Lecture Notes in Geosystems Mathematics and Computing

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Waves in Continuous Media

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Preface

Wave phenomena occur everywhere in nature and therefore are studied in many areas of science for a long time.

The mathematical wave theory emerged as an independent discipline in the mid-1970s due to numerous applications in natural science and engineering stimulating the further development of mathematical methods.

The lecture course *Waves in Continuous Media* is one of the disciplines on continuum mechanics and mathematical modeling included into the education program at the Department of Mechanics and Mathematics, Novosibirsk State University. This course was first given by Professor L. V. Ovsyannikov,[1] a distinguished scientist who obtained a number of fundamental results in the field of wave hydrodynamics. Based on Ovsyannikov's principles of selecting the material, the authors developed new variants of the course adapted to groups of master's students specialized in applied mathematics, mechanics, and geophysics.

The textbook contains a rich collection of exercises and problems which have been carefully selected and tested at practical works and seminars of courses given by the authors at Novosibirsk State University (Russia) and Aix-Marseille University (France) for many years. Most of the problems and exercises are supplied with answers and hints. Solutions of some typical problems are explained in detail, and some theoretical background material is included in order to make the book self-contained and give students the necessary tools for self-education. More than 200 problems formulated in the book allowed us to propose to each master's student an individual semester mini-project consisting in solving up to six problems. Most of them are solved by applying the theoretical approaches from the course, but the other ones demand a deeper understanding of the methods discussed in the course. During the semester, the students have also been working in research laboratories, so a set of problems specific to the research activity of the students was usually proposed.

[1]Ovsyannikov, L. V.: Wave Motions of Continuous Media. Novosibirsk State University, Novosibirsk (1985) [in Russian].

The textbook consists of three chapters. Chapters 1 and 2 present the basic notions and facts of the mathematical theory of waves illustrated by numerous examples and methods of solving typical problems. The reader learns how to recognize the hyperbolicity property; find characteristics, Riemann invariants, and conservation laws for quasilinear systems of equations; construct and analyze solutions with weak or strong discontinuities; and investigate equations with dispersion: analysis of dispersion relations, the study of large time asymptotic behavior of solutions, the construction of traveling wave solutions for models reducible to nonlinear evolution equations, etc. The majority of problems are formulated within the framework of wave models arising in gas dynamics, magnetohydrodynamics, elasticity and plasticity, linear and nonlinear acoustics, chemical adsorption, and other applications.

Chapter 3 deals with surface and internal waves in an incompressible fluid. The efficiency of mathematical methods is demonstrated on a hierarchy of approximate submodels generated from the Euler equations of homogeneous and inhomogeneous fluids. Some problems illustrate the influence of viscosity and vorticity on the wave processes.

The list of references consists mainly of monographs and textbooks recommended for further reading. Three of them are generic [1–3], while others [4–33] are more specific for each chapter. These have been selected to allow readers to understand better mathematical statements whose proofs were skipped, and find solutions of relatively hard exercises. A separate bibliography for each chapter is maintained. The reference list for Chap. 3 also contains five research articles on the theory of water waves [16, 17, 21, 26, 31] we explicitly refer to. The books for further reading are not cited in the text.

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Chapter 1
Hyperbolic Waves

1.1 Hyperbolic Systems

We consider the quasilinear system of first order equations

$$\mathbf{u}_t + A(\mathbf{u}, x, t)\mathbf{u}_x + \mathbf{b}(\mathbf{u}, x, t) = 0, \tag{1.1}$$

where the $n \times n$-matrix A and vector $\mathbf{b}$ depend on x, t and $\mathbf{u} = (u_1, \ldots, u_n)^T$. A direction $dx/dt = c$ is called *characteristic* if there exists a linear combination of equations of the form (1.1) such that each unknown function u_i is differentiable along this direction. The quantity c in the definition of a characteristic direction is an eigenvalue of the matrix A, i.e.,

$$\det(A - cI) = 0. \tag{1.2}$$

For any eigenvalue c and the corresponding left eigenvector $\mathbf{l} = (l_1, \ldots, l_n)$ of the matrix A (i.e., $\mathbf{l}A = c\mathbf{l}$) the system (1.1) implies the following condition on the *characteristic* (the curve corresponding to a characteristic direction):

$$\mathbf{l} \cdot (d_t\mathbf{u} + \mathbf{b}) = 0, \tag{1.3}$$

where $d_t = \partial_t + c\partial_x$ is the operator of differentiation along the characteristic.

