *Mathematical Notes*, vol. 28 (Princeton University Press/University of Tokyo Press, Princeton/Tokyo, 1982)
44. P. Guan, Regularity of a class of quasilinear degenerate elliptic equations. Adv. Math. **132**, 24–45 (1997)
45. P. Guan, C^2 *a priori* estimates for degenerate Monge-Ampère equations. Duke Math. J. **86**, 323–346 (1997)
46. P. Guan, N.S. Trudinger, X.-J. Wang, On the Dirichlet problem for degenerate Monge-Ampère equations. Acta Math. **182**, 87–104 (1999)
47. H. Helson, G. Szegö, A problem in prediction theory. Ann. Mat. Pura Appl. **51**, 107–138 (1960)
48. L. Hörmander, Hypoelliptic second order differential equations. Acta. Math. **119**, 141–171 (1967)
49. J. Iaia, Isometric embeddings of surfaces with nonnegative curvature in $\mathbb{R}^3$. Duke Math. J. **67**, 423–459 (1992)
50. R.L. Long, F.S. Nie, Weighted Sobolev inequality and eigenvalue estimates of Schrödinger operators, in *Harmonic Analysis* (Tianjin, 1988). Lecture Notes in Mathematics, vol. 1494 (Springer, Berlin, 1991), pp. 131–141
51. B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function. Trans. Am. Math. Soc. **165**, 207–226 (1972)
52. L. Rothschild, E.M. Stein, Hypoelliptic differential operators and nilpotent groups. Acta. Math. **137**, 247–320 (1976)
53. L. Saloff-Coste, A note on Poincaré, Sobolev, and Harnack inequalities. Int. Math. Res. Not. (2), 27–38 (1992)
54. C.-J. Xu, C. Zuily, Higher interior regularity for quasilinear subelliptic systems. Calc. Var. **5**(4), 323–343 (1997)

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On Some Pointwise Inequalities Involving Nonlocal Operators

Luis A. Caffarelli and Yannick Sire

To Dick Wheeden, with admiration and affection

Abstract The purpose of this paper is threefold: first, we survey on several known pointwise identities involving fractional operators; second, we propose a unified way to deal with those identities; third, we prove some new pointwise identities in different frameworks in particular geometric and infinite-dimensional ones.

1 Introduction

The present paper is devoted to several pointwise inequalities involving several nonlocal operators. We focus on two types of pointwise inequalities: the Córdoba-Córdoba inequality and the Kato inequality. In order to keep the presentation simple, we state the inequalities in question in the case of the fractional laplacian, i.e. $(-\Delta)^s$, in $\mathbb{R}^n$. Actually, in subsequent sections, we will generalize these inequalities to a lot of different contexts. Furthermore, we will present a unified proof for both inequalities based on some extension properties of some nonlocal operators. Our proofs are elementary and simplify the original arguments.

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S. Chanillo et al. (eds.), *Harmonic Analysis, Partial Differential Equations and Applications*, Applied and Numerical Harmonic Analysis,
DOI 10.1007/978-3-319-52742-0_1

The fractional Laplacian can be defined in various ways, which we review now. It can be defined using Fourier transform by

$$\mathcal{F}((-\Delta)^s v) = |\xi|^{2s}\, \mathcal{F}(v),$$

for $v \in H^s(\mathbb{R}^n)$. It can also be defined through the kernel representation (see the book by Landkof [12])

$$(-\Delta)^s v(x) = C_{n,s}\ \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{v(x) - v(\overline{x})}{|x - \overline{x}|^{n+2s}}\, d\overline{x}, \tag{1}$$

for instance for $v \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz space of rapidly decaying functions. Here we will only consider $s \in (0, 1)$.

The inequalities considered in the present paper are the following

Theorem 1.1 (Córdoba-Córdoba Inequality) *Let φ be a $C^2(\mathbb{R}^n)$ convex function. Assume that u and $\varphi(u)$ are such that $(-\Delta)^s u$ and $(-\Delta)^s \varphi(u)$ exist. Then the following holds*

$$(-\Delta)^s \varphi(u) \le \varphi'(u)(-\Delta)^s\, u. \tag{2}$$

The next theorem is the Kato inequality.

Theorem 1.2 (Kato Inequality) *The following inequality holds in the distributional sense*

$$(-\Delta)^s |u| \le sgn(u)(-\Delta)^s\, u. \tag{3}$$

The previous two theorems are already known: Theorem 1.1 is due to Córdoba and Córdoba (see [8, 9]). Theorem 1.2 is due to Chen and Véron (see [6]). Both original proofs are based on the representation formula given in (1). This formula holds only when the fractional laplacian is defined on $\mathbb{R}^n$. The Córdoba-Córdoba inequality is a very useful result in the study of the quasi-geostrophic equation (see [9]). This inequality has been generalized in several contexts in [10] for instance or [7]. In this line of research we propose a unified way of proving these inequalities based on some extension properties for nonlocal operators.

2 Some New Inequalities

In this section, we derive by a very simple argument several inequalities at the nonlocal level, i.e. without using any extensions, which are not available in these frameworks.

