SpringerWienNewYork

Michael Leyton

The Structure of Paintings

SpringerWienNewYork

Prof. Michael Leyton
Psychology Department, Center for Discrete Mathematics &
Theoretical Computer Science,
Rutgers University, New Brunswick, NJ 08854, USA

E-Mail: mleyton@dimacs.rutgers.edu

Printed in Austria

SpringerWienNewYork is part of
Springer Science+Business Media
springer.com

Cover Illustration: Michael Leyton, *The Supremacy of Line*, New York, 1992,
acrylic on canvas, 152 x 122 cm (60 x 48 in)
Typesetting: Camera ready by the author
Printing: Holzhausen Druck und Medien GmbH, 1140 Vienna

Printed on acid-free and chlorine-free bleached paper
SPIN:11784098

With numerous (partly coloured) Figures

Library of Congress Control Number: 2006930614

ISBN-10 3-211-35739-4 SpringerWienNewYork
ISBN-13 978-3-211-35739-2 SpringerWienNewYork

Contents

Chapter 1

Shape as Memory Storage

1.1 Introduction

This is the first in a series of books whose purpose is to give a systematic elaboration of the laws of artistic composition. We shall see that these laws enable us to build up a complete understanding of any painting – both its structure and meaning.

The reason why it is possible to build up such an understanding is as follows. In a series of books and papers, I have developed new foundations to geometry – foundations that are very different from those that have been the basis of geometry for the last 3000 years. A conceptual elaboration of these new foundations was given by my book *Symmetry, Causality, Mind* (MIT Press, 630 pages), and the mathematical foundations were elaborated by my book *A Generative Theory of Shape* (Springer-Verlag, 550 pages). The central proposal of this theory is:

SHAPE = MEMORY STORAGE.

That is: What we mean by shape is memory storage, and what we mean by memory storage is shape.

In the next section, we will see how these new foundations for geometry are directly the opposite of the foundations that have existed from Euclid to modern physics, including Einstein.

My books apply these new foundations to several disciplines: human and computer vision, robotics, software engineering, musical composition, architecture, painting, linguistics, mechanical engineering, computer-aided design and modern physics.

The new foundations unify these disciplines by showing that a result of these foundations is that geometry becomes equivalent to aesthetics. That is, the theory of aesthetics, given by the new foundations, unifies all scientific and artistic disciplines.

Now, as said above, according to the new foundations, shape is equivalent to memory storage. With respect to this, a significant principle of my books is this:

ARTWORKS ARE MAXIMAL MEMORY STORES.

My argument is that the above principle explains the structure and function of artworks. Furthermore, it explains why artworks are the most valuable objects in human history.

1.2 New Foundations to Geometry

This book will show that the new foundations to geometry explain art, whereas the conventional foundations of Euclid and Einstein do not. Thus, to understand art, we need to begin by comparing the two opposing foundations.

The reader was, no doubt, raised to consider Einstein a hero who challenged the basic assumptions of his time. In fact, Einstein's theory of relativity is simply a re-statement of the concept of *congruence* that is basic to Euclid. It is necessary to understand this, and to do so, we begin by considering an example of congruence.

Fig 1.1 shows two triangles. To test if they are congruent, you translate and rotate the upper one to try to make it coincident with the lower one. If exact coincidence is possible, you say that they are *congruent*. This allows you to regard the triangles as essentially the *same object*.

This approach has been the basis of geometry for over 2,000 years, and received its most powerful formulation in the late 19th century by Klein, in the most famous statement in all mathematics – a statement which became the basis not only of all geometry, but of all mathematics and physics: *A geometric object is an invariant (an unchanged property) under some chosen transformations*.

Let us illustrate by returning to the two triangles in Fig 1.1. Consider the upper triangle: It has a number of properties:

(1) Three sides.
(2) Points upward.
(3) Two equal angles.

Now apply a movement to make it coincident with the lower triangle. Properties (1) and (3) remain invariant (unchanged); i.e., the lower triangle also has three sides and has two equal angles. In contrast, property (2) is not invariant; i.e., the triangle no longer points upwards. Klein said that the *geometric* properties are those that remain invariant; i.e., properties (1) and (3).

Now a crucial part of my argument is this: Because properties (1) and (3) are unchanged (invariant) under the movement, *it is impossible to infer from them that the movement has taken place*. Only the non-invariant property, the direction of pointing, allows us to recover the movement. Therefore, in the terminology of my books, I say that *invariants are those properties that are memoryless*; i.e., they yield no information about the past. Because Klein proposes that a geometric object consists of invariants, Klein views geometry as the study of memorylessness.

