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R.V. Gamkrelidze V.L. Popov

E. A. Tevelev

Projective Duality and Homogeneous Spaces

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Author

Evgueni A. Tevelev
Department of Mathematics
University of Texas at Austin
78712 Austin, Texas, USA
e-mail: tevelev@math.utexas.edu

Founding editor of the Encyclopaedia of Mathematical Sciences:
R. V. Gamkrelidze

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Preface

During several centuries various reincarnations of projective duality have inspired research in algebraic and differential geometry, classical mechanics, invariant theory, combinatorics, etc. To put it simply, projective duality is the systematic way of recovering a projective variety from the set of its tangent hyperplanes. In this survey I have tried to emphasize that there are many different aspects of projective duality and that it can be studied using a wide range of methods. But at the same time I was pushing hard to minimize the technical details in the hope of writing a text that requires a knowledge of only basic facts from algebraic geometry and the theory of algebraic (or Lie) groups. Most proofs in this book are compilations from various sources.

Projective duality is defined for arbitrary projective varieties and it does not seem natural a priori to consider varieties with symmetries. However, it turns out that many important examples carry the natural action of the algebraic group. This is especially true for projective varieties that have extremal projective properties: self-dual varieties, varieties of positive defect, Severi varieties, Scorza varieties, varieties of small codegree, etc. I have tried to emphasize this phenomenon in this survey. However, one aspect is totally omitted – I decided against including results about dual varieties of toric varieties and A-discriminants. This theory is presented in the beautiful book by Gelfand, Zelevinsky, and Kapranov [GKZ2] (the author always keeps it under his pillow) and I don't feel I have anything to add to it.

Let us say a few words about the contents of this survey.

Chapter 1 is intended as a brief introduction to projective duality. All results here were already well-known in the 19th century. After giving basic definitions in Sect. 1.1 we discuss duality for plane curves in Sect. 1.2. We give parametric equations for dual curves, discuss the connection to the Legendre transform, introduce Plücker formulas, and study curves of degree 2, 3, and 4. In Sect. 1.3 we prove the Reflexivity Theorem. Here we follow the exposition in [GKZ2] and use the interpretation of a conormal variety as a Lagrangian subvariety in the cotangent bundle. We introduce the discriminant as a defining equation for the dual variety and the defect that measures how far the dual variety is from being a hypersurface. We prove that if the defect is positive then the variety is ruled. Finally, in Sect. 1.4 we study the behaviour of a dual variety under projections and introduce the standard notation of algebraic geometry related to divisors and line bundles.

In our survey we are mostly interested in the study of varieties with symmetries, and in Chap. 2 we study projective geometry of homogeneous spaces; more precisely we look at orbit closures for algebraic groups acting on projective spaces with finitely many orbits. In Sect. 2.1 we give the necessary background on algebraic groups and fix notation. In Sect. 2.2 we discuss Pyasetskii pairing, which is an interesting instance of projective duality. We give some standard and a few exotic examples. The next Sect. 2.3 contains the systematic treatment of actions related to gradings of simple Lie algebras. These actions provide a wealth of very important varieties that will be studied throughout this book. Some examples include Severi varieties, smooth self-dual varieties, smooth building blocks for varieties of positive defect, varieties of small codegree, etc. It is quite an interesting phenomenon that varieties with extremal projective properties tend to have maximal symmetries. We finish this chapter with the description of Pyasetskii pairing for actions related to gradings of GL_n called the Zelevinsky involution.

In Chap. 3 we study projectively dual varieties using calculations in local coordinates. In Sect. 3.1 we prove a formula due to Katz that expresses the dimension of a dual variety in terms of the hessian of local equations of a variety. We use it to prove a formula of Weyman and Zelevinsky that expresses the defect of a Segre embedding of a product of two varieties. In Sect. 3.2 we introduce a gadget called the second fundamental form that incorporates these calculations. We prove some results of Griffiths and Harris about the second fundamental form. We finish this section with the description of higher fundamental forms of flag varieties obtained by Landsberg.

In Chap. 4 we study projective constructions related to projective duality but also having a merit of their own. We prove a theorem of Zak and Ran that the Gauss map of a smooth variety is a normalization. We introduce secant and tangent varieties in Sect. 4.2, prove the Terracini Lemma, give a method for the calculation of multisecant varieties of homogeneous spaces, discuss the relationship discovered by Zak between the degree of a dual variety and the order of the variety, and give an overview of old and new results related to the Waring problem for forms. In Sect. 4.3 we discuss theorems of Zak related to the Hartshorne conjecture – theorems on tangencies, on linear normality and on Severi varieties. The main tool is a connectedness theorem of Fulton and Hansen. We finish in Sect. 4.4 by explaining the Cayley trick for Chow forms.

The dual variety is the image of the conormal variety which is the projectivized conormal bundle if the variety is smooth. In Chap. 5 we exploit this relation between duality and vector bundles. In Sect. 5.1 we prove a theorem of Holme and Ein about the defect of a smooth effective very ample divisor. We deduce this result from a theorem of Munoz about the dimension of the linear span of a tangential variety. We discuss the related notion of projective extendability. In Sect. 5.2 we apply Hartshorne's ample vector bundles to prove a theorem of Ein that a dual variety of a smooth complete intersection is a hypersurface. We introduce resultants and prove a classical theorem that they are well-defined. We also explain the Cayley trick for resultants. In the last Sect. 5.3 we describe the "Cayley method" developed by Gelfand, Kapranov, and Zelevinsky. The idea is to show that the dual variety is represented in the derived category by Koszul complexes of jet bundles. The discriminant is then equal to the "Cayley determinant" of a generically exact complex. As an application we deduce some classical formulas for discriminants and their degrees.

In Chap. 6 we discuss about the degree of dual varieties and resultants. We start in Sect. 6.1 by recalling Chern classes and then proving a formula of Katz, Kleiman, and Holme that expresses the degree of a dual variety in terms of Chern classes of the cotangent bundle. We give many examples and generalizations. We prove a theorem of De Concini and Weyman about the formula for the degree with non-negative coefficients. In Sect. 6.2 we discuss formulas for the codegree and ranks related to the Cayley method, such as a formula due to Lascoux.

In Chap. 7 we study varieties with positive defect. In Sect. 7.1 we focus on beautiful theorems of Ein about the normal bundle of a generic contact locus. Since this locus is a projective subspace, it is possible to use the machinery of vector bundles on projective spaces. We prove Ein's theorem that this normal bundle is symmetric and uniform, which explains among other things a parity theorem of Landman. We introduce the Beilinson spectral sequence and use it to calculate the normal bundle to a generic contact locus in small dimensions. Finally, we study dual varieties of scrolls and prove a theorem of Ein that a variety of defect at least 2 is a scroll if and only if the normal bundle to a generic contact locus splits. In Sect. 7.2 we follow [IL] and discuss linear systems of quadrics of constant rank and how they are related to dual varieties via the second fundamental form. In Sect. 7.3 we prove a theorem of Beltrametti, Fania, and Sommese that relates the defect of a projective variety and its Mori-theoretic characteristic called the nef value. We give a brief survey of necessary results from Mori theory. We finish by giving a classification of smooth varieties of positive defect up to dimension 10 obtained by many authors and initiated by Ein. Finally, in Sect. 7.4 we use this connection with Mori theory to classify all flag varieties with positive defect. This approach was developed by Snow in contrast with the original proof of Knop and Menzel that used the Katz dimension formula.

