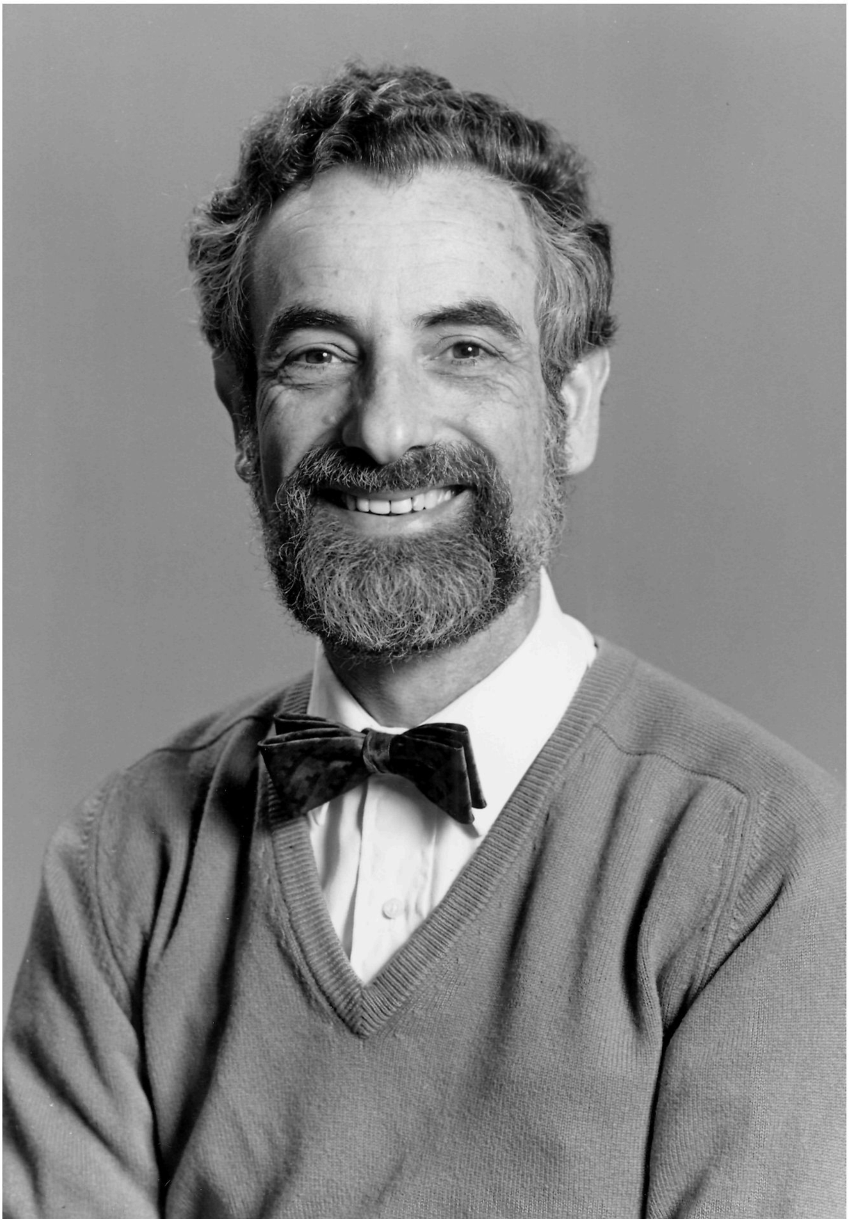
The Stability of Matter:
From Atoms to Stars

Selecta of Elliott H. Lieb



Elliott H. Lieb

The Stability of Matter: From Atoms to Stars

Selecta of Elliott H. Lieb

Edited by W. Thirring

With a Preface by F. Dyson

Fourth Edition

Springer

Professor Elliott H. Lieb
Departments of Mathematics and Physics
Jadwin Hall
Princeton University
P.O. Box 708
Princeton, New Jersey 08544-0708, USA

Professor Walter Thirring
Institut für Theoretische Physik, Universität Wien
Boltzmanngasse 5, 1090 Wien, Austria

Professor Freeman Dyson
The Institute of Advanced Study
School of Natural Sciences
Princeton, New Jersey 08540, USA

Library of Congress Control Number:
2004108033

ISBN 3-540-22212-X Fourth Edition
Springer Berlin Heidelberg New York

ISBN 3-540-42083-5 Third Edition
Springer Berlin Heidelberg New York

Springer Berlin Heidelberg New York
is a part of Springer science+business media

springeronline.com

Printed on acid-frree paper SPIN 11013839 55/3141/YL – 5 4 3 2 1 0

Preface to the Fourth Edition

This fourth edition of selecta of my work on the stability of matter contains recent work on two topics that continue to fascinate me: Quantum electrodynamics (QED) and the Bose gas.

Three papers have been added to Part VII on QED. As I mentioned in the preface to the third edition, there must be a way to formulate a *non-perturbative* QED, presumably with an ultraviolet cutoff, that correctly describes low energy physics, i.e., ordinary matter and its interaction with the electromagnetic field. The new paper VII.5, which "quantizes" the results in V.9, shows that the elementary 'no-pair' version of relativistic QED (using the Dirac operator) is unstable when many-body effects are taken into account. Stability can be restored, however, if the Dirac operator with the field, instead of the bare Dirac operator, is used to define an electron. Thus, the notion of a "bare" electron without its self-field is physically questionable.

One of the truly basic quantum-mechanical phenomena is the existence of a ground state, but to define this one must have a Hamiltonian and a variational principle. This is not easy to formulate in a truly relativistic theory using the Dirac operator, but work has been done trying to make the ground state clear for the non-relativistic Schrödinger operator coupled to the relativistic electromagnetic field. Paper VII.3 completes the work started in VII.2 by showing that the "binding condition" of VII.2 is satisfied and hence that atoms and molecules have true ground states in this theory, despite the possibility of an "infrared problem".

Paper VII.4 is a first, primitive attempt to quantify the mass renormalization problem non-perturbatively.

Part VIII on the Bose gas has four new papers. Paper VIII.7 carries forward what was begun in VIII.6 on the one-component gas ('jellium')by proving that Dyson's 1967 conjecture on the ground state of a *two*-component gas of high density charged bosons is exactly describable, asymptotically, by a mean-field equation.

The theory of the low density gas, is quite a different matter and its ground state is shown to have 100% Bose-Einstein condensation (in VIII.8) and 100% superfluidity (in VIII.9) for gases in traps (in the Gross-Pitaevskii limit). These long-conjectured results are now proved, but the 'holy grail' of proving Bose-Einstein condensation for an interacting gas in the usual thermodynamic limit remains as a challenge.