2.1 A Pointwise Inequality for Nonlocal Operators in Non-divergence Form

Nonlocal operators in non-divergence form are defined by

$$\mathcal{I}u(x) = -\int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)\,dy$$

for a kernel $K \geq 0$. Denote

$$\delta_y u(x) = -\Big(u(x+y) + u(x-y) - 2u(x)\Big).$$

Then, considering a C^2 convex function φ, one has by the fact that a convex function is above its tangent line

$$\delta_y \varphi(u)(x) = -\Big(\varphi(u(x+y)) + \varphi(u(x-y)) - 2\varphi(u(x))\Big) =$$
$$-\Big(\varphi(u(x+y)) - \varphi(u(x)) + \varphi(u(x-y)) - \varphi(u(x))\Big)$$
$$\leq \varphi'(u(x))\delta_y u(x).$$

Hence for the operator $\mathcal{I}$ one has also an analogue of the original Córdoba-Córdoba estimate.

2.2 The Case of Translation Invariant Kernels

Consider the operator

$$\mathcal{L}u(x) = \int_{\mathbb{R}^n} (u(x) - u(y))K(x-y)\,dy$$

where K is symmetric. Hence one can write

$$\mathcal{L}u(x) = \int_{\mathbb{R}^n} (u(x) - u(x-h))K(h)\,dh$$

or in other words, by a standard change of variables

$$\mathcal{L}u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \delta_h u(x)K(h)\,dh$$

We start with the following lemma, which is a direct consequence of the symmetry of the kernel

Lemma 2.1

$$\int_{\mathbb{R}^n} \mathcal{L}u(x) = 0.$$

The following lemma is consequence of straightforward computations

Lemma 2.2

$$\delta_h uv(x) = u\delta_h v + v\delta_h u +$$
$$(v(x+h) - v(x))(u(x+h) - u(x)) + (v(x-h) - v(x))(u(x-h) - u(x)).$$

Hence by the two previous lemma one has the useful identity

$$0 = \int_{\mathbb{R}^n} \mathcal{L}u^2 = 2\int_{\mathbb{R}^n} u\mathcal{L}u + 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (u(x) - u(y))^2 K(x-y)\, dxdy.$$

2.3 Some Integral Operators on Geometric Spaces

In this section, we describe new operators involving curvature terms. These operators appear naturally in harmonic analysis, as described below. They are of the form

$$\mathcal{L}u(x) = \int (u(x) - u(y))K(x,y)\, dy$$

where the non-negative kernel K is symmetric and has some geometric meaning. The integral sign runs either over a Lie group or over a Riemannian manifold. By exactly the same argument as in the previous section, one deduces trivially Córdoba-Córdoba estimates for these operators. We now describe these new operators.

2.3.1 The Case of Lie Groups

Let G be a unimodular connected Lie group endowed with the Haar measure dx. By "unimodular", we mean that the Haar measure is left and right-invariant. If we denote by $\mathcal{G}$ the Lie algebra of G, we consider a family

$$\mathbb{X} = \{X_1, \ldots, X_k\}$$

of left-invariant vector fields on G satisfying the Hörmander condition, i.e. $\mathcal{G}$ is the Lie algebra generated by the $X_i's$. A standard metric on G , called the Carnot-Caratheodory metric, is naturally associated with $\mathbb{X}$ and is defined as follows: let $\ell : [0,1] \to G$ be an absolutely continuous path. We say that ℓ is admissible if there exist measurable functions $a_1, \dots, a_k : [0,1] \to \mathbb{C}$ such that, for almost every $t \in [0,1]$, one has

$$\ell'(t) = \sum_{i=1}^{k} a_i(t) X_i(\ell(t)).$$

If ℓ is admissible, its length is defined by

$$|\ell| = \int_0^1 \left(\sum_{i=1}^{k} |a_i(t)|^2 \, dt \right)^{\frac{1}{2}}.$$

For all $x, y \in G$, define $d(x,y)$ as the infimum of the lengths of all admissible paths joining x to y (such a curve exists by the Hörmander condition). This distance is left-invariant. For short, we denote by $|x|$ the distance between e, the neutral element of the group and x, so that the distance from x to y is equal to $|y^{-1}x|$.

For all $r > 0$, denote by $B(x,r)$ the open ball in G with respect to the Carnot-Caratheodory distance and by $V(r)$ the Haar measure of any ball. There exists $d \in \mathbb{N}^*$ (called the local dimension of $(G, \mathbb{X})$) and $0 < c < C$ such that, for all $r \in (0,1)$,

$$cr^d \le V(r) \le Cr^d,$$

see [14]. When $r > 1$, two situations may occur (see [11]):

- Either there exist $c, C, D > 0$ such that, for all $r > 1$,

  $$cr^D \le V(r) \le Cr^D$$

  where D is called the dimension at infinity of the group (note that, contrary to d, D does not depend on $\mathbb{X}$). The group is said to have polynomial volume growth.
- Or there exist $c_1, c_2, C_1, C_2 > 0$ such that, for all $r > 1$,

  $$c_1 e^{c_2 r} \le V(r) \le C_1 e^{C_2 r}$$

  and the group is said to have exponential volume growth.