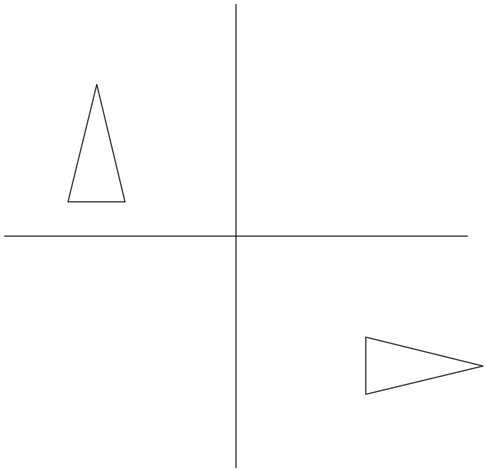

Figure 1.1: Conventional geometry.

Klein's approach became the basis of 20th century mathematics and physics. Thus let us turn to Einstein's theory of relativity. Einstein's fundamental principle says this: The objects of physics are those properties that remain invariant under changes of reference frame. Thus the name "theory of relativity" is the completely wrong name for Einstein's theory. It is, in fact, the theory of *anti-relativity*. It says that one must reject from physics any property that is relative to an observer's reference frame.

Now I argue this: Because Einstein's theory says that the only valid properties of physics are those that do not change in going from one reference frame to another, he is actually implying that physics is the study of those properties from which you cannot recover the fact that there has been a change of reference frame; i.e., they are memoryless to the change of frame.

Einstein's program spread to all branches of physics. For example, quantum mechanics is the study of invariants under the actions of measurement operators. Thus the classification of quantum particles is simply the listing of invariants arising from the energy operator.

The important thing to observe is that this is all simply an application of Klein's theory that geometry is the study of invariants. Notice that Klein's view really originates with Euclid's notion of congruence: The invariants are those properties that allow congruence.

> **The basis of modern physics can be traced back to Euclid's concern with congruence.**
>
> **We can therefore say that the entire history of geometry, from Euclid to modern physics, has been founded on the notion of** *memorylessness*.

This fundamentally contrasts with the theory of geometry developed in my books. In this theory, a geometric object is a *memory store* for action. Consider the shape of the human body. One can recover from it the history of embryological development and subsequent growth, that the body underwent. The shape is full of its history. There is very little that is congruent between the developed body and the original spherical egg from which it arose. There is very little that has remained invariant from the origin state. I argue that *shape is equivalent to the history that it has undergone*.

Let us therefore contrast the view of geometric objects in the two opposing foundations for geometry:

STANDARD FOUNDATIONS FOR GEOMETRY
(Euclid, Klein, Einstein)
A geometric object is an *invariant;* **i.e.,** *memoryless.*

NEW FOUNDATIONS FOR GEOMETRY
(Leyton)
A geometric object is a *memory store.*

Furthermore, my argument is that the latter view of geometry is the appropriate one for the computational age. A computational system is founded on the use of memory stores. Our age is concerned with the retention of memory rather than the loss of it. We try to buy computers with greater memory, not less. People are worried about declining into old age, because memory decreases.

The point is that, for the computational age, we don't want a theory of geometry based on the notion of memorylessness – the theory of the last 2,500 years. We want a theory of geometry that does the opposite: Equates shape with memory storage. This is the theory proposed and developed in my books.

Furthermore, from this fundamental link between shape and memory storage, I argue the following:

The retrieval of memory from shape is the real meaning of aesthetics.

As a result of this, the new foundations establish the following 3-way equivalence:

Geometry $\equiv$ **Memory** $\equiv$ **Aesthetics.**

In fact, my books have shown that this is the basis of artistic composition. The rules by which an artwork is structured are the rules that will enable the artwork to act as a memory store.

The laws of artistic composition are the laws of memory storage.

Let us also consider a simple analogy. A computer has a number of memory stores. They can be inside the computer, or they can be attached as external stores. My claim is that artworks are external memory stores for human beings. In fact, they are the most powerful memory stores that human beings possess.

1.3 The World as Memory Storage

So let us begin. We start by defining memory in the simplest possible way:

Memory = Information about the past.

Consequently, we will define a memory store in the following way:

Memory store = Any object that yields information about the past.

In fact, I argue that the entire world around us is memory storage, i.e., information about the past. We extract this information from the objects we see. There are many sources of memory. Let us consider some examples. It is worth reading them carefully to fully understand them.

(1) SCARS: A scar on a person's face is, in fact, a memory store. It gives us information about the past: It tells us that, in the past, the surface of the skin was cut. Therefore, past events, i.e., process-history, is stored in a scar.

(2) DENTS: A dent in a car door is also a memory store; i.e., it gives us information about the past: It tells us that, in the past, the door underwent an impact from another object. Therefore, process-history is stored in a dent.

(3) GROWTHS: Any growth is a memory store, i.e., it yields information about the past. For example, the shape of a person's face gives us information that a history of growth has occurred, e.g., the nose and cheekbones grew outward, the wrinkles folded inward, etc. The shape of a tree gives us very accurate information about how it grew. Both, a face and a tree, inform us of a past history. Each is therefore a memory store of process-history.