The system (1.1) is *hyperbolic* if all eigenvalues c_i of the matrix A are real (in this case, they can be ordered: $c_1 \leqslant c_2 \leqslant \ldots \leqslant c_n$) and there exist n linearly independent real left eigenvectors of the matrix A.

Figure 1.1 shows the location of characteristics emanating from a given point M in the (x, t)-plane. A hyperbolic system of equations is equivalent to a system of n relations on characteristics. A system is hyperbolic if and only if the normal Jordan form of the matrix A is diagonal. We indicate the sufficient hyperbolicity

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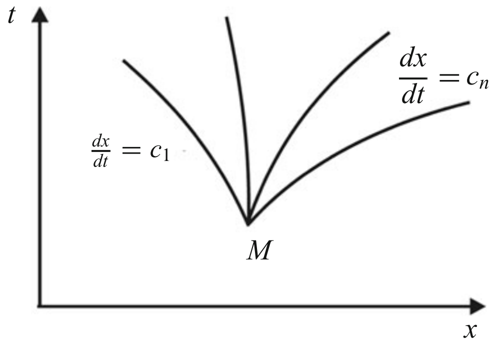

Fig. 1.1 Location of characteristics emanating from a given point M in the (x, t)-plane

conditions:

(a) the matrix A is symmetric,
(b) all roots of Eq. (1.2) are real and distinct.

In case (b), where the matrix A has no multiple eigenvalues, the system (1.1) is called *strictly hyperbolic*.

Example 1.1 The process of chemical adsorption used for separating substances in a liquid or gas mixture by the chromatography method is described by the equations

$$\partial_t (\mathbf{u} + \mathbf{f}(\mathbf{u})) + v\partial_x \mathbf{u} = 0, \tag{1.4}$$

where $\mathbf{u} = (u_1, \ldots, u_n)^T$ are the concentrations of the separated substances passing through the adsorption column, $\mathbf{f}(\mathbf{u}) = (f_1(u), \ldots, f_n(u))^T$ are the concentrations of the substances adsorbed by the adsorbent, and $v = \text{const} > 0$ is the mixture velocity. Let the vector-valued function $\mathbf{f}(\mathbf{u})$, called the *adsorption isotherm*, be such that all eigenvalues of the Jacobi matrix

$$\mathbf{f}'(\mathbf{u}) = \frac{\partial(f_1, \ldots, f_n)}{\partial(u_1, \ldots, u_n)}$$

are real, positive, and distinct:

$$0 < \lambda_1 < \ldots < \lambda_n.$$

Then Eqs. (1.4) can be transformed to the form (1.1) with the matrix $A(\mathbf{u}) = v(I + \mathbf{f}'(\mathbf{u}))^{-1}$ and vector $\mathbf{b} = 0$. Since

$$A - cI = ((v - c)I - c\mathbf{f}'(\mathbf{u}))(I + \mathbf{f}'(\mathbf{u}))^{-1},$$

the eigenvalues of the matrix A are connected with the eigenvalues of the matrix $\mathbf{f}'(\mathbf{u})$ by the identity

$$c_j = \frac{v}{1+\lambda_j} \quad (j = 1, 2, \ldots, n).$$

Consequently, the system (1.4) is strictly hyperbolic; moreover, all its characteristic velocities are positive and do not exceed the mixture velocity. The noncoincidence of the velocities $c_i \neq c_j$ $(i \neq j)$ is the basis of the chromatography method.

If there exist scalar functions $r(\mathbf{u})$ and $\mu(\mathbf{u}, x, t)$ such that

$$\frac{\partial r}{\partial u_i} = \mu l_i \quad (i = 1, \ldots, n),$$

then the relation (1.3) is equivalent to the equation

$$d_t r(\mathbf{u}) = -\mu \mathbf{l} \cdot \mathbf{b},$$

where $r(\mathbf{u})$ is called a *Riemann invariant*. The motivation of this definition becomes clear in the case $\mathbf{l} \cdot \mathbf{b} = 0$, where the Riemann invariant r is constant along characteristics. Riemann invariants always exist for the system (1.1) consisting of one or two equations and for the system (1.1) with constant matrix A of arbitrary order n (in the second case, $r = \mathbf{l} \cdot \mathbf{u}$). In the general case $n \geqslant 3$, Riemann invariants do not necessarily exist. In the case $n = 3$, the identity $\mathbf{l} \cdot \operatorname{curl} \mathbf{l} = 0$ is necessary and sufficient for the existence of a Riemann invariant for the characteristic $dx/dt = c$ corresponding to a simple eigenvalue c of the matrix A with eigenvector $\mathbf{l}(\mathbf{u}) = (l_1, l_2, l_3)$.