Other interesting facts, experimental and theoretical, about dilute gases in traps continue to emerge. Recent experiments show that gases in elongated traps behave like one-dimensional systems, and the mathematical proof is announced in VIII.10. A more detailed discussion of these papers is in VIII.1.

Three other selecta have also been published: *Inequalities*, edited by Michael Loss and Mary Beth Ruskai; *Statistical Mechanics* and *Condensed Matter Physics* *and Exactly Soluble Models*, both edited by Bruno Nachtergaele, Jan Philip Solovej and Jakob Yngvason. Section VIII of the Condensed Matter Physics selecta contains other papers on the Bose gas, including a recent review article.

It is a pleasure to thank Wolf Beiglböck and Walter Thirring for their support and encouragement and Sabine Lehr and Brigitte Reichel-Mayer for their patient help with the production through four editions.

Princeton, August 2004 *Elliott Lieb*

Preface to the Third Edition

The second edition of this "selecta" of my work on the stability of matter was sold out and this presented an opportunity to add some newer work on the quantum-mechanical many-body problem. In order to do so, and still keep the volume within manageable limits, it was necessary to delete a few papers that appeared in the previous editions. This was done without sacrificing content, however, since the material contained in the deleted papers still appears, in abbreviated form, at least, in other papers reprinted here.

Sections VII and VIII are new. The former is on *quantum electrodynamics* (QED), to which I was led by consideration of stability of the non-relativistic many-body Coulomb problem, as contained in the first and second editions. In particular, the fragility of stability of matter with classical magnetic fields, which requires a bound on the fine-structure constant even in the non-relativistic case (item V.4), leads to the question of stability in a theory with quantized fields. There are many unresolved problems of QED if one attempts to develop a non-perturbative theory – as everyone knows. A non-perturbative theory is essential, however, if one is going to understand the stability of the many-body problem, which is the stability of ordinary matter. Some physicists will say that a non-perturbative QED does not exist – and this might be true in the absence of cutoffs – but an effective theory with cutoffs of a few Mev *must* exist since matter exists.

At the present time physicists believe fully in the non-relativistic Schrödinger equation and even write philosophical tracts about it. This equation is the basis of condensed matter physics and, while it is understood that relativistic effects exist, they are not large and can be treated as mild perturbations. The original Schrödinger picture is alive and well as a very good approximation to reality, and shows no signs of internal inconsistency. For this reason, the 'stability of matter' question, which takes the equation literally and seriously without approximation, is accepted as a legitimate question about physical reality.

The radiation field, on the other hand, especially the quantized radiation field, is almost always treated perturbatively in one way or another. The Schrödinger equation, including coupling to the quantized radiation field treated perturbatively, describes ordinary matter with great accuracy, for it is the basis of chemistry and solid state physics.

One can ask if there is a theory, analogous to the Schrödinger theory, that incorporates particles and electromagnetic fields together without having to resort to perturbation theory. If it were really impossible to have a self-consistent, non-perturbative theory of matter and the low frequency radiation field it would not be unreasonable to call this a crisis in physical theory. It has always been assumed that at each level of physical reality there is a self consistent theory that describes phenomena well at that level. For example, we have chemistry, which is well described at the level of molecules, phase transitions and thermodynamics, which is presumably well described by statistical mechanics, etc. At the present time we have the beginnings of a consistent theory of matter and radiation at the nonrelativistic level, but a corresponding relativistic theory, is less developed.

Section VII describe some primitive attempts to address some of the questions that have to be resolved. One problem concerns renormalization. Starting from a relativistic Dirac theory, one finds that the radiation field forces a renormalization of the electron mass. This is well known, but what is not so well known is that the nonrelativistic theory forces an even bigger renormalization. Thus, if one starts from a relativistic theory, performs the renormalization, and then asks for the nonrelativistic limit of such a theory, it appears that one has to make further renormalizations – which it should not be necessary to do.

The new Section VIII is devoted to a subject that was a favorite of mine from the beginning of my career: *many-boson systems*. Some of the papers reprinted in the first and second editions on the stability of charged boson systems have been moved to this section. A very brief summary of all this material, as well as a list of my earliest work on this topic is given in an introduction VIII.1. Three new papers (VIII.2, 3, 4) are about the solution to a very old problem, namely the calculation of the ground state energy of a system of Bose particles with short-range forces at low density. The stability question is trivial, but the calculation of physical quantities in this intensely quantum-mechanical regime requires another sort of physical insight that goes back half a century but which was never rigorously validated.

Currently, the study of low density, low temperature Bose gases in "traps" is an active area of experimental physics. The extension of the homogeneous gas results to inhomogeneous gases in traps is in VIII.3. Similar subtleties arise for charged bosons in a neutralizing background ("jellium"), and this was resolved recently in VIII.7. The new material is summarized in VIII.4.

Certainly, much more remains to be understood about the ground states of bosonic systems. The two-component charged Bose gas, which is related to the jellium problem (see VIII.6) should be better understood, quantitatively. And, of course, there is the ancient quest for a proof of the existence of Bose condensation in interacting systems.

Princeton, March 2001 *Elliott Lieb*

Preface to the Second Edition

The first edition of "The Stability of Matter: From Atoms to Stars" was sold out after a time unusually short for a selecta collection and we thought it appropriate not just to make a reprinting but to include eight new contributions. They demonstrate that this field is still lively and keeps revealing unexpected features. Of course, we restricted ourselves to developments in which Elliott Lieb participated and thus the heroic struggle in Thomas-Fermi theory where the accuracy has been pushed from $Z^{7/3}$ to $Z^{5/3}$ is not included. A rich landscape opened up after Jakob Yngvason's observation that atoms in magnetic fields also are described in suitable limits by a Thomas-Fermi-type theory. Together with Elliott Lieb and Jan Philip Solovej it was eventually worked out that one has to distinguish 5 regions. If one takes as a dimensionless measure of the magnetic field strength B the ratio Larmor radius/Bohr radius one can compare it with $N \sim Z$ and for each of the domains

(i) $B \ll N^{4/3}$,
(ii) $B \sim N^{4/3}$,
(iii) $N^{4/3} \ll B \ll N^3$,
(iv) $B \sim N^3$,
(v) $B \gg N^3$

a different version of magnetic Thomas-Fermi theory becomes exact in the limit $N \to \infty$. In two dimensions and a confining potential ("quantum dots") the situation is somewhat simpler, one has to distinguish only

(i) $B \ll N$,
(ii) $B \sim N$,
(iii) $B \gg N$

and thus there are three semiclassical theories asymptotically exact. These fine distinctions make it clear how careful one has to be when people claim to have derived results valid for high magnetic fields. There is plenty of room for confusion if they pertain to different regions.

