MATHEMATICS AND STATISTICS SERIES

STOCHASTIC MODELS IN SURVIVAL ANALYSIS AND RELIABILITY SET

Volume 2

Stochastic Risk Analysis and Management

Boris Harlamov

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Stochastic Models in Survival Analysis and Reliability Set

coordinated by
Catherine Huber-Carol and Mikhail Nikulin

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First published 2017 in Great Britain and the United States by ISTE Ltd and John Wiley & Sons, Inc.

ISTE Ltd
27-37 St George's Road
London SW19 4EU
UK

www.iste.co.uk

John Wiley & Sons, Inc.
111 River Street
Hoboken, NJ 07030
USA

www.wiley.com

Library of Congress Control Number: 2016961651

British Library Cataloguing-in-Publication Data
A CIP record for this book is available from the British Library
ISBN 978-1-78630-008-9

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Mathematical Bases

1.1. Introduction to stochastic risk analysis

1.1.1. *About the subject*

The concept of risk is diverse enough and is used in many areas of human activity. The object of interest in this book is the theory of collective risk. Swedish mathematicians Cramér and Lundberg established stochastic models of insurance based on this theory.

Stochastic risk analysis is a rather broad name for this volume. We will consider mathematical problems concerning the Cramér-Lundberg insurance model and some of its generalizations. The feature of this model is a random process, representing the dynamics of the capital of a company. These dynamics consists of alternations of slow accumulation (that may be not monotonous, but continuous) and fast waste with the characteristic of negative jumps.

All mathematical studies on the given subject continue to be relevant nowadays thanks to the absence of a compact analytical description of such a process. The stochastic analysis of risks which is the subject of interest has special aspects. For a long time, the most interesting problem within the framework of the considered model was ruin, which is understood as the capital of a company reaching a certain low level. Such problems are usually more difficult than those of the value of process at fixed times.

Stochastic Risk Analysis and Management, First Edition. Boris Harlamov.

1.1.2. *About the ruin model*

Let us consider the dynamics of the capital of an insurance company. It is supposed that the company serves several clients, which bring in insurance premiums, i.e. regular payments, filling up the cash desk of the insurance company. Insurance premiums are intended to compensate company losses resulting from single payments of great sums on claims of clients at unexpected incident times (the so-called insured events). They also compensate expenditures on maintenance, which are required for the normal operation of a company. The insurance company's activity is characterized by a random process which, as a rule, is not stationary. The company begins business with some initial capital. The majority of such undertakings come to ruin and only a few of them prosper. Usually they are the richest from the very beginning. Such statistical regularities can already be found in elementary mathematical models of dynamics of insurance capital.

The elementary mathematical model of dynamics of capital, the Cramér-Lundberg model, is constructed as follows. It uses a random process R_t $(t \geq 0)$

$$Rt = u + p\,t - \sum_{n=1}^{Nt} U_n, \qquad [1.1]$$

where $u \geq 0$ is the initial capital of the company, $p > 0$ is the growth rate of an insurance premium and $p\,t$ is the insurance premium at time t. $(U_n)_{n=1}^{\infty}$ is a sequence of suit sizes which the insurance company must pay immediately. It is a sequence of independent and identically distributed (i.i.d.) positive random variables. We will denote a cumulative distribution function of U_1 (i.e. of all remaining) as $B(x) \equiv P(U_1 \leq x)$ $(x \geq 0)$. The function (N_t) $(t \geq 0)$ is a homogeneous Poisson process, independent of the sequence of suit sizes, having time moments of discontinuity at points $(\sigma_n)_{n=1}^{\infty}$. Here, $0 \equiv \sigma_0 < \sigma_1 < \sigma_2 < \ldots$; values $T_n = \sigma_n - \sigma_{n-1}$ $(n \geq 1)$ are i.i.d. random variables with a common exponential distribution with a certain parameter $\beta > 0$.

Figure 1.1 shows the characteristics of the trajectories of the process.