Problem 1.1 Find characteristics and Riemann invariants for the system describing shallow water flows over the flat bottom

$$\begin{aligned} &h_t + (uh)_x = 0, \\ &u_t + uu_x + gh_x = 0, \end{aligned} \tag{1.5}$$

where $h(x, t)$ is the layer depth, $u(x, t)$ is the horizontal fluid velocity, and g is the acceleration of gravity.

Solution We compose the matrix of coefficients of the original system of equations

$$A - cI = \begin{pmatrix} u - c & h \\ g & u - c \end{pmatrix}.$$

Then we find the characteristic velocities $c^{\pm} = u \pm \sqrt{gh}$. The system is hyperbolic in the domain $h > 0$. For the characteristic $dx/dt = c^{+}$ the left eigenvector, defined up to an arbitrary scalar factor, has the form $\mathbf{l} = (\sqrt{g}, \sqrt{h})$. Consequently, to find the Riemann invariant $r(h, u)$, we should find a solution to the system of equations

$$\frac{\partial r}{\partial h} = \mu\sqrt{g},$$
$$\frac{\partial r}{\partial u} = \mu\sqrt{h},$$

where $\mu(h, u)$ is an unknown integrating factor. Excluding this factor, we obtain the linear first order partial differential equation for r

$$\frac{\partial r}{\partial u} - \sqrt{\frac{h}{g}}\frac{\partial r}{\partial h} = 0.$$

From the equation of characteristics $du = -\sqrt{g/h}dh$ we find the first integral $r = u + 2\sqrt{gh}$. Since there is a certain functional arbitrariness in the definition of a Riemann invariant, the obtained first integral can be taken for the sought invariant. The characteristic $dx/dt = c^-$ is studied in a similar way. □

Answer:

$$\frac{dx}{dt} = u + \sqrt{gh}: \quad u + 2\sqrt{gh} = \text{const}\,,$$
$$\frac{dx}{dt} = u - \sqrt{gh}: \quad u - 2\sqrt{gh} = \text{const}\,.$$

1.2 Propagation of Weak Discontinuities

The Cauchy problem for the system (1.1) is to find a solution for $t > t_0$ provided that $u_i(x, t_0) = u_{i0}(x)$ is given at $t = t_0$.

Theorem 1.1 (uniqueness) *Assume that the system* (1.1) *is hyperbolic and the coefficient matrix A and vector* $\mathbf{b}$ *are continuously differentiable. Let a continuously differentiable solution* $\mathbf{u}(x, t)$ *be defined in the characteristic triangle X_1MX_n (cf. Fig.* 1.2*). If* $\overline{\mathbf{u}}$ *is another continuously differentiable solution to the system* (1.1) *in X_1MX_n and* $\overline{\mathbf{u}} = \mathbf{u}$ *on the segment X_nX_1, then* $\overline{\mathbf{u}} = \mathbf{u}$ *in the entire characteristic triangle X_1MX_n.*

This theorem implies the existence of wave fronts defined by the characteristics X_1M and X_nM and defining the *domain of determinacy* of the solution to the Cauchy problem by the initial data solely at the *domain of dependence* X_nX_1 of the point M.

Let a domain D be divided by a smooth curve $\Gamma : x = \chi(t)$ into two subdomains D_- and D_+ (cf. Fig. 1.3).