The question of stability of matter in in a magnetic field B has also been further cleaned up. Already the partial results in V.3 and V.4 of the first edition showed where the problem resides. Whereas diamagnetism poses no problem to stability since there a magnetic field pushes the energy up, the paramagnetism of the electron's spin magnetic moment can lower the energy arbitrarily much with increasing B. Only if one includes the field energy $+\frac{1}{8\pi}\int B^2$ one gets a bound uniform in B. This has now been shown to be true for arbitrarily many electrons and nuclei provided α and $Z\alpha^2$ are small enough. Though the sharp constants have not been determined, for $\alpha = 1/137$ and $Z \leq 1050$ (nonrelativistic) stability is guaranteed. Two recent works pertaining to this question have now been included:

1. A transparent and relatively simple proof of stability for Coulomb systems with relativistic kinetic energy is now available.
2. The problem has been studied with the Dirac Hamiltonians for the individual electrons in an external magnetic field. Here the question of how to fill the Dirac sea arises and it turns out that the correct way is to use the Hamiltonian including the magnetic field. Once this is done the Coulomb interaction does not introduce an instability provided the charges are below certain limits. On the contrary if one fills the Dirac sea of the free electrons and then introduces the magnetic field then one gets instability. Amusingly the relativistic theory also allows the above nonrelativistic limit on Z to be improved to $Z \leq 2265$.

Unfortunately QED has not yet matured mathematically to a state where these questions can be answered in a full-fledged relativistic quantum field theory and only in patchworks of relativistic corrections some answers are obtained. Thus this field is by no means exhausted and will keep challenging future generations.

Vienna, November 1996 *Walter Thirring*

Preface to the First Edition

With this book, Elliott Lieb joins his peers Hermann Weyl and Chen Ning Yang. Weyl's *Selecta* was published in 1956, Yang's *Selected Papers* in 1983. Lieb's "Selecta", like its predecessors, gives us the essence of a great mathematical physicist concentrated into one convenient volume. Weyl, Yang and Lieb have much more in common than the accident of this manner of publication. They have in common a style and a tradition. Each of them is master of a formidable mathematical technique. Each of them uses hard mathematical analysis to reach an understanding of physical laws. Each of them enriches both physics and mathematics by finding new mathematical depths in the description of familiar physical processes.

The central theme of Weyl's work in mathematical physics was the idea of symmetry, linking physical invariance-principles with the mathematics of group-theory. One of Yang's central themes is the idea of a gauge field, linking physical interactions with the mathematics of fibre-bundles. The central theme of Lieb's papers collected in this book is the classical Thomas-Fermi model of an atom, linking the physical stability of matter with the mathematics of functional analysis. In all three cases, a rather simple physical idea provided the starting-point for building a grand and beautiful mathematical structure. Weyl, Yang and Lieb were not content with merely solving a problem. Each of them was concerned with understanding the deep mathematical roots out of which physical phenomena grow.

The historical development of Lieb's thinking is explained in the review articles in this volume, items 65, 92 and 136 in Lieb's publication list. I do not need to add explanatory remarks to Lieb's lucid narrative. I suppose the reason I was asked to write this preface is because Lenard and I found a proof of the stability of matter in 1967. Our proof was so complicated and so unilluminating that it stimulated Lieb and Thirring to find the first decent proof, included here (paper 85 in the publication list). Why was our proof so bad and why was theirs so good? The reason is simple. Lenard and I began with mathematical tricks and hacked our way through a forest of inequalities without any physical understanding. Lieb and Thirring began with physical understanding and went on to find the appropriate mathematical language to make their understanding rigorous. Our proof was a dead end. Theirs was a gateway to the new world of ideas collected in this book.

Princeton, March 1990 *Freeman Dyson*

Contents

Part IV. Thomas–Fermi and Related Theories

Part V. Stability of Matter

Part VI. The Thermodynamic Limit for Real Matter with Coulomb Forces

Part VII. Quantum Electrodynamics

Part VIII. Bosonic Systems

Introduction

1

"Once we have the fundamental equation (*Urgleichung*[1]) we have the theory of everything" is the creed of some physicists. They go on to say that "then physics is complete and we have to seek other employment". Fortunately, other scientists do not subscribe to this credo, for they believe that it is not the few Greek letters of the *Urgleichung* that are the essential physics but rather that physics consists of all the consequences of the basic laws that have to be unearthed by hard analysis. In fact, sometimes it is the case that physics is not so much determined by the specific form of the fundamental laws but rather by more general mathematical relations. For instance, the KAM-theorem that determines the stability of planetary orbits does not depend on the exact $1/r$ law of the gravitational potential but it has a number theoretic origin. Thus a proper understanding of physics requires following several different roads: One analyzes the general structure of equations and the new concepts emerging from them; one solves simplified models which one hopes render typical features; one tries to prove general theorems which bring some systematics into the gross features of classes of systems and so on. Elliott Lieb has followed these roads and made landmark contributions to all of them. Thus it was a difficult assignment when Professor Beiglböck of Springer-Verlag asked me to prepare selecta on one subject from Lieb's rich publication list[2]. When I finally chose the papers around the theme "stability of matter" I not only followed my own preference but I also wanted to bring the following points to the fore:

(a) It is sometimes felt that mathematical physics deals with epsilontics irrelevant to physics. Quite on the contrary, here one sees the dominant features of real matter emerging from deeper mathematical analysis.

(b) The *Urgleichung* seems to be an ever receding mirage which leaves in its wake laws which describe certain more or less broad classes of phenomena. Perhaps the widest class is that associated with the Schrödinger equation with $1/r$-potentials, which appears to be relevant from atoms and molecules to bulk matter and even cosmic bodies. Thus the papers reproduced here do not deal with mathematical games but with the very physics necessary for our life.

(c) In mathematics we see a never ending struggle for predominance between geometry and analysis, the fashions swinging between extremes. Not too long ago the intuitive geometrical way in which most physicists think was scorned by

[1] *Urgleichung* (which is a word coined by Heisenberg) has now been replaced by the TOE (theory of everything) of the advocates of superstring theory.
[2] Henceforth quoted as [Lieb, ...].

mathematicians and only results from abstract analysis were accepted. Today the pendulum has swung the other way and the admired heroes are people with geometric vision, whereas great analysts like J. von Neumann tend to be thought of as degenerate logicians. As a physicist one should remain neutral with regard to internal affairs of mathematics, but it is always worthwhile to steer against the trend. Stability of matter illustrates beautifully that the great masters have forged for us the very analytical tools which we need to extract the physics from the fundamental equations.

In the selected papers the deeper results are carefully derived and what is presupposed on the reader's part are standard mathematical relations like convexity inequalities. Thus, it is to be hoped that this volume is not only of historical interest but also a useful source for workers in quantum mechanics.