(4) SCRATCHES: A scratch on a table is information about the past. It informs us that, in the past, the surface had contact with a sharp moving object. Therefore, past events, i.e., process-history, is stored in a scratch.

(5) CRACKS: A crack in a vase is a memory store, i.e., it yields information about the past. It informs us that, in the past, the vase underwent some impact. Therefore, process-history is stored in a crack.

I argue that the world is, in fact, layers and layers of memory storage. One can see this for instance by looking at the relationships between the examples just listed. For example, consider item (1) above, a scar on a person's face. This is memory of scratching. This sits on a person's face, item (3), which is memory of growth. Thus the memory store for scratching – the scar – sits on top of the memory store for growth – the face.

As another example, consider item (5): a crack in a vase. The crack is due to the history of hitting, but the vase on which it occurs is the result of formation from clay on the potter's wheel. Indeed the shape of the vase tells us much about how it was formed. The vertical height is memory of the process that pushed the clay upwards; and the outline of the vase, curving in and out, is memory of the changing pressure of the potter's hands. Therefore the memory store for hitting – the crack – sits on top of the memory store for clay-manipulation – the vase.

According to this theory, therefore, the entire world is memory storage. Each object around us is a memory store of the history of processes that formed it. A central part of my new foundations for geometry is that they establish the rules by which it is possible to extract memory from objects.

1.4 The Fundamental Laws

According to the new foundations, memory storage can take an infinite variety of forms. For example, scars, dents, growths, scratches, twists, cracks, are all memory stores because they all yield information about past actions. However, mathematical arguments given in my books, show that, on a deep level, all memory stores have only one form. This is given by my fundamental laws of memory storage:

FIRST FUNDAMENTAL LAW OF MEMORY STORAGE
(Leyton, 1992)
Memory is stored only in asymmetries.

SECOND FUNDAMENTAL LAW OF MEMORY STORAGE
(Leyton, 1992)
Memory is erased by symmetries.

That is, information about the past can be recovered only from asymmetries. And correspondingly, information about the past is erased by symmetries.

Let us begin with a simple example. Consider the sheet of paper shown on the left in Fig 1.2. Even if one had never seen that sheet before, one would conclude that it had undergone twisting. The reason is that the asymmetry in the sheet yields information about the past. In other words, from the asymmetry, one can *recover* the past history. That is, the *asymmetry acts as a memory store for the past action* – as stated in my First Fundamental Law of Memory Storage (above).

Now let us un-twist the paper, thus obtaining the straight sheet given on the right in Fig 1.2. Suppose we show this straight sheet to any person on the street. Would they be able to infer from it the fact that it had once been twisted? The answer is "No." The reason is that the symmetry of the straight sheet has wiped out the ability to recover the preceding history. This means that the *symmetry erases the memory store* – as stated in my Second Fundamental Law of Memory Storage (above).

Figure 1.2: A twisted sheet is a source of information about the past. A non-twisted sheet is not.

This means that symmetry is the absence of information about the past. In fact, from the symmetry, one concludes that the straight sheet had always been like this. For example, when you take a sheet of paper from a box of paper you have just bought, you do not assume that it had once been twisted or crumpled. Its very straightness (symmetry) leads you to conclude that it had always been straight.

The two diagrams in Fig 1.2 illustrate the two fundamental laws of memory storage given above. These two laws are the very basis of my foundations for geometry. I formulate these two laws in the following way:

LAW 1. ASYMMETRY PRINCIPLE.

An asymmetry in the present is understood as having originated from a past symmetry.

and

LAW 2. SYMMETRY PRINCIPLE.

A symmetry in the present is understood as having always existed.

At first, it might seem as if there are many exceptions to these two laws. In fact, my books show that all the apparent exceptions are due to incorrect descriptions of situations. These laws cannot be violated for deep mathematical reasons.

Now, recall my claim is that artworks are maximal memory stores. My books show:

The Fundamental Laws of Memory Storage = The Fundamental Laws of Art.

We will see that these laws reveal the *complete* structure of any painting. Furthermore, they map out its entire meaning.

Let us now start to develop a familiarity with the two laws. What will be seen, over and over again, is that the way to use the two laws is to go through the following simple procedure: First partition the presented situation into its asymmetries and its symmetries. Then use the Asymmetry Principle (Law 1) on the asymmetries, and the Symmetry Principle (Law 2) on the symmetries. Note that the application of the Asymmetry Principle will return the asymmetries to symmetries. And the application of the Symmetry Principle will preserve the symmetries.

What does one obtain when one applies this procedure to a situation? The answer is this: One obtains the *past*!