In Chap. 8 we study dual varieties and discriminants of several special homogeneous spaces. We start in Sect. 8.1 by showing how to use standard results of representation theory such as the Borel–Weil–Bott theorem, the BGG homomorphism, identities with Schur functors, and formulas of the Schubert calculus to find the codegree of Grassmannians or full and partial flag varieties. We give a list of formulas for the degree of hyperdeterminants and sketch the proof of a theorem of Zak about varieties of codegree 3. In Sect. 8.2 we generalize the theorem of Matsumura and Monsky about automorphisms of smooth hypersurfaces to automorphisms of smooth very ample divisors on flag varieties. In Sect. 8.3 we study commutative algebras without identities from the "discriminantal" point of view. As a corollary we prove that the algebra of diagonal matrices does not have quasiderivations. In Sect. 8.4 we study anticommutative algebras (nets of skew-symmetric forms). We show that they have beautiful geometric properties related to cubic surfaces, Del Pezzo surfaces, representation theory of S_5, etc. In Sect. 8.5 we show that the discriminant in a simple Lie algebra defined by analogy with the disciminant of a linear operator is equal to the discriminant of the minimal orbit, the so-called adjoint variety. Finally, in Sect. 8.6 we study related questions about schemes of zeros of irreducible homogeneous vector bundles. In particular, we address a question of classifying irreducible homogeneous vector bundles with a trivial line subbundle, find the maximal dimension of an isotropic subspace of a generic symmetric or skew-symmetric form, and study properties of the related Moore–Penrose involution.

In Chap. 9 we study self-dual varieties, i.e. varieties isomorphic to their projectively dual variety. In Sect. 9.1 we consider smooth self-dual varieties. The complete list of these varieties is (conjecturally) surprisingly short. All known varieties are flag varieties so we start by considering this case, where everything follows from the classification of flag varieties of positive defect. After a brief introduction to the Hartshorne conjecture we sketch the proof of the amazing theorem of Ein that gives the complete list of self-dual varieties in the range that is allowed by the Hartshorne conjecture. We also prove a finiteness theorem of Muñoz that uses the distribution of primes to give restrictions on the Beilinson spectral sequence. We finish in Sect. 9.2 by describing results of Popov about self-dual nilpotent orbits.

In the final Chap. 10 we study how the topology of the variety is reflected in singularities of the dual variety. We start in Sect. 10.1 by proving the class formula and its variant due to Landman that relates the degree of the dual variety and the Euler characteristic of the variety and its hyperplane sections. In the singular case this formula was proved by Ernström, but the Euler characteristic has to be substituted by the degree of the Chern–Mather class. In Sect. 10.2 we prove theorems of Dimca, Nemethi, Aluffi and others that multiplicities of the dual variety are given by Milnor numbers (or classes). To give an example we follow Aluffi and Cukierman and calculate multiplicities of the dual variety to a smooth surface. Finally, we give some results of Weyman and Zelevinsky about singularities of hyperdeterminants.

I would like to thank F. Zak for very inspiring discussions of projective geometry and R. Muñoz for his insights about positive defect varieties. I am grateful to S. Keel, V. Popov, A. Kuznetsov, P. Katsylo, D. Saltman, E. Vinberg for many helpful remarks and encouragement. P. Aluffi, F. Cukierman, N.C. Leung, A.J. Sommese and many others sent me lots of comments about the preliminary version of this survey.

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Moscow–Edinburgh–Austin, *Jenia Tevelev*
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Contents

1

Introduction to Projective Duality

This chapter is intended as a brief introduction to projective duality. All results here were already well-known in the 19th century. After giving basic definitions in Sect. 1.1 we discuss duality for plane curves in Sect. 1.2. We give parametric equations for dual curves, discuss the connection to the Legendre transform, introduce Plücker formulas, and study curves of degree 2, 3, and 4. In Sect. 1.3 we prove the Reflexivity Theorem. Here we follow the exposition in [GKZ2] and use the interpretation of a conormal variety as a Lagrangian subvariety in the cotangent bundle. We introduce the discriminant as a defining equation for the dual variety and the defect that measures how far the dual variety is from being a hypersurface. We prove that if the defect is positive then the variety is ruled. Finally, in Sect. 1.4 we study the behaviour of a dual variety under projections and introduce the standard notation of algebraic geometry related to divisors and line bundles.

1.1 Projectively Dual Varieties

For any finite-dimensional complex vector space V we denote by $\mathbb{P}(V)$ its projectivization (the set of 1-dimensional subspaces). If $V = \mathbb{C}^{N+1}$ then $\mathbb{P}^N = \mathbb{P}(\mathbb{C}^{N+1})$ is the standard complex projective space.

Let $V^\vee$ be the dual vector space of linear forms on V. Points of the dual projective space $\mathbb{P}(V)^\vee = \mathbb{P}(V^\vee)$ correspond to hyperplanes in $\mathbb{P}(V)$. Conversely, to any point p of $\mathbb{P}(V)$, we can associate a hyperplane in $\mathbb{P}(V)^\vee$, namely the set of all hyperplanes in $\mathbb{P}(V)$ passing through p. Therefore $\mathbb{P}(V)^{\vee\vee}$ is naturally identified with $\mathbb{P}(V)$. More generally, there exists the projective duality between projective subspaces in $\mathbb{P}(V)$ and $\mathbb{P}(V)^\vee$: for any $L \subset \mathbb{P}(V)$, its *projectively dual subspace* $L^* \subset \mathbb{P}(V)^\vee$ parametrizes all hyperplanes that contain L.

Quite remarkably, this projective duality extends to the involutive correspondence between arbitrary subvarieties in $\mathbb{P}^N$ and $(\mathbb{P}^N)^\vee$.

Let $X \subset \mathbb{P}^N$ be an irreducible projective variety with $\dim X = n$. We denote by $T_{x,X}$ the tangent space at any smooth point $x \in X_{sm}$. We can also define the *embedded projective tangent space* $\hat{T}_{x,X} \subset \mathbb{P}^N$ as follows:

$$\hat{T}_{x,X} = \mathbb{P}(T_{v,\mathrm{Cone}(X)}),$$

where $\mathrm{Cone}(X) \subset V$ is the cone over X, v is any non-zero point on the line x, and we consider $T_{v,\mathrm{Cone}(X)}$ as a linear subspace of V (it does not depend on the choice of v).