2 Extensivity

We all take for granted that two liters of gasoline contain twice as much energy as one liter. In thermodynamics this fact has been dogmatized by the statement that the energy is an extensive quantity. This means that if $E(N)$ is the energy of the N-particle system then $\lim_{N\to\infty} E(N)/N$ is supposed to exist, or that the energy per particle approaches a limit in the many-body problem. This is what is meant by "stability of matter" and it was tacitly assumed by people working in this field even though at first sight it seems rather improbable for a system of particles interacting via two-body potentials. The Hamiltonian

$$H_N = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \sum_{i>j} v(x_i - x_j) \tag{1}$$

contains a double sum and one might expect $E(N) \sim N^2$ rather than $E(N) \sim N$. Historically the existence of $\lim_{N\to\infty} \frac{1}{N} E(N)$ was first seriously studied in the framework of classical statistical mechanics by Van Hove [1] where he had to assume a potential with a hard core. However, real matter, which consists of electrons and nuclei, obeys the laws of quantum mechanics and the dominant interactions are electrostatic or, for cosmic bodies, gravitational. In the latter case the N-dependence of the quantum mechanical ground state energy E_N was first established by Levy-Leblond [2] who proved that in the gravitational case, $v(x_i - x_j) = -\kappa m_i m_j / |x_i - x_j|$, $E_N \sim N^3$ for bosons and $E_N \sim N^{7/3}$ for fermions. Consequently, as was pointed out by Fisher and Ruelle [3], the situation for purely electrostatic interactions, $v(x_i - x_j) = e_i e_j / |x_i - x_j|$, was not so clear. A new chapter in science was opened up when this question was finally settled by Dyson and Lenard [4] who derived in a seminal paper the fundamental result that in this case actually $E_N \sim N$ if all particles of one sign of charge are fermions. If this is not the case and there are bosons of both signs of charge then they could show that E_N is somewhere between $N^{5/3}$ and $N^{7/5}$. It is fairly obvious that there is no chance for stability if the potential is all attractive, but it was rather shocking that electrostatics with its screening property might lead to instability. One might be inclined to consider this as a pathology of the Coulomb potential $1/r$ with its long range and its singularity and that things would be stable once these troublemakers are removed. That this is not the case is illustrated by the following [5].

Proposition (2)

For the Hamiltonian

$$H_N = \sum_{i=1}^{N} \frac{p_N^2}{2m} + \sum_{i>j} e_i e_j v(|x_i - x_j|), \qquad e_i = \pm e, \qquad \sum_i e_i = 0,$$

the quantum mechanical ground state energy E_N satisfies

(i) $E_N \sim N$ for fermions, $E_N \sim N^{7/5}$ for bosons if

$$v(r) = \frac{1}{r},$$

(ii) $E_N \sim N$ for fermions and bosons if

$$v(r) = \frac{1 - e^{-\mu r}}{r},$$

(iii) $E_N \sim N^2$ for fermions and bosons if

$$v(r) = re^{-\mu r}.$$

Thus it becomes clear that $E_N \sim N$ happens only under exceptional circumstances and one has to inquire what goes wrong in the case that $E_N \sim N^\gamma$, $\gamma > 1$. It is sometimes said that in this case the thermodynamic limit does not exist and therefore thermodynamics no longer applies. Though there is some truth in this statement, the argument is somewhat superficial and a more detailed analysis is warranted. One might object that in reality one deals with finite systems and the existence of $\lim_{N\to\infty}$ is only of mathematical interest. This is not quite true because the existence of the limit simply says that for a large but finite system the corresponding quantities are already close to their limiting value and the system can be described by the idealized situation $N \to \infty$. But for this to be true the kind of limit that is taken is irrelevant. For instance, in the gravitational case one can show the existence of $\lim_{N\to\infty} N^{-7/3} E_N$ for fermions (or $\lim_{N\to\infty} N^{-3} E_N$ for bosons). For stars ($N \sim 10^{57}$) one is certainly close to the limit, and their energy is predicted equally well by this kind of $N^{7/3}$ limit as in the stable situation. However I shall now discuss three dominating features which set the case $E_N \sim N$ apart from $E_N \sim N^\gamma$, $\gamma > 1$, and justify the terminology "stability of matter".

2.1 Relativistic Collapse

The relativistic expression for the kinetic energy $\sqrt{p^2 + m^2}$ weakens the zero point pressure to the extent that the "relativistic" quantum mechanical Hamiltonian

$$H_N^r = \sum_i \sqrt{p_i^2 + m_i^2} + \sum_{i>j} v(x_i - x_j) \tag{3}$$

might cease to be bounded from below if N is sufficiently big. Although this fact was recognized at the beginning of quantum mechanics [6], its dramatic consequence that a star of more than about twice the solar mass will collapse was doubted by leading astrophysicists. Only the experimental discovery of pulsars in the sixties provided overwhelming evidence for this prediction of quantum mechanics and therefore its relevance for cosmic bodies. Thus, there seems to be an essential difference between the relativistic and the nonrelativistic Hamiltonian. Whereas (1) with $1/r$-potentials is always bounded from below – and instability only implies that its lower bound E_N is not proportional to N – relativistically E_N may be $-\infty$ for all $N > N_c$ = some critical particle number ("relativistic collapse"). Related to this is the fact that nonrelativistically the size of the ground state may shrink with N but it always stays finite, whereas relativistically for $N > N_c$ the system keeps contracting until effects not contained in (3) take over. The two phenomena are related by the following fact which illustrates why instability is catastrophic [7].

Proposition (4)
If in (1) $v(x)$ scales like $v(\lambda x) = \lambda^{-1} v(x)$ then nonrelativistic instability implies relativistic collapse.

The converse statement is false since nonrelativistically matter is stable for arbitrary values of e, whereas relativistically the two-body Hamiltonian already becomes unbounded from below for $\alpha > \pi/2 = \alpha_c(2)$. Once it was found by Daubechies and Lieb [Lieb, 150] that these stability limits for α become more severe in the many-body situation, the question of stability of Coulomb matter with relativistic kinetic energy appeared more serious because $\lim_{N\to\infty} \alpha_c(N)$ could conceivably be zero. That this is not so for fermions was first shown by Conlon [8] who proved for nuclear charge $Z = 1$ $\alpha_c(N) > 10^{-200}$ for all N, a value which was subsequently improved by Fefferman and de la Llave [9]. Finally, Lieb and Yau [Lieb, 186] obtained the optimal result which is stability if $\alpha Z \leq 2/\pi$ and $\alpha \leq 1/94$. By Proposition (4) in the other cases (Coulomb matter with positive and negative bosons or with gravitation) the relativistic collapse is unavoidable.