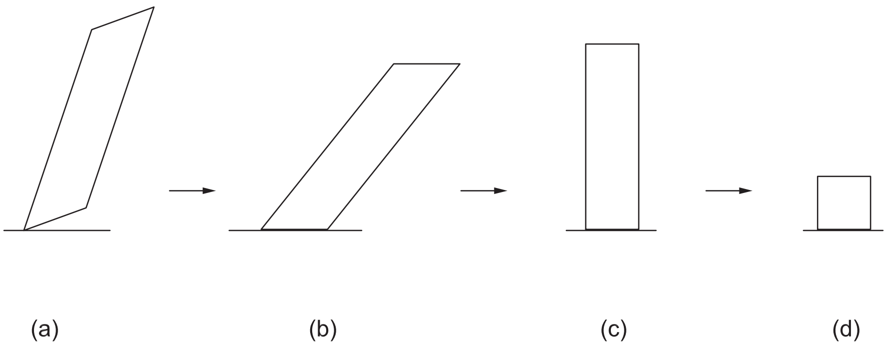

Figure 1.3: The history inferred from a rotated parallelogram.

Now recall that memory is information about the past, so this procedure is the procedure for the extraction of memory. That is, it converts objects into memory stores.

Since this procedure will be used throughout the book, it will now be stated succinctly as follows:

PROCEDURE FOR RECOVERING THE PAST

(1) Partition the situation into its asymmetries and symmetries.
(2) Apply the Asymmetry Principle to the asymmetries.
(3) Apply the Symmetry Principle to the symmetries.

An extended example will now be considered that will illustrate the power of this procedure, as follows: In a set of psychological experiments that I carried out in the psychology department in Berkeley in 1982, I found that, when subjects are presented with a rotated parallelogram, as shown in Fig 1.3a, they refer it in their heads to a non-rotated parallelogram, Fig 1.3b, which they then refer in their heads to a rectangle, Fig 1.3c, which they then refer in their heads to a square, Fig 1.3d. It is important to understand that the subjects are presented with only the first shape. The rest of the shapes are actually generated by their own minds, as a response to the presented shape.

Close examination reveals that what the subjects are doing is *recovering the history* of the rotated parallelogram. That is, they are saying that, prior to its current state, the rotated parallelogram, Fig 1.3a, was non-rotated, Fig 1.3b, and prior to this it was a rectangle, Fig 1.3c, and prior to this it was a square, Fig 1.3d.

The following should be noted about this sequence. The sequence from *right to left* – that is, going from the square to the rotated parallelogram – represents the direction of *forward time*; i.e., the history starts in the *past* (the square) and ends with the *present* (the rotated parallelogram). Conversely, the sequence from *left to right* – that is, going from the rotated parallelogram to the square – represents the direction of *backward time*. Thus, what the subjects are doing, when their minds generate the sequence of shapes from the rotated parallelogram to square, is this: They are **running time backwards!**

We shall now see that the subjects create this sequence by using the Asymmetry Principle and the Symmetry Principle, i.e., the two above laws for the extraction of memory. Recall that the way one uses the two laws is to apply the simple three-stage Procedure for Recovering the Past, given above: (1) Partition the presented situation into its asymmetries and symmetries, (2) apply the Asymmetry Principle to the asymmetries, and (3) apply the Symmetry Principle to the symmetries.

Thus to use this procedure on the rotated parallelogram, let us begin by identifying the asymmetries in that figure. It is important first to note an important fact:

Asymmetries are the same thing as distinguishabilities.

In the rotated parallelogram, there are three distinguishabilities:

> (1) The distinguishability between the orientation of the shape and the orientation of the environment – indicated by the difference between the bottom edge of the shape and the horizontal line which it touches.
>
> (2) The distinguishability between adjacent angles in the shape: they are different sizes.
>
> (3) The distinguishability between adjacent sides in the shape: they are different lengths.

It is clear that what happens in the sequence, from the rotated parallelogram to the square, is that these three distinguishabilities are removed successively backwards in time. The removal of the first distinguishability, that between the orientation of the shape and the orientation of the environment, results in the transition from the rotated parallelogram to the non-rotated one. The removal of the second distinguishability, that between adjacent angles, results in the transition from the non-rotated parallelogram to the rectangle, where the angles are equalized. The removal of the third distinguishability, that between adjacent sides, results in the transition from the rectangle to the square, where the sides are equalized.

Therefore, each successive step in the sequence is a use of the Asymmetry Principle, which says that an asymmetry must be returned to a symmetry backwards in time.

Having identified the asymmetries in the rotated parallelogram and applied the Asymmetry Principle to each of these, we now identify the symmetries in the rotated parallelogram and apply the Symmetry Principle to each of these. First we need an important fact:

Symmetries are the same thing as indistinguishabilities.

In the rotated parallelogram, there are two indistinguishabilities:

> (1) The opposite angles are indistinguishable in size.
>
> (2) The opposite sides are indistinguishable in length.