Definition 1.1 A hyperplane $H \subset \mathbb{P}^N$ is a *tangent hyperplane* of X if $\hat{T}_{x,X} \subset H$ for some $x \in X_{sm}$. The closure of the set of all tangent hyperplanes is called the *projectively dual variety* $X^* \subset (\mathbb{P}^N)^\vee$.

We shall prove later that

Theorem 1.2 (Reflexivity Theorem) $X^{**} = X$.

1.2 Dual Plane Curves

The most classical example of a dual variety is the dual curve C^* of a plane curve $C \subset \mathbb{P}^2$. By definition, generic points of C^* are the tangents to C at smooth points. In this case the Reflexivity Theorem has a simple meaning: the tangent line $\hat{T}_p$ at a smooth point $p \in C$ is the limit of secants $\overline{pq}$ for $q \in C$ as $q \to p$. The point in $\mathbb{P}^2$ that corresponds to the tangent to $C^* \subset (\mathbb{P}^2)^\vee$ at a non-singular point $\hat{T}_p$ is the limit of the intersection points of the tangents $\hat{T}_p$ and $\hat{T}_q$ as $q \to p$. Of course, this point is p.

1.2.1 Parametric Equations

It is easy to write down a parametric representation of the dual curve C^* using a given parametric representation of C. Let x, y, z be homogeneous coordinates on $\mathbb{P}^2$ and p, q, r the dual homogeneous coordinates on $(\mathbb{P}^2)^\vee$. We choose an affine chart $\mathbb{C}^2 = \{z \neq 0\} \subset \mathbb{P}^2$ with affine coordinates x, y (so we set z to be 1). The dual chart $(\mathbb{C}^2)^\vee \subset (\mathbb{P}^2)^\vee$ with coordinates p, q is obtained by setting the coordinate r of $(\mathbb{P}^2)^\vee$ to be -1. Then $(\mathbb{C}^2)^\vee$ consists of all lines in $\mathbb{P}^2$ that do not pass through the origin $(0,0) \in \mathbb{C}^2 \subset \mathbb{P}^2$. Every such line either has an affine equation $px + qy = 1$ or is the line "at infinity" with coordinates $p = q = 0$. Suppose that a local parametric equation of C has the form $x = x(t)$, $y = y(t)$, where t is a local parameter on C, and $x(t)$, $y(t)$ are analytic functions. The dual curve C^* is parametrized by $p = p(t)$, $q = q(t)$, where $p(t)x + q(t)y = 1$ is the tangent line to C at $(x(t), y(t))$. It follows that

$$p(t) = \frac{-y'(t)}{x'(t)y(t) - x(t)y'(t)}, \quad q(t) = \frac{x'(t)}{x'(t)y(t) - x(t)y'(t)}.$$

Applying this formula twice one can prove the Reflexivity Theorem.

1.2.2 Legendre Transformation

Projective duality and the Legendre transformation of classical mechanics are closely related to each other. Let us recall this classical definition in the case of real functions in one variable. Details and generalizations can be found in [Ar].

Definition 1.3 Let $y = f(x)$ be a smooth convex real function, $f''(x) > 0$. The *Legendre transformation* of f is a function $g(p)$ defined as follows. Let $x = x(p)$ be a point at which the graph $y = f(x)$ has slope p. Then

$$g(p) = px(p) - f(x(p)).$$

Equivalently, $x(p)$ is the unique point where the function $F(p,x) = px - f(x)$ has a maximum with respect to x and $g(p) = F(p, x(p))$.

The Legendre transformation is easily seen to be involutive. To link it with projective duality, we need caustic curves. Let us express projective duality entirely in terms of the projective plane $\mathbb{P}^2$. A tangent line to a curve C at some point x is the line which is infinitesimally close to C near x. A point of $(\mathbb{P}^2)^\vee$ is a line $l \subset \mathbb{P}^2$. A curve in $(\mathbb{P}^2)^\vee$ is therefore a 1-parameter family of lines in $\mathbb{P}^2$. For example, a line in $(\mathbb{P}^2)^\vee$ is a pencil x^* of all lines in $\mathbb{P}^2$ passing through a given point $x \in \mathbb{P}^2$. The dual curve C^* is a 1-parameter family of tangent lines to C. Now let us take an arbitrary curve $C' \subset (\mathbb{P}^2)^\vee$ (a 1-parameter family of lines in $\mathbb{P}^2$) and then find a geometric interpretation of the dual curve $C'^* \subset \mathbb{P}^2$. Take some line $l \in C'$. The condition that x^* is tangent to C' at l means that l is a member of a family C' and other lines from C' near l are infinitesimally close to the pencil of lines x^*. This is usually expressed by saying that x is a *caustic point* for the family of lines C'. The set of all caustic points of C' is called the *caustic curve* of C'. This is nothing else but the projectively dual curve C'^*.

By the Reflexivity Theorem, any curve $C \subset \mathbb{P}^2$ is the caustic curve of the family of its tangent lines. Another consequence is that any 1-parameter family of lines in $\mathbb{P}^2$ is a family of tangent lines to some curve C.

The caustic could be found (locally) using the Legendre transformation:

Theorem 1.4 *Consider a family of lines $y = px - g(p)$. Its caustic curve has an equation $y = f(x)$, where f is the Legendre transformation of g.*

Proof. The tangent line to $y = f(x)$ at a point $x(p)$ with slope p is equal to

$$y = p(x - x(p)) + f(x(p)) = px - g(p).$$

It remains to use the involutivity of the Legendre transformation. $\square$

1.2.3 Plücker Formulas

Even if a plane curve $C \subset \mathbb{P}^2$ is smooth, $C^* \subset ((\mathbb{P}^2)^\vee$ is almost always singular. There is a natural map

$$C \to C^*, \quad p \mapsto \hat{T}_p.$$

This map is a resolution of singularities. Moreover, even if C is not smooth, the curves C and C^* are birationally equivalent, in particular they have the same geometric genus g. Indeed, consider the conormal variety

$$I_C \subset \mathbb{P}^2 \times (\mathbb{P}^2)^\vee, \quad I_C = \overline{\{(p,l)\,|\,p \in C_{sm},\ l \in C^*{}_{sm},\ l = \hat{T}_p\}}.$$

Obviously, I_C projects birationally on C. Therefore, by the Reflexivity Theorem, I_C also projects birationally on C^*. So C and C^* are birationally equivalent. If C is smooth then, of course, $I_C = C$.

A line l tangent to C in at least two points is a singular point of C^*. It is known as a *multiple tangent*. If l has exactly two tangency points on C and the intersection multiplicity at each of them is 2, then l is a *bitangent*. A bitangent corresponds to an ordinary double point (node) of C^*.

If $l = \hat{T}_p$ intersects C at p with multiplicity $m \geq 3$, it is again a singular point of C^*. If $m = 3$, and l is not tangent to C at any other point, then p is called an *inflection point* (or flex) of C. l is a cuspidal point (or cusp) of C^*.