2.2 Thermodynamic Stability

Shortly after $H_N \geq -N \cdot const$ was established, Lieb and Lebowitz showed the existence of the thermodynamic functions in the limit $N \to \infty$ for Coulomb matter [Lieb, 43]. This does not follow automatically from stability because the long range of the Coulomb potential poses additional problems. They had to demonstrate that this is sufficiently screened so that separated portions of matter are sufficiently isolated. They not only generalized the work of Van Hove et al. [1,3] to the realistic situation but also considered the microcanonical ensemble and showed that the specific heat was positive. Van Hove had shown that classically the limiting system had positive compressibility which Lieb and Lebowitz also verified for the quantum Coulomb case. These conditions for thermodynamic stability are concisely expressed by a convexity property. Denote by $H_{N,V}$ the N-particle Hamiltonian (1) in a volume V, by $\mathcal{H}_S$ an e^S-dimensional subspace of the Hilbertspace in which $H_{N,V}$ acts, and $\mathrm{Tr}_{\mathcal{H}_S}$ the trace in this subspace. Then for a finite system the energy as function of particle number, entropy and volume is

$$E(N,S,V) = \inf_{\mathcal{H}_S} \mathrm{Tr}_{\mathcal{H}_S} H_{N,V}. \tag{5}$$

In the thermodynamic limit one considers

$$\varepsilon(N,S,V) = \lim_{\lambda\to\infty} \lambda^{-1} H(\lambda N, \lambda S, \lambda V) \tag{6}$$

and thermodynamic stability requires that ε is jointly convex in its arguments. Lest the reader might think that one gets thermodynamic stability for free one has to note that it was shown by Hertel and Thirring [10] that in the gravitational case the temperature-dependent Thomas-Fermi theory becomes exact in a certain limit such that in this case $\varepsilon(N,S,V)$ not only exists but is calculable. Since this system is not stable, ε has to be defined by

$$\varepsilon(N,S,V) = \lim_{\lambda\to\infty} \lambda^{-7/3} H(\lambda N, \lambda S, \lambda^{-1} V). \tag{7}$$

It turned out that this system was also thermodynamically unstable; it showed a region of negative specific heat [11,12,13]. Such a phenomenon had been previously discovered in some models [14,15,16] and was always suspected by astrophysicists. That it was no accident that it occurred for the nonextensive Hamiltonian was clarified by Landsberg [17] by means of the following.

Theorem (8)

Let $x \to f(x)$ be a map from a convex set of $\mathbf{R}^n$ into $\mathbf{R}$. Then any two of the conditions

Homogeneity:	$H : f(\lambda x) = \lambda f(x), \quad \lambda \in \mathbf{R}^+$
Subadditivity:	$S : f(x_1 + x_2) \le f(x_1) + f(x_2)$
Convexity:	$C : f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2)$

imply the third.

If we take for f the function $\varepsilon(N,S,V)$ then H is obviously satisfied in the stable situation (6) but not in the unstable case (7). Thus H is a condition for stability against implosion. On the other hand, the subadditivity S means a stability against explosion; one gains energy by putting two parts together. Finally, C is thermodynamic stability. The theorem says that any two of these stability notions imply the third or, if one fails to hold, the others cannot both be true. In the cases we considered, S holds. This is clear for attractive interactions like gravity whereas for repulsive potentials (e.g. all charges having the same sign) the system would be explosive and S is violated. However, for electrically neutral systems there is always a van der Waals attraction [Lieb, 166] so that S is satisfied, which means that H and C become equivalent. Thus it turns out that our stability condition in terms of extensitivity is equivalent to thermodynamic stability provided the system is not explosive.

2.3 The Existence of Quantum Field Theory

Systems with negative specific heat cannot coexist with other systems. They heat up and give off energy until they reach a state of positive specific heat. This is what thermodynamic considerations tell us [15] but it was doubted [18] that in these cases thermodynamics is applicable and reflects the dynamics of the system. However, recent computer studies [19,20] of the dynamics of unstable systems have revealed that they behave exactly the way one expected by determining the dominant feature in phase space. In these studies one solved the classical equations of motion for 400 particles on a torus with

$$v(x_i - x_j) = -\exp[-|x_i - x_j|^2/b^2]$$

or

$$v(x_i - x_j) = e_i e_j (x_i - x_j)^2 \exp[-|x_i - x_j|^2/b^2].$$

It turned out that irrespective of the initial state, if the total energy is sufficiently small a hot cluster of size b with N_c, say 150 particles, developed, the rest being a homogeneous atmosphere. The temperature increased proportional to N_c. Thus there is no hope for a limiting dynamics for $N \to \infty$. In particular in quantum field theory, where N is not restricted, one cannot expect that a Hamiltonian H with a smooth potential that, however, leads to an unstable system leads to a dynamics in the Heisenberg representation via the usual formula

$$a(t) = e^{iHt} a e^{-iHt}.$$

Since states with unlimited N will lead to unlimited temperatures and thus unlimited velocities there cannot be a state independent dynamics. These ideas about the relevance of stability for the existence of quantum field theory were brought forward first by Dyson [21] and have been substantiated by proofs only in the converse direction. If one achieves stability by a momentum cut-off then there exists a dynamics in the Heisenberg representation for the quantum field theory [22].

Thus, if one wants a many-body system to behave in the way we are used to, the first question to be answered is its stability. Instability seems to be the rule rather than the exception and actually we owe our lives to the gravitational instability. Not only did Boltzmann's heat death not take place but, on the contrary, the universe, which originally is supposed to have been in an equilibrium state, developed hot clusters, the stars. Thus we can enjoy the sunshine which is rich in energy and lean in entropy, the kind of diet we need.

3

We have seen how a seemingly inconspicuous inequality turned into a key notion for understanding many-body physics. One of the great contributors to this development was Elliott Lieb. He usually did not say the first word on any of these issues but, with various collaborators, the last. His work (with B. Simon) [Lieb, 97] which put Thomas-Fermi (T.F.) theory on a firm basis opened the way for a better understanding of stability of matter. Since they proved that there is no chemical binding in T.F. theory, stability of matter becomes obvious once one knows that the T.F. energies are lower bounds for the corresponding quantum problem. That this is so is still a conjecture but it was shown by E. Lieb (with W. Thirring) [Lieb, 85] that this is true if one changes the T.F.-energies by some (N-independent) numerical factor.

If the electrons were bosons there remained the question whether the lower bound $\sim -N^{5/3}$ or the upper bound $\sim -N^{7/5}$ given by Dyson and Lenard [4] reflects the true behavior. This question was settled in an unexpected way. Lieb showed that if the masses of the nuclei are ∞ then $-N^{5/3}$ is correct [Lieb, 118] whereas if they are finite he supplied (with J. Conlon and H.-T. Yau) [Lieb, 188] a lower bound $\sim -N^{7/5}$, thereby showing the essential difference between these two cases. This is all the more surprising since in the justification of the Born-Oppenheimer approximation, which says that there is not much difference between big and infinite nuclear mass, usually no reference is made to the fermionic nature of the electrons.