The Symmetry Principle requires that these two symmetries in the rotated parallelogram must be preserved backwards in time. And indeed, this turns out to be the case. That is, the first symmetry, the equality between opposite angles, in the rotated parallelogram, is preserved backwards through the entire sequence: i.e., each subsequent shape, from left to right, has the property that opposite angles are equal. Similarly, the other symmetry, the equality between opposite sides in the rotated parallelogram, is preserved backwards through the entire sequence: i.e., each subsequent shape, from left to right, has the property that opposite sides are equal.

Thus what we have seen in this example is this: The sequence from the rotated parallelogram to the square is determined by two rules: the Asymmetry Principle which returns asymmetries to symmetries, and the Symmetry Principle which preserves the symmetries. These two rules allow us to *recover the past*, i.e., *run time backwards*.

1.5 The Meaning of an Artwork

The preceding section gave what my books have shown are the two Fundamental Laws of Memory Storage, which were also formulated as the Asymmetry Principle and Symmetry Principle. Furthermore, since my claim is that artworks are maximal memory stores, I have also argued that these two laws are *the two most fundamental laws of art*.

According to my foundations for geometry, the history recovered from a memory store is the set of *processes* that produced the current state of the store. The reason is that the foundations constitute a *generative theory*. This is why the book in which I elaborated the mathematical foundations is called *A Generative Theory of Shape* (Springer-Verlag). The idea is that: *shape is defined by the set of processes that produced it*.

Thus, what is being recovered from shape, i.e., from the memory store, is its *process-history.*

According to the new foundations, this gives the *meaning* of an artwork. That is, as argued in my book *Symmetry, Causality, Mind* (MIT Press):

THE MEANING OF AN ARTWORK

The meaning of an artwork is the process-history recovered from it.

We shall see that an important consequence of this is the following: Because the new foundations for geometry allow us to systematically recover the process-history that produced a memory store, we have this:

> **The new foundations to geometry allow us to systematically map out the entire meaning of an artwork.**

1.6 Tension

In this section, the Fundamental Laws of Memory Storage, given in section 1.4, are used to begin a theory of tension in artworks. Any artist knows that an artwork is defined by its structure of tension. Yet remarkably, no one has ever given a theory of tension in artworks. In contrast, this book will give a *complete* theory of tension. The following will be one of the basic proposals made in this book:

TENSION $\equiv$ MEMORY STORAGE.

The reason why this will be argued is because the following will also be proposed:

Tension is the recovery of the past.

In other words, given the present state, tension is what allows one to recover the past state. Therefore tension must correspond to the rules for the recovery of the past from the present. But the new foundations say that the two fundamental rules for this recovery are the Asymmetry Principle and Symmetry Principle. Therefore, I will now propose the following:

FIRST FUNDAMENTAL LAW OF TENSION.

> **Tension is the use of the Asymmetry Principle. That is, tension occurs from a present asymmetry to its past symmetry.**

To explain: The Asymmetry Principle states that any asymmetry in the present is understood as having arisen from a past symmetry. The above law says that tension is the relation from the present asymmetry to the inferred past symmetry.

The truth of this law will be demonstrated many times in this book. However, as an immediate illustration, let us return to the rotated-parallelogram example of section 1.4. We saw that the rotated parallelogram has three asymmetries, i.e., three distinguishabilities:

> (1) The distinguishability between the orientation of the shape and the orientation of the environment – indicated by the difference between the bottom edge of the shape and the horizontal line which it touches.
>
> (2) The distinguishability between adjacent angles in the shape: they are different sizes.
>
> (3) The distinguishability between adjacent sides in the shape: they are different lengths.

The Asymmetry Principle states that each asymmetry is understood as having arisen from a past symmetry. This means that there are exactly three uses of the Asymmetry Principle on the rotated parallelogram, one for each asymmetry.

Now, the First Fundamental Law of Tension, stated above, says that tension is the use of the Asymmetry Principle. This means that there are exactly three types of tension in the rotated parallelogram – one for each use of the Asymmetry Principle. Furthermore, the law allows us to precisely define what these three tensions are. They are:

(1) A tension that tries to reduce the difference between the orientation of the shape and the orientation of the environment; i.e., tries to make the two orientations equal.

(2) A tension that tries to reduce the difference between the sizes of the adjacent angles; i.e., tries to make the sizes of the angles equal.

(3) A tension that tries to reduce the difference between the lengths of the adjacent sides; i.e., tries to make the lengths of the sides the same.

That is, each tension tries to turn a distinguishability into an indistinguishability, i.e., each is an example of returning an asymmetry to symmetry.

Simple as this example is, it illustrates the basic power of the First Fundamental Law of Tension, as follows:

CONSEQUENCE OF THE FIRST FUNDAMENTAL LAW OF TENSION.