Now we can introduce a class of "generic" curves with singularities, which is preserved by the projective duality. Namely, we say that C is generic if both C and C^* have only double points and cusps as their singularities. Suppose that C is generic in this sense. Let d, g, κ, δ, b, f be the degree, the geometric genus, the number of cusps, the number of double points, the number of bitangents, and the number of flexes of C. Let d^*, g^*, κ^*, δ^*, b^*, f^* be the corresponding numbers for C^* (d^* is also called the *class* of C). Then by the Reflexivity Theorem we have the following

Proposition 1.5 $g = g^*$, $\kappa = f^*$, $\delta = b^*$, $b = \delta^*$, $f = \kappa^*$.

It turns out that there is another remarkable set of equations linking these numbers. It was discovered by Plücker and Clebsch. The proof can be found in [GH].

Theorem 1.6

$$g = \frac{1}{2}(d^* - 1)(d^* - 2) - b - f,$$

$$g = \frac{1}{2}(d - 1)(d - 2) - \delta - \kappa,$$

$$d = d^*(d^* - 1) - 2b - 3f,$$

$$d^* = d(d - 1) - 2\delta - 3\kappa.$$

1.2.4 Curves of Small Degree

Conics

Let $C \subset \mathbb{P}^2$ be a smooth conic given by the equation

$$(Ax, x) = \sum_{i,j=1}^{3} a_{ij} x_i x_j = 0,$$

where $A = (a_{ij})$ is a non-degenerate symmetric 3×3 matrix. The tangent line to C at $x_0 \in C$ is given by $(Ax_0, x) = 0$. Hence the point $\xi \in (\mathbb{P}^2)^\vee$ corresponding to this tangent line has homogeneous coordinates Ax_0, which implies that $(A^{-1}\xi, \xi) = 0$. Therefore $C^* \subset (\mathbb{P}^2)^\vee$ is also a smooth conic defined by A^{-1}. Conics are the only smooth plane curves having smooth duals.

Cubics

Let $C \subset \mathbb{P}^2$ be a smooth cubic curve. By the Bézout Theorem, C^* has only flexes as singularities and the Plücker formulas are always applicable. Hence C has no bitangents and exactly 9 flexes. The dual curve C^* is a very special curve of degree 6 with 9 cusps and no double points. A beautiful determinantal formula for it was found by Schläfli. Let x_1, x_2, x_3 be homogeneous coordinates in $\mathbb{P}^2$ and p_1, p_2, p_3 the dual coordinates in $(\mathbb{P}^2)^\vee$. Let $f(x_1, x_2, x_3) = 0$ be the homogeneous equation of C and $F(p_1, p_2, p_3) = 0$ the homogeneous equation of C^*. Consider the polynomial

$$V(p, x) = \begin{vmatrix} 0 & p_1 & p_2 & p_3 \\ p_1 & \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ p_2 & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ p_3 & \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3 \partial x_3} \end{vmatrix}.$$

Clearly $V(p, x)$ has degree 2 in x. Then Schläfli's formula is as follows:

$$F(p_1, p_2, p_3) = \begin{vmatrix} 0 & p_1 & p_2 & p_3 \\ p_1 & \frac{\partial^2 V}{\partial x_1 \partial x_1} & \frac{\partial^2 V}{\partial x_1 \partial x_2} & \frac{\partial^2 V}{\partial x_1 \partial x_3} \\ p_2 & \frac{\partial^2 V}{\partial x_2 \partial x_1} & \frac{\partial^2 V}{\partial x_2 \partial x_2} & \frac{\partial^2 V}{\partial x_2 \partial x_3} \\ p_3 & \frac{\partial^2 V}{\partial x_3 \partial x_1} & \frac{\partial^2 V}{\partial x_3 \partial x_2} & \frac{\partial^2 V}{\partial x_3 \partial x_3} \end{vmatrix}.$$

The proof can be found in [GKZ2].

Quartics

Let $C \subset \mathbb{P}^2$ be a generic smooth quartic curve. By the Plücker formulas, C has genus 3, 24 flexes, and 28 bitangents. There are two visual descriptions of these bitangents. The first classical approach is to realize the quartic curve as a 'shade' of a generic cubic surface $S \subset \mathbb{P}^3$. Namely, suppose that $x \in S$ is a generic point. Consider the projective plane $\mathbb{P}^2$ of lines in $\mathbb{P}^3$ passing through x. Then lines $l \in \mathbb{P}^2$ that are tangent to S form a quartic curve C. The 28 bitangents are given by projections of 27 lines on S plus one extra line, the projection of $\hat{T}_{x,S}$.

Another classical description is less well-known: Consider a generic 3-dimensional linear system L of quadrics in $\mathbb{P}^3$. Singular members of this linear system give rise to a quartic curve in $\mathbb{P}(L) = \mathbb{P}^2$. It can be shown that any generic quartic curve can be obtained this way. By the Bézout Theorem, L has $2^3 = 8$ base points in $\mathbb{P}^3$. For any two base points p, q let $l_{pq} \subset \mathbb{P}(L)$ be the line formed by all quadrics that contain not only points p and q, but also the line $[pq]$ connecting them. Thus we have 28 lines $l_{pq} \subset \mathbb{P}(L)$. Let us show that these lines are bitangents to C.

Fix a basis $\{e_1, e_2, e_3, e_4\}$ in $\mathbb{C}^4$ such that e_1 belongs to p and e_2 belongs to q. Then quadrics from l_{pq} have the form

$$G = \begin{pmatrix} 0 & A^T \\ A & B \end{pmatrix},$$

and therefore

$$\det G = (\det A)^2.$$

This means exactly that l_{pq} intersects C in two double points, i.e. l_{pq} is a bitangent to C.

1.3 Reflexivity Theorem

Now let us formulate the general version of the Reflexivity Theorem.

Theorem 1.7

(i) *For any irreducible projective variety* $X \subset \mathbb{P}^N$, *we have* $X^{**} = X$.
(ii) *More precisely, if* $z \in X_{sm}$ *and* $H \in X^*{}_{sm}$, *then* H *is tangent to* X *at* z *iff* z, *regarded as a hyperplane in* $(\mathbb{P}^N)^\vee$, *is tangent to* X^* *at* H.

In the proof of the Reflexivity Theorem we follow [GKZ2] and deduce this result from the classical theorem of symplectic geometry. Other proofs, including the subtleties of a prime characteristic case, can be found in [Se], [M], [Wa].

1.3.1 Proof of the Relexivity Theorem

We start with some standard definitions.

Definition 1.8 If X is a smooth algebraic variety, then T_X denotes the tangent bundle of X. If $Y \subset X$ is a smooth algebraic subvariety, then T_Y is a subbundle in $T_X|_Y$. The quotient is called the normal bundle of Y in X, denoted by $N_{Y,X}$ (or simply N_Y if X is clear). By taking dual bundles we obtain the cotangent bundle $T_X^\vee$ and the conormal bundle $N_{Y,X}^\vee$. The conormal bundle can be naturally regarded as a subvariety of $T_X^\vee$.