Once one has established the gross features of Coulombic matter reflected by T.F. theory one can worry about finer details not described by it. Of these I will single out two to which Lieb has significantly contributed. One is the question of negative ions (which do not exist in pure T.F. theory but exist if the von Weizsäcker correction to the kinetic energy is added [Lieb, 130]). Ruskai [24] was the first to show that $N(Z)$, the maximal number of electrons bound by a nucleus (in the real quantum theory), is finite. Lieb gave an elegant proof that $N(Z) < 2Z + 1$ [Lieb, 157]. Furthermore he showed (with I. Sigal, B. Simon and W. Thirring) [Lieb, 185] that if the electrons are fermions then $\lim_{Z\to\infty} N(Z)/Z = 1$. (If the electrons were bosons this limit would be 1.21 [Lieb, 160], [23].) The latter result on real electrons has recently been improved by Fefferman and Seco but the proof of the conjecture $N(Z) < 2 + Z$ is still a challenge for the future.

A typical feature of quantum mechanics which is not included in T.F. theory or any single electron theory (or density functional theory for that matter) is the attraction between neutral objects. There Lieb (with W. Thirring) [Lieb, 166] showed that the Schrödinger equation predicts a potential which is below $-c/(R + R_0)^6$ where R is the distance between their centers and R_0 a length related to their size.

From Proposition (4) it follows that the relativistic Hamiltonian with gravitation becomes unbounded from below if $N > N_c$ = some critical particle number. It remained to be demonstrated that it stays bounded for $N < N_c$ and to determine N_c. Lieb (with W. Thirring) [Lieb, 158] showed that this was actually the case and gave some bounds for N_c which show the expected behavior $\kappa^{-3/2}$ for fermions and κ^{-1} for bosons (κ = gravitational constants in units $\hbar = c = m = 1$). Later he sharpened (with H.-T. Yau) [Lieb, 177] these results and showed that, for $\kappa \to 0$, N_c for fermions is exactly the Chandrasekhar limit.

The papers selected here show how powerful modern functional analysis is in determining the gross features of real matter from the basic quantum mechanical equations. Whereas a century's effort of the greatest mathematicians on the classical N-body problem with $1/r$ potentials only produced results for the simplest special cases, in quantum mechanics the contours of the general picture have emerged with much greater clarity.

The papers are grouped in various subheadings, the first paper being a recent review which will serve as a convenient introduction.

Vienna, November 1990 *Walter Thirring*

References

[1] L. Van Hove, Physica *15*, 137 (1950).

[2] J.M. Lèvy-Leblond, J. Math. Phys. *10*, 806 (1969).

[3] M. Fisher, D. Ruelle, J. Math. Phys. *7*, 260 (1966).

[4] F.J. Dyson, A. Lenard, J. Math. Phys. *8*, 423 (1967).

[5] W. Thirring, *A Course in Mathematical Physics*, Vol IV, Springer, Wien, New York, 1980.

[6] S. Chandrasekhar, Month. Not. Roy. Ast. Soc. *91*, 456 (1931).

[7] W. Thirring, Naturwissenschaften *73*, 605 (1986).

[8] J. Conlon, Commun. Math. Phys. *94*, 439 (1984).

[9] Ch. Fefferman, R. de la Llave, I. Rev. Math. Iberoamerican *2*, 119 (1986).

[10] P. Hertel, W. Thirring, J. Math. Phys. *10*, 1123 (1969).

[11] P. Hertel, W. Thirring, in *Quanten und Felder*, ed. H.P. Dürr, Vieweg, Braunschweig, 1971.

[12] A. Pflug, Commun. Math. Phys. *22*, 83 (1980).

[13] J. Messer, J. Math. Phys. *22*, 2910 (1981).

[14] D. Lynden-Bell, R. Wood, Month. Not. *138*, 495 (1968).

[15] W. Thirring, Z. f. Physik *235*, 339 (1970).

[16] P. Hertel, W. Thirring, Ann. of Phys. *63*, 520 (1971).

[17] P. Landsberg, J. Stat. Phys. *35*, 159 (1984).

[18] H. Jensen, private communications to W. Thirring.

[19] A. Compagner, C. Bruin, A. Roelse, Phys. Rev. *A 39*, 5989 (1989).

[20] H. Posch, H. Narnhofer, W. Thirring, in *Simulations of Complex Flows*, NATO A.S.I., ed. M. Mareschal (1990) and "Dynamics of Unstable Systems", UWThPh-1990-2, submitted to Phys. Rev. A

[21] F.J. Dyson, Phys. Rev. *85*, 613 (1952).

[22] H. Narnhofer, W. Thirring, "On Quantum Field Theories with Galilei-Invariant Interactions", Phys. Rev. Lett. *54*, 1863 (1990).

[23] B. Baumgartner, J. Phys. *A 17*, 1593 (1984).

[24] M.B. Ruskai, Commun. Math. Phys. *82*, 457 (1982).

Part I
Review

THE STABILITY OF MATTER: FROM ATOMS TO STARS

ELLIOTT H. LIEB

Why is ordinary matter (e.g., atoms, molecules, people, planets, stars) as stable as it is? Why is it the case, if an atom is thought to be a miniature solar system, that bringing very large numbers of atoms together (say 10^{30}) does not produce a violent explosion? Sometimes explosions do occur, as when stars collapse to form supernovae, but normally matter is well behaved. In short, what is the peculiar mechanics of the elementary particles (electrons and nuclei) that constitute ordinary matter so that the material world can have both rich variety and stability?

The law of motion that governs these particles is the quantum (or wave) mechanics discovered by Schrödinger [SE] in 1926 (with precursors by Bohr, Heisenberg, Sommerfeld and others). Everything we can sense in the material world is governed by this theory and some of its consequences are quite dramatic, e.g., lasers, transistors, computer chips, DNA. (DNA may not appear to be very quantum mechanical, but notice that it consists of a very long, thin, complex structure whose overall length scale is huge compared to the only available characteristic length, namely the size of an atom, and yet it is stable.) But we also see the effects of quantum mechanics, without realizing it, in such mundane facts about stability as that a stone is solid and has a volume which is proportional to its mass, and that bringing two stones together produces nothing more exciting than a bigger stone.

The mathematical proof that quantum mechanics gives rise to the observed stability is not easy because of the strong electric forces among the elementary constituents (electrons and nuclei) of matter. The big breakthrough came in the mid sixties when Dyson and Lenard [DL] showed, by a complicated proof, that stability

Received by the editors August 15, 1989.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 81H99, 81M05, 85A15; Secondary 81C99, 82A15.

This paper was presented as the Sixty-Second Josiah Willard Gibbs Lecture on January 11, 1989 at the 95th annual meeting of the American Mathematical Society in Phoenix, Arizona.