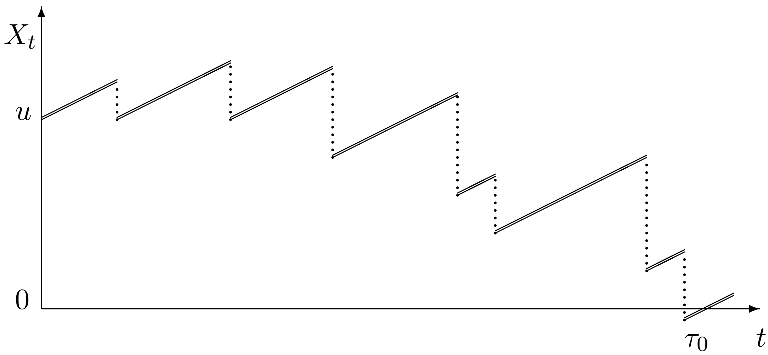

Figure 1.1. *Dynamics of capital*

This is a homogeneous process with independent increments (hence, it is a homogeneous Markov process). Furthermore, we will assume that process trajectories are continuous from the right at any point of discontinuity.

Let τ_0 be a moment of ruin of the company. This means that at this moment, the company reaches into the negative half-plane for the first time (see Figure 1.1). If this event does not occur, this moment is set as equal to infinity.

The first non-trivial mathematical results in risk theory were connected with the function:

$$\psi(u) = P_u(\tau_0 < \infty) \quad (u \geq 0),$$

i.e. a probability of ruin on an infinite interval for a process with the initial value u. Interest is also represented by the function $\psi(u,t) = P_u(\tau_0 \leq t)$. It is called the ruin function on "finite horizon".

Nowadays many interesting outcomes have been reported for the Cramér-Lundberg model and its generalizations. In this volume, the basic results of such models are presented. In addition, we consider its generalizations, such as insurance premium inflow and distribution of suit sizes.

This is concentrated on the mathematical aspects of a problem. Full proofs (within reason) of all formulas, and volume theorems of the basic course are presented. They are based on the results of probability theory which are assumed to be known. Some of the information on probability theory is shortly presented at the start. In the last chapter some management problems in insurance business are considered.

1.2. Basic methods

1.2.1. *Some concepts of probability theory*

1.2.1.1. *Random variables*

The basis of construction of probability models is an abstract probability space $(\Omega, \mathcal{F}, P)$, where Ω is a set of elementary events; $\mathcal{F}$ is a sigma-algebra of subsets of the set Ω, representing the set of those random events, for which it makes sense to define the probability within the given problem; P is a probability measure on set Ω, i.e. non-negative denumerably additive function on $\mathcal{F}$. For any event $A \in \mathcal{F}$, the probability, $P(A)$, satisfies the condition $0 \leq P(A) \leq 1$. For any sequence of non-overlapping sets $(A_n)_1^\infty$ $(A_n \in \mathcal{F})$ the following equality holds:

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n),$$

and $P(\Omega) = 1$. Random events A_1 and A_2 are called independent if $P(A_1, A_2) \equiv P(A_1 \cap A_2) = P(A_1)P(A_2)$. This definition is generalized on any final number of events. Events of infinite system of random events are called mutually independent if any of its final subsystem consists of independent events.

A random variable is a measurable function $\xi(\omega)$ $(\omega \in \Omega)$ with real values. It means that for any real x, the set $\{\omega : \xi(\omega) \leq x\}$ is a random event and hence, probability of it exists, designated as $F_\xi(x)$. Thus, the cumulative distribution function, F_ξ, is defined as follows :

$$F_\xi(x) = P(\xi \le x) \quad (-\infty < x < \infty).$$

It is obvious that this function does not decrease when x increases. In this volume, we will deal with absolutely continuous distributions and discreet distributions (sometimes with their mixtures).

For an absolutely continuous distribution, there exists its distribution density $f_\xi(x) = dF_\xi(x)/dx$ for all $x \in (-\infty, \infty)$ such that

$$\int_{-\infty}^{\infty} f_\xi(x)\, dx = 1.$$

For discreet distributions, there exists a sequence of points (atoms) $(x_n)_1^\infty$ for which non-negative probabilities $p(x_n) = P(\xi = x_n)$ are defined as:

$$\sum_{n=1}^{\infty} p(x_n) = 1.$$

The random variable is called integer if it has a discreet distribution with atoms in the integer points of a numerical axis, denoted by $\mathbb{Z}$.