Suppose that $X \subset \mathbb{P}^N$ is an irreducible subvariety.

Definition 1.9 Consider the set $I_X^0 \subset \mathbb{P}^N \times (\mathbb{P}^N)^\vee$ of pairs (x, H) such that $x \in X_{sm}$ and H is the hyperplane tangent to X at x. Let I_X be the Zariski closure of I_X^0. Then I_X is called the *conormal variety* of X.

The first projection $\mathrm{pr}_1 : I_X^0 \to X_{sm}$ makes I_X^0 a projective bundle over X_{sm} whose fibers are projective subspaces of dimension $N - n - 1$. Therefore I_X is an irreducible variety of dimension $N - 1$. The dual variety X^* is the image of the second projection $\mathrm{pr}_2 : I_X \to (\mathbb{P}^N)^\vee$. Hence X^* is an irreducible variety. Let us show that $I_X^0 = \mathbb{P}(N_{X_{sm},\mathbb{P}^N}^\vee)$. Indeed, the choice of a hyperplane $H \subset \mathbb{P}^N$ tangent to X_{sm} at x is equivalent to the choice of a hyperplane $T_{x,H}$ in the tangent space $T_{x,\mathbb{P}^N}$, which contains $T_{x,X}$. The equation of this hyperplane is an element of $N_{X_{sm},\mathbb{P}^N}^\vee$ at x.

The Reflexivity Theorem can be reformulated as follows:

$$I_X = I_{X^*}. \tag{1.1}$$

Let $\mathbb{P}^N = \mathbb{P}(V)$ and $(\mathbb{P}^N)^\vee = \mathbb{P}(V^\vee)$. We take

$$Y = \mathrm{Cone}(X) \subset V, \quad Y^* = \mathrm{Cone}(X^*) \subset V^\vee.$$

Denote by $\mathrm{Lag}(Y)$ the closure of $N_{Y_{sm},V}^\vee$ in $T_V^\vee$ (we are about to show that $\mathrm{Lag}(Y)$ is a Lagrangian subvariety).

$T_V^\vee$ is canonically identified with $V \times V^\vee$. Denote by $\mathrm{pr}_1, \mathrm{pr}_2$ the projections of this product to its factors. Then Y^* coincides with $\mathrm{pr}_2(\mathrm{Lag}(Y))$. Therefore, (1.1) can be reformulated as follows:

$$\mathrm{Lag}(Y) = \mathrm{Lag}(Y^*), \tag{1.2}$$

where we identify $T_V^\vee$ and $T_{V^\vee}^\vee$ with $V \times V^\vee$.

Recall that a smooth algebraic variety M is called symplectic if it has a symplectic structure, i.e. a differential 2-form ω such that

- ω is closed, $d\omega = 0$.
- For any $p \in M$, the restriction of ω on $T_{p,M}$ is non-degenerate.

In this case $\dim M$ is necessarily even. An irreducible closed subvariety $\Lambda \subset M$ is called *Lagrangian* if $\dim \Lambda = \dim M/2$ and the restriction of ω to Λ_{sm} vanishes as a 2-form (is totally isotropic).

The cotangent bundle $T^\vee_X$ of a smooth algebraic variety X carries a canonical symplectic structure defined as follows. Let $(x_1, \ldots, x_n)$ be a local coordinate system in X. Let ξ_i be the fiberwise linear function on $T^\vee_X$ given by the pairing of a 1-form with the vector field $\partial/\partial x_i$. Then $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ forms a local coordinate system in $T^\vee_X$. The form ω is defined by

$$\omega = \sum_{i=1}^{n} d\xi_i \wedge dx_i. \tag{1.3}$$

It easy to give an equivalent definition of ω without any coordinate systems. Let $\pi : T^\vee_X \to X$ be the canonical projection. Let $p \in T^\vee_X$ and $v \in T_{p,T^\vee_X}$ be a vector tangent to $T^\vee_X$ at p. Then $\nu(v) = p(\pi_* v)$ is a canonical 1-form on $T^\vee_X$ and $\omega = d\nu$.

An example of a Lagrangian subvariety in $T^\vee_X$ can be obtained as follows. Let $Y \subset X$ be an irreducible subvariety and let

$$\mathrm{Lag}(Y) = \overline{N^\vee_{Y_{sm},X}}$$

be the closure in $T^\vee X$. Clearly, $\mathrm{Lag}(Y)$ is a conical subvariety (invariant under dilations of fibers of $T^\vee_X$).

Theorem 1.10

(i) $\mathrm{Lag}(Y)$ *is a Lagrangian subvariety.*
(ii) *Any conical Lagrangian subvariety has the form* $\mathrm{Lag}(Y)$ *for some irreducible subvariety* $Y \subset X$.

Proof. Let us show that $\mathrm{Lag}(Y)$ is Lagrangian. Clearly,

$$\dim \mathrm{Lag}(Y) = \dim Y + (\dim X - \dim Y) = \dim T^\vee_X/2.$$

So we need only verify that $\omega|_{\mathrm{Lag}(Y)} = 0$. It suffices to restrict to points over Y_{sm}. Let $x_1, \ldots, x_n$ be a local coordinate system on X such that Y is locally defined by equations $x_1 = \ldots = x_r = 0$. Then the fibers of the conormal bundle over points of Y are generated by 1-forms $dx_1, \ldots, dx_r$. Hence $\xi_{r+1} = \ldots = \xi_n = 0$ on $N^\vee_{Y_{sm},X}$ and by (1.3) we see that $\omega = 0$ on $N^\vee_{Y_{sm},X}$. Therefore $\mathrm{Lag}(Y)$ is indeed Lagrangian.

Suppose now that $\Lambda \subset T^\vee_X$ is a conical Lagrangian subvariety. We take $Y = \pi(\Lambda)$, where $\pi : T^\vee_X \to X$ is the projection, and claim that $\Lambda = \mathrm{Lag}(Y)$. It suffices to show that $\Lambda \subset \mathrm{Lag}(Y)$, because Λ and $\mathrm{Lag}(Y)$ are irreducible varieties of the same dimension. In turn, to prove that $\Lambda \subset \mathrm{Lag}(Y)$, it suffices to check that, for any $y \in Y_{sm}$, the fiber $\pi^{-1}(y) \cap \Lambda$ is contained in the conormal space $(N^\vee_{Y,X})_y$. Let ξ be any covector from $\pi^{-1}(y) \cap \Lambda$. Since $T^\vee_{y,X}$ is a vector space, we can regard ξ as a "vertical" tangent vector to $T^\vee_X$ at $y \in X \subset T^\vee_X$, where we identify X with the zero section of π. Since Λ is conical, $\xi \in T_{\xi,\Lambda}$. Since Λ is Lagrangian, ξ is orthogonal with respect to ω to any tangent vector from $T_{\xi,\Lambda}$ and, hence, to any tangent vector $v \in T_{y,Y}$. But by (1.3) it is easy to see that this is equivalent to $\xi \in (N^\vee_{Y,X})_y$. $\square$