Work partially supported by U.S. National Science Foundation grant PHY-85-15288-A03.

is, indeed, a consequence of quantum mechanics. (Part of their motivation came from earlier work by Van Hove, Lee and Yang, van Kampen, Wils, Mazur, van der Linden, Griffiths, Dobrushin, and especially Fisher and Ruelle who formulated the problem and showed how to handle certain well chosen, but unrealistic forces.) This was a milestone but there was room left for improvement since their results had certain drawbacks and did not cover all possible cases; for instance, it turns out that quantum mechanics, which was originally conceived to understand atoms, is also crucial for understanding why stars do not collapse. Another problem was that they proved what is called here stability of the second kind while the existence of the thermodynamic limit (Theorem 3 below), which is also essential for stability, required further work [LL]. The full story has now, two decades later, mostly been sorted out, and that is the subject of this lecture. The answer contains a few surprises, some of which are not even discussed in today's physics textbooks.

No physics background will be assumed of the reader, so Part I reviews some basic facts. Part II contains a synopsis of the aspects of quantum mechanics needed here. Part III treats the simplest system—the hydrogen atom, and Part IV introduces the strange Pauli exclusion principle for many electrons and extends the discussion to large atoms. Part V deals with the basic issue of the stability of matter (without relativistic effects) while Part VI treats hypothetical, but interesting, matter composed of bosons. Part VII treats the problems introduced by the special theory of relativity. Finally, Part VIII applies the results of Part VII to the structure of stars.

PART I. THE PHYSICAL FACTS AND THEIR PREQUANTUM INTERPRETATION

While it is certainly possible to present the whole story in a purely mathematical setting, it is helpful to begin with a brief discussion of the physical situation.

The first elementary constituent of matter to be discovered was the **electron** (J.J. Thomson, 1897). This particle has a **negative electric charge** (denoted by $-e$) and a **mass**, m. It is easy to produce a beam of electrons (e.g., in a television tube) and use it to measure the ratio e/m quite accurately. The measurement of e alone is much trickier (Millikan 1913). The electron can be considered to be a point, i.e., it has no presently discernible geometric structure. Since matter is normally electrically neutral (otherwise we would feel electric fields everywhere), there must also be another constituent with positive charge. One of the early ideas about this positive object was that it is a positively charged ball of a radius about equal to the radius of an atom, which is approximately 10^{-8} cm. (This atomic radius is known, e.g., by dividing the volume of a solid, which is the most highly compressed form of matter, by the number of atoms in the solid.) The electrons were thought to be stuck in this charged ball like raisins in a cake; such a structure would have the virtue of being quite stable, almost by fiat. This nice picture was destroyed, however, by Rutherford's classic 1903 experiment which showed that the positive entities were also essentially points. (He did this by scattering positively charged helium nuclei through thin metal foils and by showing that the distribution of scattering angles was the same as for the Kepler problem in which the trajectories are hyperbolas; in other words, the scatterers were effectively points—not extended objects.)

The picture that finally emerged was the following. Ordinary matter is composed of two kinds of particles: the point electrons and positively charged **nuclei**. There are many kinds of nuclei, each of which is composed of positively charged **protons** and chargeless **neutrons**. While each nucleus has a positive radius, this radius (about 10^{-13} cm) is so small compared to any length we shall be considering that it can be taken to be zero for our purposes. The simplest nucleus is the single proton (the nucleus of hydrogen) and it has charge $+e$. The number of protons in a nucleus is denoted by z and the values $z = 1, 2, \dots, 92$, except for $z = 43, 61, 85$, are found in nature. Some of these nuclei, e.g., all $84 \leq z \leq 92$, are unstable (i.e., they eventually break apart spontaneously) and we see this instability as naturally occurring radioactivity, e.g., radium. Nuclei with the missing z values 43, 61, 85, as well as those with $92 < z \leq 109$ have all been produced artificially, but they decay more or less quickly [AM]. Thus, the charge of a naturally occurring nucleus can be $+e$ up to $+92e$ (except for 43, 61, 85), but, as mathematicians often do, it is interesting to ask questions about "the asymptotics as $z \to \infty$" of some problems. Moreover, in almost all cases we shall consider here, the physical constraint that z is an integer need not and will not be imposed. The other constituent—the neutrons—will be of no importance to us until we come to stars. They merely add to the mass of the nucleus, for they are electrically neutral. For each given z several possible neutron numbers actually occur in nature; these different nuclei with a common z are called isotopes of each other. For example, when $z = 1$ we have the **hydrogen nucleus** (1 proton) and the deuterium nucleus (1 proton and 1 neutron)

Bull. Amer. Math. Soc. *22*, 1-49 (1990)

which occur naturally, and the tritium nucleus (1 proton and 2 neutrons) which is artificial and decays spontaneously in about 12 years into a helium nucleus and an electron, but which is important for hydrogen bombs. Isolated neutrons are also not seen naturally, for they decay in about 13 minutes into a proton and an electron.

Finally, the **nuclear mass**, M, has to be mentioned. It satisfies $zM_p \leq M \leq 3zM_p$ where $M_p = 1837m$ is the mass of a proton. Since the nuclear mass is huge compared to the electron mass, m, it can be considered to be infinite for most purposes, i.e., the nuclei can be regarded as fixed points in $\mathbf{R}^3$, although the location of these points will eventually be determined by the requirement that the total energy of the electron-nucleus system is minimized. A similar approximation is usually made when one considers the solar system; to calculate the motion of the planets the sun can be regarded as fixed.

The forces between these constituents of matter (electrons and nuclei) is given by **Coulomb's inverse square law** of electrostatics: If two particles have charges q_1 and q_2 and locations x_1 and x_2 in $\mathbf{R}^3$ then F_1—the force on the first due to the second—is minus F_2—the force on the second due to the first—and is given by

$$-F_2 = F_1 = q_1 q_2 \frac{(x_1 - x_2)}{|x_1 - x_2|^3}. \tag{1.1}$$

(Later on, when stars are discussed, the gravitational force will have to be introduced.) If $q_1 q_2 < 0$ then the force is **attractive**; otherwise it is **repulsive**. This force can also be written as minus the gradient (denoted by ∇) of a **potential energy function**

$$W(x_1, x_2) = q_1 q_2 \frac{1}{|x_1 - x_2|}, \tag{1.2}$$

that is

$$F_1 = -\nabla_1 W \quad \text{and} \quad F_2 = -\nabla_2 W. \tag{1.3}$$

If there are N electrons located at $\underline{X} = (x_1, \dots, x_N)$ with $x_i \in \mathbf{R}^3$, and k nuclei with positive charges $\underline{Z} = (z_1, \dots, z_k)$ and located at $\underline{R} = (R_1, \dots, R_k)$ with $R_i \in \mathbf{R}^3$, the **total-potential energy function** is then