If $\mathbb{R}$ is the set of all real numbers, φ is a measurable function on $\mathbb{R}$, and ξ is a random variable, then superposition $\psi(\omega) \equiv \varphi(\xi(\omega))$ $(\omega \in \Omega)$ is a random variable too. Various compositions of random variables are possible, which are also random variables. Two random variables ξ_1 and ξ_2 are called independent, if for any x_1 and x_2 events $\{\xi_1 \le x_1\}$ and $\{\xi_2 \le x_2\}$ are independent.

Expectation (average) $E\xi$ of a random variable ξ is the integral of this function on Ω with respect to the probability measure P, i.e.:

$$E\xi = \int_\Omega \xi(\omega)\, P(d\omega) \equiv \int \xi\, dP$$

(an integral of Lebesgue). By a cumulative distribution function, this integral can be noted as an integral of Stieltjes:

$$E\xi = \int_{-\infty}^{\infty} x\, dF_{\xi}(x),$$

and for a random variable ξ with absolute continuous distribution, it can be represented as integral of Riemann:

$$E\xi = \int_{-\infty}^{\infty} x f_{\xi}(x)\, dx.$$

For a random variable ξ with a discreet distribution, it is possible to write an integral in the form of the sum:

$$E\xi = \sum_{n=1}^{\infty} x_n p(x_n).$$

When evaluating an expectation, it is necessary to be careful in case the integral from the module of this random variable is equal to infinity. Sometimes it useful to distinguish three cases: an integral equal to plus infinity, an integral equal to minus infinity and an integral does not exist.

Let us note that it is possible to consider separately a cumulative distribution function out of connection with random variables generating them and probability spaces. However, for any non-decreasing, continuous from the right, function F such that $F(x) \to 0$ as $x \to -\infty$ and $F(x) \to 1$ as $x \to \infty$ (the cumulative distribution function of any random variable possesses these properties), it is possible to construct a probability space and with random variable on this space, which has F as its cumulative distribution function on this probability space. Therefore, speaking about a cumulative distribution function, we will always mean some random variable within this distribution. It allows us to use equivalent expressions such as “distribution moment”, “moment of a random variable”, “generating function of a distribution” and “generating function of a random variable”.

The following definitions are frequently used in probability theory. The moment of nth order of a random variable ξ is an integral $E\xi^n$ (if it exists). The central moment of nth order of a random variable ξ is an integral $E(\xi - E\xi)^n$ (if it exists). The variance (dispersion) $D\xi$ of a random variable ξ is its central moment of second order.

The generating function of a random variable is the integral $E\exp(\alpha\xi)$, considered as a function of α. Interest represents those generating functions which are finite for all α in the neighborhood of zero. In this case, there is one-to-one correspondence between the set of distributions and the set of generating functions. This function has received the name because of its property "to make" the moments under the formula:

$$E\xi^n = \left.\frac{d^n E\exp(\alpha\xi)}{d\alpha^n}\right|_{\alpha=0}.$$

A random n-dimensional vector is the ordered set of n random variables $\xi = (\xi_1, \dots, \xi_n)$. Distribution of this random vector (joint distribution of its random coordinates) is a probability measure on space $\mathbb{R}^n$, defined by n-dimensional cumulative distribution function:

$$F_\xi(x_1, \dots, x_n) = P(\xi_1 \le x_1, \dots, \xi_n \le x_n) \quad (x_i \in \mathbb{R}, i = 1, \dots, n).$$

As the generating function of a random vector is called function of n variables $E\exp(\alpha, \xi)$, where $\alpha = (\alpha_1, \dots, \alpha_n)$ $(\alpha_i \in \mathbb{R})$ and $(\alpha, \xi) = \sum_{i=1}^n \alpha_i\xi_i$. The mixed moment of order $m \ge 2$ of a random vector ξ is called $E(\xi_1^{m_1} \cdot \dots \cdot \xi_n^{m_n})$, where $m_i \ge 0$, $\sum_{i=1}^n m_i = m$. Covariance of random variables ξ_1 and ξ_2 is called central joint moment of the second order:

$$\mathrm{cov}(\xi_1, \xi_2) = E(\xi_1 - E\xi_1)(\xi_2 - E\xi_2).$$

1.2.1.2. *Random processes*

In classical probability theory, random process on an interval $T \subset \mathbb{R}$ is called a set of random variables $\xi = (\xi_t)_{t\in T}$, i.e. function of two