Now we can prove the Reflexivity Theorem:

Proof of the Reflexivity Theorem

We are going to verify (1.2). The identification $T_V^\vee = V \times V^\vee = T_{V^\vee}^\vee$ preserves the canonical symplectic structure up to a sign. Therefore $\mathrm{Lag}(Y)$ is still Lagrangian as a subvariety of $T_{V^\vee}^\vee$. Moreover, since $\mathrm{Lag}(Y) \subset V \times V^\vee$ is invariant under dilations of V and $V^\vee$, $\mathrm{Lag}(Y)$ is a conical Lagrangian variety of $T_{V^\vee}^\vee$. Therefore, by Theorem 1.10, $\mathrm{Lag}(Y) = \mathrm{Lag}(Z)$, where Z is the projection of $\mathrm{Lag}(Y)$ on $V^\vee$. But this projection coincides with Y^*. Hence $\mathrm{Lag}(Y) = \mathrm{Lag}(Y^*)$. □

1.3.2 Defect and Discriminant

We should expect that in 'typical' cases X^* is a hypersurface. Having this in mind, we give the following definition:

Definition 1.11 $\operatorname{def} X := \operatorname{codim}_{(\mathbb{P}^N)^\vee} X^* - 1$ is called the *defect* of X.

If $\operatorname{def} X = 0$ then X^* is defined by an irreducible polynomial Δ_X.

Definition 1.12 Δ_X is called the *discriminant* of X.

Δ_X is defined up to a scalar multiple. If $\operatorname{def} X > 0$ then we set $\Delta_X = 1$.

Let us give an equivalent definition of Δ_X. Suppose that $x_1, \ldots, x_{n+1}$ are some local coordinates on $\mathrm{Cone}(X) \subset V$. Any $f \in V^*$, a linear form on V, being restricted to $\mathrm{Cone}(X)$ becomes an algebraic function in $x_1, \ldots, x_{n+1}$. Then Δ_X is just an irreducible polynomial, which vanishes at $f \in V^*$ whenever the function $f(x_1, \ldots, x_{n+1})$ has a multiple root, that is, vanishes at some $v \in \mathrm{Cone}(X)$, $v \neq 0$, together with all first derivatives $\partial f / \partial x_i$.

Example 1.13 Consider the projective space $\mathbb{P}^N = \mathbb{P}(V)$ with homogeneous coordinates $z_0, \ldots, z_N$, and let $X \subset \mathbb{P}^N$ be the rational normal curve

$$(x^N : x^{N-1}y : x^{N-2}y^2 : \ldots : xy^{N-1} : y^N), \ x, y \in \mathbb{C}, \ (x, y) \neq (0, 0)$$

(the image of the Veronese embedding $\mathbb{P}^1 \subset \mathbb{P}^N$). Any linear form $f(z) = \sum a_i z_i$ is uniquely determined by its restriction to $\mathrm{Cone}(X)$, which is a binary form $f(x, y) = \sum a_i x^{N-i} y^i$. Therefore $f \in \mathrm{Cone}(X^*)$ if and only if $f(x, y)$ vanishes at some point $(x_0, y_0) \neq (0, 0)$ (so $(x_0 : y_0)$ is a root of $f(x, y)$) with its first derivatives (so $(x_0 : y_0)$ is a multiple root of $f(x, y)$). It follows that Δ_X is the classical discriminant of a binary form.

The following Theorem is an easy corollary of the Reflexivity Theorem. It allows us to find singular points of hyperplane sections of smooth projective varieties.

Theorem 1.14 *Suppose that $X \subset \mathbb{P}^N$ is smooth and $X^* \subset (\mathbb{P}^N)^\vee$ is a hypersurface. Let $z_0, \ldots, z_N$ be homogeneous coordinates on $\mathbb{P}^N$ and $a_0, \ldots, a_N$ the dual homogeneous coordinates on $(\mathbb{P}^N)^\vee$. Suppose that $f = (a_0 : \ldots : a_N)$ is a smooth point of X^*. Then the hyperplane section $\{f = 0\}$ of X has a unique singular point with coordinates given by*

$$\left(\frac{\partial \Delta_X}{\partial a_0}(f) : \ldots : \frac{\partial \Delta_X}{\partial a_N}(f)\right).$$

Proof. Let $H \subset \mathbb{P}^N$ be the hyperplane corresponding to f. By the Reflexivity Theorem, H is tangent to X at z if and only if the hyperplane in $(\mathbb{P}^N)^\vee$ corresponding to z is tangent to X^* at f. Since X^* is smooth at f, such a point z is unique and is given by $z_i = \frac{\partial \Delta_X}{\partial a_i}(f)$. □

Consider the diagram of projections

$$X \xleftarrow{\mathrm{pr}_1} I_X \xrightarrow{\mathrm{pr}_2} X^*.$$

Theorem 1.15

(i) *If X is smooth then I_X is smooth.*
(ii) *If X^* is a hypersurface then* pr_2 *is birational.*
(iii) *If X is smooth and X^* is a hypersurface then* pr_2 *is a resolution of singularities.*

Proof. The map pr_1 is a projective bundle over X_{sm}. Therefore, if X is smooth then I_X is smooth. If X^* is a hypersurface then $\dim X^* = \dim I_X = N - 1$. Since pr_2 is generically a projective bundle, it is birational. Finally, (iii) follows from (i) and (ii). □

Typically the dual variety $X^* \subset (\mathbb{P}^N)^\vee$ is a hypersurface. Namely, we shall see that if $\operatorname{def} X > 0$ then X is a ruled variety.

Definition 1.16 A projective variety X is called *ruled* in projective subspaces of dimension r if for any $x \in X$ there exists a projective subspace L such that $x \in L \subset X$ and $\dim L = r$.

Definition 1.17 For any projective variety $X \subset \mathbb{P}^N$ and a hyperplane H, the singular locus of $X \cap H$ is called the *contact locus* of X and H.

Theorem 1.18 *Suppose that* $\operatorname{def} X = r \geq 1$. *Then*

(i) *X is ruled in projective subspaces of dimension r.*
(ii) *If X is smooth then the contact locus $(X \cap H)_{sing}$ is a projective subspace of dimension r for any $H \in (X^*)_{sm}$. The union of these projective subspaces is dense in X.*

Proof. By the Reflexivity Theorem 1.7, (i) is equivalent to the following: if $\operatorname{codim} X = r + 1$ then X^* is ruled in projective subspaces of dimension r. By a standard closedness argument it is sufficient to check this property only for some Zariski open dense subset of X^*. The condition for a hyperplane H to be tangent to X at $x \in X_{sm}$ is that H contains $\hat{T}_{x,X}$. For a given x, all H with this property form a projective subspace of dimension r. But the set of hyperplanes of X^* tangent to X at some smooth point obviously contains a Zariski open subset of X^*.