$$W(\underline{X}) = -A(\underline{X}) + B(\underline{X}) + U \tag{1.4}$$

with

$$A(\underline{X}) = e^2 \sum_{i=1}^{N} V(x_i) \tag{1.5}$$

$$V(x) = \sum_{j=1}^{k} z_j |x - R_j|^{-1} \tag{1.6}$$

$$B(\underline{X}) = e^2 \sum_{1 \le i < j \le N} |x_i - x_j|^{-1} \tag{1.7}$$

$$U = e^2 \sum_{1 \le i < j \le k} z_i z_j |R_i - R_j|^{-1}. \tag{1.8}$$

The A term is the electron-nucleus attractive potential energy, with $eV(x)$ being the electric potential of the nuclei. B is the electron-electron repulsive energy and U is the repulsive energy of the nuclei. A, B, U and V depend on $\underline{R}$ and $\underline{Z}$, which are fixed and therefore do not appear explicitly in the notation. It is then the case that the force on the i th particle is

$$F_i = -\nabla_i W. \tag{1.9}$$

In the case of an **atom**, $k = 1$ by definition. The case $k > 1$ will be called the **molecular** case, but it includes not only the molecules of the chemist but also solids, which are really only huge molecules.

So far this is just classical electrostatics and we turn next to classical dynamics. Newton's law of motion is (with a dot denoting $\frac{d}{dt}$, where t is the time)

$$m\ddot{x}_i = F_i. \tag{1.10}$$

This law of motion, which is a system of second order differential equations, is equivalent to the following system of first order equations. Introduce the **Hamiltonian function** which is the function on the **phase space** $\mathbf{R}^{6N} = (\mathbf{R}^3 \times \mathbf{R}^3)^N$ given by

$$H(\underline{P}, \underline{X}) = \frac{1}{2m} \sum_{i=1}^{N} p_i^2 + W(\underline{X}). \tag{1.11}$$

The notation $\underline{P} = (p_1, \dots, p_N)$ with p_i in $\mathbf{R}^3$ is used, and the quantity

$$T = \frac{1}{2m} \sum_{i=1}^{N} p_i^2 \tag{1.12}$$

Bull. Amer. Math. Soc. *22*, 1-49 (1990)

is called the **kinetic energy**. The equations of motion (1.10) are equivalent to the following first order system in $\mathbf{R}^{6N}$

$$(1.13) \qquad \begin{aligned} v_i \equiv \dot{x}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial x_i}. \end{aligned}$$

The **velocity** of the ith electron is v_i and p_i is called its **momentum**: $p_i = mv_i$ by the first equation in (1.13).

From (1.13) it will be seen that $H(\underline{P}, \underline{X})$ is constant throughout the motion, i.e., $dH(\underline{P}(t), \underline{X}(t))/dt = 0$. This fixed number is called the **energy** and is denoted by E; it depends, of course, on the trajectory, and it is important to note that it can take all values in $(-\infty, \infty)$.

Another interesting fact about the flow defined by (1.13), but one which will not be important for us, is that it preserves Lebesgue measure $dx_1 \cdots dx_N dp_1 \cdots dp_N$ on $\mathbf{R}^{6N}$; this is Liouville's theorem and it follows from the fact that the vector field that defines the flow, $(\partial H/\partial p_1, \ldots, \partial H/\partial p_N, -\partial H/\partial x_1, \ldots, -\partial H/\partial x_N)$, is divergence free. This theorem is one important reason for introducing the Hamiltonian formalism, for it permits a geometric interpretation of classical mechanics and is crucial for ergodic theory and statistical mechanics. The analogue in quantum mechanics turns out to be that quantum mechanical time evolution is given by a one parameter *unitary* group in Hilbert space (see (2.18))—but time evolution will not concern us here.

Consider the simplest possible case, neutral *hydrogen*, with $z = 1$ (a proton) and one electron ($N = 1$ and $k = 1$). With the proton fixed at the origin (i.e., $R_1 = 0$) the Hamiltonian is $p^2/2m - Ze^2|x|^{-1}$ and classical bound orbits (i.e., orbits which do not escape to infinity) of the electron are well known to be the ellipses of Kepler with the origin as a focus. These can pass as close as we please to the proton. Indeed, in the degenerate case the orbit is a radial line segment and in such an orbit the electron passes through the nucleus. One measure of average closeness of the electron to the nucleus in an orbit is the energy E, which is always negative for a bound orbit. Moreover E can be arbitrarily negative because the electron can be arbitrarily close to the nucleus and also have arbitrarily small kinetic energy T. A consequence of this fact is that the hydrogen atom would be physically unstable; in a gas of many atoms another particle or atom could collide with our atom and absorb energy from it. After many such collisions our electron could find itself in a tiny orbit around the nucleus and our atom would no longer be recognizable as an object whose radius is supposed to be 10^{-8} cm. Each atom would be an infinite source of energy which could be transmitted to other atoms or to radiation of electromagnetic waves.

The problem was nicely summarized by Jeans [J] in his 1915 textbook.

> "There would be a very real difficulty in supposing that the (force) law $1/r^2$ held down to zero values of r. For the force between two charges at zero distance would be infinite; we should have charges of opposite sign continually rushing together and, when once together, no force would be adequate to separate them... Thus the matter in the universe would tend to shrink into nothing or to diminish indefinitely in size."

The inability to account for stable atoms in terms of classical trajectories of pointlike charged particles was the major problem of prequantum physics. Since the existence of atoms and molecules was largely inferential in those days (nowadays we can actually "see" atoms with the tunneling electron microscope), the inability to account for their structure even led some serious people to question their existence—or at least to question the nice pictures drawn by chemists. The main contribution of quantum mechanics was to provide a quantitative theory that "explains" why the electron cannot fall into the nucleus. In brief, when the electron is close to the nucleus its kinetic energy—which could be zero classically—is forced to increase in such a way that the total energy (1.11) goes to $+\infty$ as the average distance $|x|$ goes to zero. This property is known as the **uncertainty principle**.

PART II. QUANTUM MECHANICS IN A NUTSHELL

Schrödinger's answer to the problem of classical mechanics was the following. While an electron is truly a point particle, its state at any given time cannot be described by a point $x \in \mathbf{R}^3$ and a momentum $p \in \mathbf{R}^3$ (or velocity $v = \frac{1}{m}p$) as in the classical view. Instead *the state of an electron is a (complex valued) function* ψ *in* $L^2(\mathbf{R}^3)$. Any ψ will do provided it satisfies the normalization condition

$$\|\psi\|_2^2 = \int_{\mathbf{R}^3} |\psi(x)|^2\, dx = 1. \tag{2.1}$$

(Actually, this statement is not accurate; an electron has a property called **spin**, and the mathematical expression of this fact is