There is one tension for each asymmetry. That is, the asymmetries are the sources of tension.

This turns out to be a powerful tool in the analysis of artistic composition, as follows:

CONSEQUENCE OF THE FIRST FUNDAMENTAL LAW OF TENSION.

The First Fundamental Law allows one to systematically elaborate all the tensions in a figure; i.e., elaborate the asymmetries and establish their symmetrizations.

The law will be illustrated many times in the book.

1.7 Tension in Curvature

The ideas developed in the previous sections will now be used to carry out an analysis of what I will argue is one of the major forms of tension in an artwork: *curvature*. We shall see that this gives enormous insight into artistic composition.

Let us state precisely what the goal will be: In accord with the theory of this book – i.e., that art is memory storage – we will develop a theory of how *history is recovered from curved shapes*. Since this is the history of past *processes* that produced the present shape, we will refer to it as *process-history*. It will be seen that the recovered process-history will yield the *tension* structure of such shapes.

The next few sections will be concerned with closed smooth shapes such as that shown in Fig 1.4. The shape is *closed*, in that it does not have any ends; and it is *smooth*, in that it does not have any sharp corners. Later on, the techniques developed for such shapes will be generalized to arbitrary shapes.

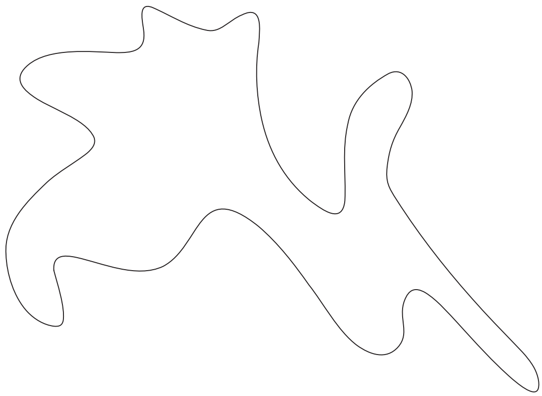

Figure 1.4: A closed smooth curve.

Our concern will be to solve the following problem: When presented with a shape like Fig 1.4, how can one infer the preceding history that produced that shape? In other words, we will be trying to solve what my books call the **history-recovery problem** for that shape.

1.8 Curvature Extrema

We now begin an analysis of how curvature creates tension in an artwork. First, it is necessary to understand the meaning of *curvature*. In the case of curves in the two-dimensional plane, curvature is easy to define. Quite simply, curvature is the *amount of bend*.

Thus, consider the downward sequence of lines shown in Fig 1.5. The line at the top has no bend. Therefore one says that it has zero curvature. The next line downwards has more bend, and thus one says that it has more curvature. The line below this has even more bend, and so one says that it has even more curvature.

Now, the curve at the bottom of Fig 1.5, should be examined carefully. It exhibits a property that is going to be crucial to the entire discussion. The property is this: There is a point, shown as E on the curve, that has more curvature (bend) than the other points on the curve. Let us examine this more closely:

There is a simple way to judge how much curvature there is at some point of a curve. Imagine that you are driving a car along a road shaped exactly like the curve. The amount of curvature at any point on the road is the amount that *the steering wheel is turned*. Obviously, for a sharp bend in the road, the steering wheel must be turned a considerable amount. This is because a sharp bend has a lot of curvature. In contrast, for a straight section of road, the steering wheel should not be turned at all; it should point directly ahead. This is because a straight section of road has no curvature.

Let us now return to the bottom curve shown in Fig 1.5. If one drives around this

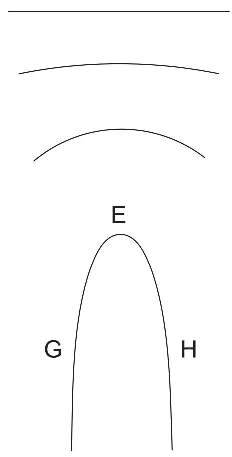

Figure 1.5: Successively increasing curvature.

curve, it is clear that, at point E, the wheel would have to be turned a considerable amount: That is, point E involves a sharp bend in the road.

However, contrast this with driving through point G shown on the curve. The wheel, in this region, should remain relatively straight, because the road there involves almost no bend, i.e., no curvature. The same applies to point H on the other side.

Thus, let us try to see what happens when one drives along the entire curve. Suppose one starts at the left end. Initially, the steering wheel is straight for quite a while. But then, as one gets closer to E, one must start turning the wheel, until at E, the amount of turn reaches a *maximum*. After one passes through E, however, one slowly begins to straighten the wheel again. And, in the final part of the road, the wheel becomes almost straight.

Because point E has the extreme amount of curvature, it is called a **curvature extremum**. Curvature extrema are going to be very important in the following discussion. We shall see that their role in an artwork is crucial.