(ii) is proved by the same argument. □

Example 1.19 Suppose that $X \subset \mathbb{P}^N$ is a non-linear curve. Then $\operatorname{def} X = 0$. Indeed, X obviously could not contain a projective subspace $\mathbb{P}^k$ for $k > 0$.

Example 1.20 Let us give the minimal possible example of a smooth variety with positive defect. Let $V = \mathrm{Mat}_{2,3}$ be the space of 2×3 matrices. Then $V^\vee = \mathrm{Mat}_{3,2}$ and the pairing is given by the trace of the product. Let $X \subset \mathbb{P}(V)$ and $X^* \subset \mathbb{P}(V)^\vee$ be the projectivizations of the varieties of matrices of rank less than 2. Then X and X^* are smooth, projectively dual to each other, and both isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$. Therefore $\operatorname{def} X = \operatorname{codim} X^* - 1 = 1$. For any $H \in X^*$, the intersection $X \cap H$ is the union of a smooth quadric surface and a plane. Their intersection, i.e. $(X \cap H)_{sing}$, is a line. See also 2.11.

1.4 Projections and Linear Normality

1.4.1 Projections

Let $P = \mathbb{P}(V)$ be an N-dimensional projective space and $L \subset P$ a projective subspace of dimension k, $L = \mathbb{P}(U)$, $U \subset V$. The quotient projective space $P/L = \mathbb{P}(V/U)$ has, as points, $(k+1)$-dimensional projective subspaces in P containing L. The *projection with center* L is the map

$$\pi_L : P \setminus L \to P/L$$

which takes any point $x \in P\setminus L$ to the $(k+1)$-dimensional projective subspace spanned by x and L. It corresponds to the projection $V \to V/U$. P/L can be identified with any $(N-k-1)$-dimensional projective subspace $K \subset P$ not intersecting L. Then π_L sends $x \in P \setminus L$ to the unique intersection point of K and the $(k+1)$-dimensional projective subspace spanned by x and L. Clearly, the dual projective space $(P/L)^\vee$ is canonically embedded in $P^\vee$ as the set of hyperplanes containing L, and so it coincides with L^*, the dual variety of L.

Theorem 1.21

(i) *Let $X \subset P = \mathbb{P}(V)$ be an irreducible subvariety not intersecting a subspace $L = \mathbb{P}(U)$ and such that* $\dim X < \dim P/L$. *Then*

$$(\pi_L(X))^* \subset L^* \cap X^*.$$

The discriminant $\Delta_{\pi_L(X)}$ (as a polynomial on $(V/U)^\vee \subset V^\vee$) is a factor of the restriction of Δ_X to $(V/U)^\vee$.

(ii) *Suppose further that* $\pi_L : X \to \pi_L(X)$ *is an isomorphism. Then*

$$(\pi_L(X)^* = L^* \cap X^*, \text{ and } \Delta_{\pi_L(X)} = \Delta_X|_{(V/U)^\vee}.$$

Proof. A hyperplane in P/L is just a hyperplane in P containing L. It is clear that, if a hyperplane $H \subset P/L$ is tangent to $\pi_L(X)$ at some smooth point $y = \pi_L(x)$, where $x \in X$ is also smooth, then H as a hyperplane in P is tangent to X at x. This proves (i).

Suppose now that π_L is an isomorphism. The same argument as above shows that, if $X_0^* \subset X^*$ (resp. $\pi_L(X)_0^* \subset \pi_L(X)^*$) is a dense subset of hyperplanes tangent to X (resp. $\pi_L(X)$) at some smooth point, then $\pi_L(X)_0^* = L^* \cap X_0^*$, since π_L induces an isomorphism $X_{sm} \to \pi_L(X)_{sm}$. It suffices to show that $L^* \cap X^* = \overline{L^* \cap X_0^*}$. Here the assumption that π_L is an isomorphism is crucial, because otherwise critical points of π_L may produce extra components of $L^* \cap X^*$.

Consider a germ of a curve $(x(t), H(t))$ in the conormal variety I_X such that $x(t) \in X_{sm}$ for $t \neq 0$ and $H(0)$ contains L. We claim that there is a germ $(x(t), H'(t))$ such that $H'(0) = H(0)$ and and $H'(t)$ contains L for any t. Since π_L is an isomorphism, it induces an isomorphism of Zariski tangent spaces. Therefore $\hat{T}_{x(0),X}$ does not intersect L. Let $T_0 \subset \hat{T}_{x(0),X}$ be a limit position of embedded tangent Zariski spaces $T_t = \hat{T}_{x(t),X}$ as $t \to 0$. Then T_t does not intersect L for any t. Therefore we may consider a family T_t' of $(\dim L + \dim X + 1)$-dimensional projective subspaces such that for any t we have $L \subset T_t'$ and for $t \neq 0$ the subspace T_t' is tangent to X at $x(t)$. Namely, T_t' is a projective subspace spanned by L and T_t. Since $\dim X < \dim P/L$, we can embed T_t' into a family of hyperplanes H_t' with same properties. □

1.4.2 Degenerate Varieties

Definition 1.22 An irreducible subvariety $X \subset \mathbb{P}^N$ is called *non-degenerate* if X is not contained in any hyperplane H.

The next theorem shows that we may restrict ourselves to non-degenerate varieties while studying dual varieties and discriminants.

Theorem 1.23 *Let* $X \subset \mathbb{P}^N$ *be an irreducible subvariety.*

(i) *Assume that* X *is contained in a hyperplane* $H = \mathbb{P}^{N-1}$. *If* $X^{*\prime}$ *is the dual variety of* X, *when we consider* X *as a subvariety of* $\mathbb{P}^{N-1}$, *then* X^* *is the cone over* $X^{*\prime}$ *with vertex* p *corresponding to* H.
(ii) *Conversely, if* X^* *is a cone with vertex* p, *then* X *is contained in the corresponding hyperplane* H.

Proof. If $H' \neq H$ is a tangent hyperplane of X then $H \cap H'$ is a tangent hyperplane of X in $\mathbb{P}^{N-1}$. Conversely, if T is a tangent hyperplane of X in $\mathbb{P}^{N-1}$ then each hyperplane H' in $\mathbb{P}^N$ containing T is tangent to X. Therefore X^* is the cone over $X^{*\prime}$. This proves (i).

By the Reflexivity Theorem, we also have (ii). Namely, each hyperplane which is tangent to X^* at a smooth point necessarily contains p. Therefore $X = X^{**}$ is contained in the hyperplane corresponding to p. □

In terms of discriminants, Theorem 1.23 can be reformulated as follows. Consider a surjection $\pi : V \to U$. Then we have an embedding $i : \mathbb{P}(U^\vee) \hookrightarrow \mathbb{P}(V^\vee)$. Let $X \subset \mathbb{P}(U^\vee)$. Then Δ_X is a polynomial function on U. If we consider X as a subvariety in $\mathbb{P}(V^\vee)$ then $\Delta_{i(X)}$ is a function on V. By Theorem 1.23 these polynomial functions are related as follows:

$$\Delta_{i(X)}(f) = \Delta_X(\pi(f)).$$

In other words, $\Delta_{i(X)}$ does not depend on some of the arguments and forgetting these arguments gives Δ_X.