When G has polynomial volume growth, it is plain to see that there exists $C > 0$ such that, for all $r > 0$,

$$V(2r) \le CV(r), \tag{4}$$

which implies that there exist $C > 0$ and $\kappa > 0$ such that, for all $r > 0$ and all $\theta > 1$,

$$V(\theta r) \le C\theta^{\kappa} V(r). \tag{5}$$

On a Lie group as previously described, one introduces the Kohn sub-laplacian

$$\Delta_G = \sum_{i=1}^{k} X_i^2.$$

On any Lie group G, it is natural by functional calculus to define the fractional powers $(-\Delta_G)^s$, $s \in (0,1)$ of the Kohn sub-laplacian $-\Delta_G$. It has been proved in [13, 15] (see also [16]) that for Lie groups with polynomial volume

$$\|(-\Delta_G)^{s/2} u\|^2_{L^2(G)} \le C \int_{G\times G} \frac{|u(x)-u(y)|^2}{V(|y^{-1}x|)|y^{-1}x|^{2s}}\, dx\, dy.$$

It is therefore natural to consider the operator which is the Euler-Lagrange operator of the Dirichlet form in the R.H.S. of the previous equation given by

$$\mathcal{L}u(x) = \int_G \frac{u(x)-u(y)}{V(|y^{-1}x|)|y^{-1}x|^{2s}}\, dy.$$

It defines a new Gagliardo-type norm, suitably designed for Lie groups (of any volume growth). By the structure itself of this norm, one can prove as before a Córdoba-Córdoba inequality.

2.3.2 The Case of Manifolds

Let M be a complete Riemannian manifold of dimension n. Denote $d(x,y)$ the geodesic distance from x to y. Similarly to the previous case it is natural to introduce the new operators, Euler-Lagrange of suitable Gagliardo norms, given by

$$\mathcal{L}u(x) = \int_M \frac{u(x)-u(y)}{d(x,y)^{n+2s}}\, dy$$

These new operators also satisfy Córdoba-Córdoba estimates (see [15] for an account in harmonic analysis where these quantities pop up).

3 A Review of the Extension Property

3.1 The Extension Property in $\mathbb{R}^n$

We first introduce the spaces

$$H^s(\mathbb{R}^n) = \left\{v \in L^2(\mathbb{R}^n) \ : \ |\xi|^s(\mathcal{F}v)(\xi) \in L^2(\mathbb{R}^n)\right\},$$

where $s \in (0,1)$ and $\mathcal{F}$ denotes Fourier transform. For $\Omega \subset \mathbb{R}^{n+1}_+$ a Lipschitz domain (bounded or unbounded) and $a \in (-1,1)$, we denote

$$H^1(\Omega, y^a) = \left\{u \in L^2(\Omega, y^a\,dx\,dy) \ : \ |\nabla u| \in L^2(\Omega, y^a\,dx\,dy)\right\}.$$

Let $a = 1-2s$. It is well known that the space $H^s(\mathbb{R}^n)$ coincides with the trace on $\partial\mathbb{R}^{n+1}_+$ of $H^1(\mathbb{R}^{n+1}_+, y^a)$. In particular, every $v \in H^s(\mathbb{R}^n)$ is the trace of a function $u \in L^2_{\text{loc}}(\mathbb{R}^{n+1}_+, y^a)$ such that $\nabla u \in L^2(\mathbb{R}^{n+1}_+, y^a)$. In addition, the function u which minimizes

$$\min\left\{\int_{\mathbb{R}^{n+1}_+} y^a\,|\nabla u|^2\ dxdy \ : \ u|_{\partial\mathbb{R}^{n+1}_+} = v\right\} \tag{6}$$

solves the Dirichlet problem

$$\begin{cases} L_a u := \operatorname{div}(y^a\nabla u) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ u = v & \text{on } \partial\mathbb{R}^{n+1}_+. \end{cases} \tag{7}$$

By standard elliptic regularity, u is smooth in $\mathbb{R}^{n+1}_+$. It turns out that $-y^a u_y(\cdot, y)$ converges in $H^{-s}(\mathbb{R}^n)$ to a distribution $h \in H^{-s}(\mathbb{R}^n)$ as $y \downarrow 0$. That is, u weakly solves

$$\begin{cases} \operatorname{div}(y^a\nabla u) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -y^a\partial_y u = h & \text{on } \partial\mathbb{R}^{n+1}_+. \end{cases} \tag{8}$$

Consider the Dirichlet to Neumann operator

$$\begin{aligned} \Gamma_a : H^s(\mathbb{R}^n) &\to H^{-s}(\mathbb{R}^n) \\ v \mapsto \Gamma_a(v) &= h := -\lim_{y\to 0^+} y^a\partial_y u = \frac{\partial u}{\partial \nu^a}, \end{aligned}$$