1.9 Symmetry in Complex Shape

Since every aspect of the theory will be founded on the notion of symmetry, it is necessary to look at how symmetry is defined in a complex shape. In particular, one must understand how reflectional symmetry is defined in complex shape, as follows:

Defining reflectional symmetry on a simple shape is easy. Consider the triangle show in Fig 1.6. It is a simple shape. One establishes symmetry in this shape merely by placing a mirror on the shape, in such a position that it reflects one half of the figure onto the other. The line, along which the mirror lies, is called the symmetry axis. It is shown as the vertical dashed line in the figure.

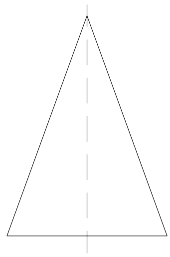

Figure 1.6: A simple shape having a straight mirror symmetry.

In contrast, consider a complex shape like that shown earlier in Fig 1.4 (p13). We cannot place a mirror on it so that it will reflect one half onto the other. Nevertheless, we shall see now that such a shape does contain a very subtle form of reflectional symmetry, and this is central to the way the mind defines the structure of tension in the figure.

Consider the two curves, c_1 and c_2, shown in Fig 1.7. The goal is to find the symmetry axis between the two curves. Observe that one cannot take a mirror and reflect one curve onto the other. For example, the top curve shown is more curved than the bottom one. Therefore a mirror will not send the top one onto the bottom one.

The way one proceeds is as follows: Insert a circle between the two curves as shown in Fig 1.8. It must touch the two curves simultaneously. For example, in this figure, we see the circle touching the upper curve at A, while simultaneously touching the lower curve at B.

Next, drag the circle along the two curves, always making sure that the circle touches the upper and lower curve simultaneously. As can be seen, one might have to expand or contract the circle so that it can touch the two curves at the same time.

Finally, as the circle moves, keep track of a particular point, Q, shown in Fig 1.8. This point is on the circle, half way between the two touch points A and B. As the circle moves along the two curves, it leaves a trajectory of points Q. This trajectory is indicated by the dotted line. The dotted line is then called the *symmetry axis* between the two curves.

Comment: For those who are familiar with symmetry axes based on the circle, one should note that the axis of Blum [1] was based on the circle center, the axis of Brady [2] was based on the chord midpoint between A and B, and the axis described above is based on the arc midpoint between A and B. This last analysis was invented by me in Leyton [16] and has particular topological properties that make it highly suitable for the inference of process-history. I therefore called it *Process-Inferring Symmetry Analysis (PISA)*. [1]

[1] In fact, the full definition of PISA involves extra conditions discussed in my previous books.

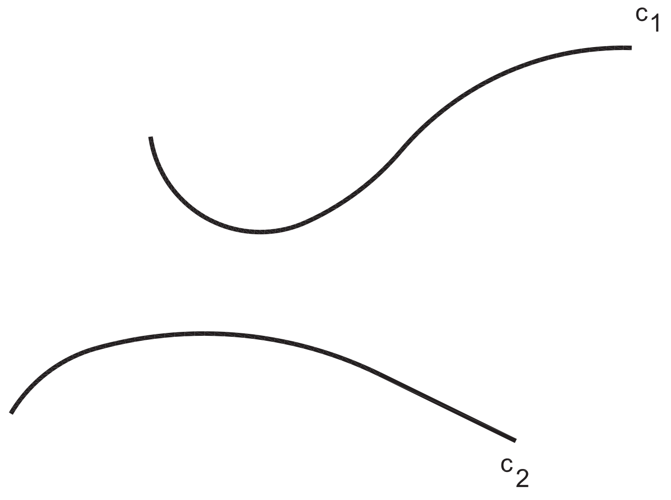

Figure 1.7: How can one construct a symmetry axis between these to curves?

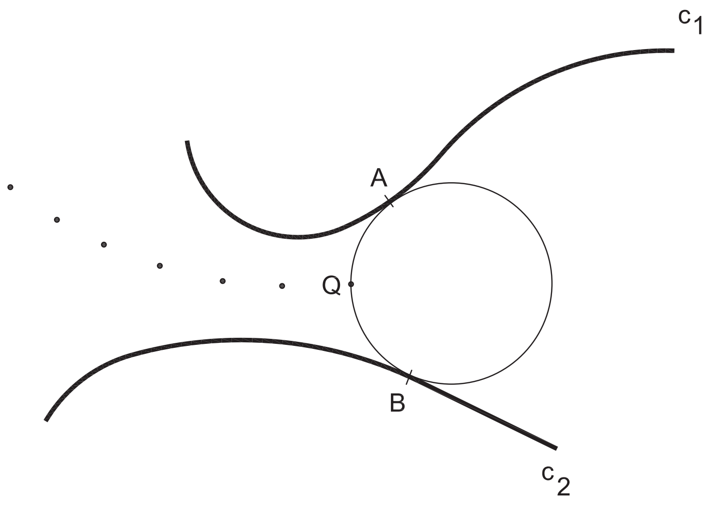

Figure 1.8: The points Q define the symmetry axis.