1.4.3 Linear Normality

Theorems 1.21 and 1.23 show that in order to study dual varieties and discriminants it suffices to consider only projective varieties X that are non-degenerate and not equal to a non-trivial projection. These projective varieties are called linearly normal. To give a more intrinsic definition of linearly normal varieties we need to recall the correspondence between invertible sheaves, linear systems, and projective embeddings.

An invertible sheaf on an algebraic variety X is simply the sheaf of sections of some algebraic line bundle. For example, the structure sheaf of regular functions $\mathcal{O}_X$ corresponds to the trivial line bundle. Usually we shall not distinguish notationally between invertible sheaves and line bundles. Invertible sheaves form the group $\mathrm{Pic}(X)$ with respect to the tensor product. A Cartier divisor on X is a family (U_i, g_i), $i \in I$, where U_i are open subsets of X covering X, and g_i are rational functions on U_i such that g_i/g_j is regular on each intersection $U_i \cap U_j$. The functions g_i are called local equations of the divisor. More precisely, a Cartier divisor is an equivalence class of such data. Two collections (U_i, g_i) and (U_i', g_i') are equivalent if their union is still a divisor. Cartier divisors can be added by multiplying their local equations. Thus they form a group, denoted by $\mathrm{Div}(X)$.

If each local equation g_i is regular on U_i, then we say that the divisor D is effective, and we write $D \geq 0$. The subschemes $\{g_i = 0\}$ of the U_i can then be glued together into a subscheme of X, also denoted by D. Therefore, effective Cartier divisors can be identified with locally principal subschemes of X (locally given by one equation). Any non-zero rational function $f \in \mathbb{C}(X)$ determines a principal Cartier divisor (f). Principal divisors form a subgroup of $\mathrm{Div}(X)$.

Let $\mathcal{K}_X$ denote the sheaf of rational functions on X, $\mathcal{K}_X(U) = \mathbb{C}(U)$. To every Cartier divisor $D = (U_i, g_i)_{i \in I}$ we can attach a subsheaf $\mathcal{O}_X(D) \subset \mathcal{K}_X$. Namely, on U_i it is defined as $g_i^{-1}\mathcal{O}_{U_i}$. On the intersection $U_i \cap U_j$ the sheaves $g_i^{-1}\mathcal{O}_{U_i}$ and $g_j^{-1}\mathcal{O}_{U_j}$ coincide, since g_i/g_j is invertible. Therefore these sheaves can be pasted together into a sheaf $\mathcal{O}_X(D) \subset \mathcal{K}_X$. For instance, $\mathcal{O}_X(0) = \mathcal{O}_X$ and $\mathcal{O}_X(D_1 + D_2) = \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$. The sheaves $\mathcal{O}_X(D)$ are invertible. In fact, multiplication by g_i defines an isomorphism ("trivialization") $\mathcal{O}_X(D)|_{U_i} \simeq \mathcal{O}_{U_i}$. This gives a surjective homomorphism $\mathrm{Div}(X) \to \mathrm{Pic}(X)$. Its kernel consists of principal divisors.

A non-zero section of $\mathcal{O}_X(D)$ is a rational function f on X such that fg_i is regular on the U_i for any i, in other words, such that the divisor $(f) + D$ is effective. If D itself is effective, the sheaf $\mathcal{O}_X(D)$ has a canonical section s_D, which corresponds to the constant function 1. This can also be reformulated as follows. Let $s \in H^0(X, \mathcal{L})$ be a non-trivial global section of an invertible sheaf $\mathcal{L}$. After choosing some trivializations $\phi_i : \mathcal{L}_{U_i} \simeq \mathcal{O}_{U_i}$ on a covering (U_i), we obtain an effective divisor $(U_i, \phi_i(s_i))$, the scheme of zeros of s, which we denote by $Z(s)$. For instance, if D is effective then $Z(s_D) = D$. So basically any effective divisor is defined by one equation $s = 0$. However, s is not a function, but a section of an invertible sheaf. If s' is another non-zero global section of $\mathcal{L}$ then the divisors $Z(s')$ and $Z(s)$ differ by the divisor of a rational function s/s'. One also says that they are linearly equivalent.

The sheaf $\mathcal{O}_X(-D)$, for D effective, is an ideal sheaf in $\mathcal{O}_X$. It defines D as a subscheme.

Example 1.24 Let us recall the construction of invertible sheaves on projective spaces $\mathbb{P}(V)$. All these sheaves have the form $\mathcal{O}(d)$, where $\mathcal{O}(d)$ is the sheaf of homogeneous functions of degree d on $\mathbb{P}(V)$. More precisely, let $\pi : V \setminus \{0\} \to \mathbb{P}(V)$ be the canonical projection. If $U \subset \mathbb{P}(V)$ is a Zariski open set, then the sections of $\mathcal{O}(d)$ over U are, by definition, regular functions f on $\pi^{-1}(U) \subset V$, which are homogeneous of degree d: $f(\lambda v) = \lambda^d f(v)$. It is well-known that these sheaves are invertible and any invertible sheaf has the form $\mathcal{O}(d)$ for some d. For example, $\mathcal{O}(-1)$, as a line bundle, is the tautological line bundle. Its fiber over a point of $\mathbb{P}(V)$ represented by a 1-dimensional subspace l is l itself. If $d < 0$ then $H^0(\mathbb{P}, \mathcal{O}(d)) = 0$. If $d \geq 0$ then $H^0(\mathbb{P}, \mathcal{O}(d)) = \mathrm{Sym}^d V^\vee$, homogeneous polynomials of degree d. For any non-zero $s \in H^0(\mathbb{P}, \mathcal{O}(d))$, the corresponding effective divisor $Z(s)$ is just the hypersurface defined by the polynomial s. In particular, hyperplanes in $\mathbb{P}(V)$ correspond to global sections of $\mathcal{O}(1)$.

Suppose now that X is an irreducible subvariety in $\mathbb{P}^N$. Then $\mathcal{O}(d)$ can be restricted on X, giving a sheaf $\mathcal{O}_X(d)$. In particular, we have a restriction homomorphism of global sections:

$$V^\vee = H^0(\mathbb{P}^N, \mathcal{O}(1)) \xrightarrow{\mathrm{res}} H^0(X, \mathcal{O}_X(1)).$$

Clearly this map is not injective if and only if X is degenerate. So suppose now that X is non-degenerate. In general, res is not surjective either. Its image is a vector subspace $W \subset H^0(X, \mathcal{O}_X(1))$ with the following obvious property: for any $x \in X$ there exists a section $s \in W$ such that $s(x) \neq 0$. Divisors of the form $Z(s)$, $s \in W$, are just hyperplane sections $X \cap H$ for various hyperplanes H.