1.10 Symmetry-Curvature Duality

The previous section defined the symmetry axis for arbitrary smooth curves. The section preceding that considered curvature extrema. Curvature extrema and symmetry axes are two entirely different structural aspects of a shape. A curvature extremum is a point lying *on* a curve. In contrast, not only does a symmetry axis lie *off* a curve, it is in fact a relation *between* two curves.

Given the very different nature of curvature extrema and symmetry axes, mathematicians did not previously suspect that there was a relationship between the two. However, in the 1980s, I proved a theorem which shows that there is an extremely strong relationship between them – in fact, a duality. I called the theorem, the Symmetry-Curvature Duality Theorem. Since I published the theorem, it has been applied by scientists in over 40 disciplines, from DNA tracking to chemical engineering:

SYMMETRY-CURVATURE DUALITY THEOREM.
Leyton (1987)

> **Any section of smooth curve, with one and only one curvature extremum, has one and only one symmetry axis. This axis is forced to terminate at the extremum itself.**

To illustrate this theorem, consider the curve shown in Fig 1.9. It is part of a much larger curve. The part shown here has three curvature extrema labeled sequentially: m_1, M, and m_2.

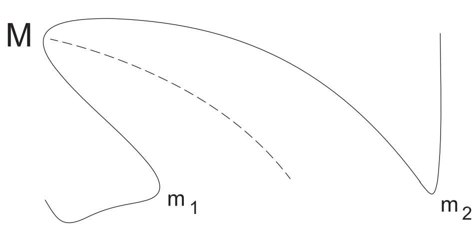

Figure 1.9: Illustration of the Symmetry-Curvature Duality Theorem.

Now, consider only the section of curve *between* the two extrema m_1 and m_2. This section is shaped like a *wave*. Most crucially, it has only *one* curvature extremum, M.

The question to be asked is this: How many symmetry axes does this section of curve possess? The above theorem gives us the answer. It says: Any section of curve with only one curvature extremum has only one symmetry axis. Thus we conclude that the section of curve containing only the extremum M can have only one axis.

The next question to be asked is this: Where does this symmetry axis go? Could it, for example hit the upper side or lower side of the wave? Again, the theorem provides us with the answer. It says that the axis is forced to terminate at the tip of the wave, i.e., the extremum M itself – as shown in Fig 1.9.

This theorem is enormously valuable in understanding the structure of any complex curve: Simply break down the curve into sections, each with only one curvature extremum. The theorem then tells us that each of these sections has only one symmetry axis, and that the axis terminates at the extremum.

Fig 1.10 illustrates this decompositional procedure. The curve has sixteen extrema. Thus, the theorem says that there must be sixteen symmetry axes associated with and terminating at those extrema. These axes are shown as the dashed lines on the figure.

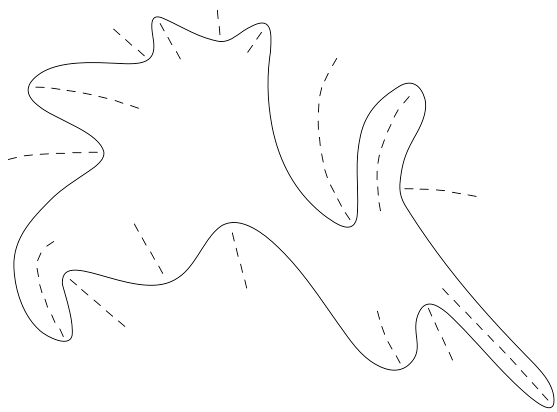

Figure 1.10: Sixteen extrema imply sixteen symmetry axes.

1.11 Curvature Extrema and the Symmetry Principle

Recall that the problem we are trying to solve is this: When presented with a shape like Fig 1.10, how can one convert it into a memory store, i.e., recover from it the process-history that produced it. Section 1.4 gave my two fundamental laws of memory storage, i.e., for the recovery of process-history from shape. These laws are the Asymmetry Principle, which states that any asymmetry in the present shape is assumed to have arisen from a past symmetry; and the Symmetry Principle, which states that any symmetry in the present shape is assumed to have always existed. Both principles must be applied to the shape. Let us first use the Symmetry Principle.

The Symmetry Principle demands that one must preserve symmetries in the shape, backwards in time. What are the symmetries? The previous section established significant symmetries in the shape: the symmetry axes illustrated in Fig 1.10, predicted by the Symmetry-Curvature Duality Theorem: i.e., those axes corresponding to the curvature extrema.

Now use the Symmetry Principle, which states that any symmetry must be preserved backward in time. In particular, it demands that the symmetry axes must be preserved backwards in time.