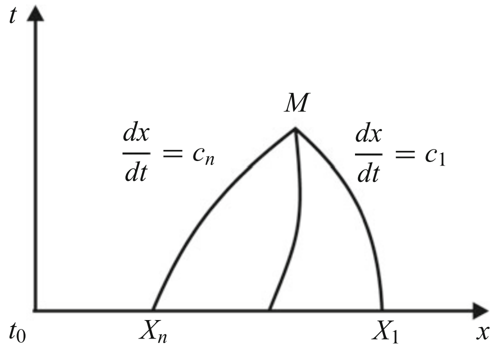

Fig. 1.2 The solution to the Cauchy problem for the system (1.1) is uniquely determined by the initial data on the interval X_nX_1 inside the curvilinear characteristic triangle X_1MX_n

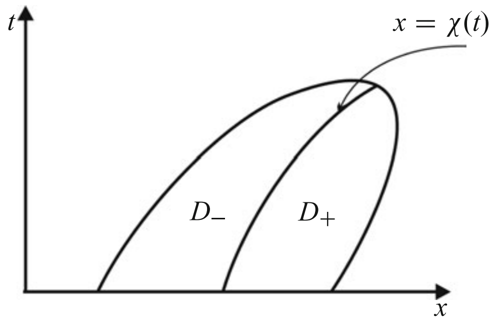

Fig. 1.3 The weak discontinuities always propagate along the characteristic curves

We assume that the solution to the hyperbolic system is continuous in the closed domain $\overline{D}$ and continuously differentiable in the closures $\overline{D}_-$ and $\overline{D}_+$. Moreover, the derivative $\partial_x \mathbf{u} = \mathbf{v}$ of the solution on Γ can have a discontinuity of the first kind with jump $[\mathbf{v}] = \mathbf{v}_+ - \mathbf{v}_-$. By the continuity of the solution $\mathbf{u}$, the jump of its tangent derivative $d_t\mathbf{u} = \partial_t \mathbf{u} + \chi'(t)\partial_x \mathbf{u}$ on Γ vanishes, which implies the following expressions for the jumps of derivatives:

$$[\partial_x \mathbf{u}] = [\mathbf{v}],$$
$$[\partial_t \mathbf{u}] = -\chi'[\mathbf{v}].$$

Consequently, from the system (1.1) we have

$$(A - \chi' I)[\mathbf{v}] = 0.$$

Thus, the derivative of the solution can be discontinuous only on the characteristic; moreover, the jump is a right eigenvector of the matrix A. In the case of a simple eigenvalue of the matrix A, the amplitude of the weak discontinuity is characterized by a scalar σ such that $[\mathbf{v}] = \sigma \mathbf{r}$, where $\mathbf{r} = (r_1, \ldots, r_n)^T$ is the corresponding right eigenvector. The quantity σ satisfies the ordinary differential equation along the characteristic

$$(\mathbf{l} \cdot \mathbf{r}) \frac{d\sigma}{dt} + P\sigma + Q\sigma^2 = 0, \tag{1.6}$$

where $\mathbf{l}$ is the corresponding left eigenvector, whereas P and Q are known functions. In particular,

$$Q = \sum_{i,j,k=1}^{n} l_i \left(\frac{\partial a_{ij}}{\partial u_k} \right) r_k r_j,$$

where a_{ij} are components of the matrix A. Without loss of generality we can assume that $(\mathbf{l} \cdot \mathbf{r}) = 1$. The relation (1.6) is the Riccati equation. It is called the *transport equation for the amplitude of the weak discontinuity*.

Problem 1.2 Consider the system of equations describing the isentropic motion of a polytropic gas, written in terms of the Riemann invariants r and l,

$$\begin{cases} r_t + (u + c) r_x = 0, \\ l_t + (u - c) l_x = 0, \end{cases} \qquad r = u + \frac{2}{\gamma - 1} c, \quad l = u - \frac{2}{\gamma - 1} c,$$

with the initial conditions

$$u(x, 0) = \begin{cases} 0, & x \geqslant a, \\ c_0 (x - a)/(l_0 + a - x), & x < a, \end{cases} \qquad c(x, 0) = c_0,$$

where $a = \text{const}$, $c_0 = \text{const}$, and $l_0 = \text{const}$ ($c_0 > 0$, $l_0 > 0$). Compute the jump $[u_x]$ of the derivative on the characteristic $x = c_0 t + a$ at time t.

Solution By the uniqueness theorem for the Cauchy problem, $u(x, t) \equiv 0$ and $c(x, t) \equiv c_0$ for $x \geqslant c_0 t + a$. Further, for the Riemann invariant l along the characteristic $dx/dt = u + c$ we have

$$[l_t] + (u - c)[l_x] = 0,$$

$$[l_t] + (u + c)[l_x] = 0,$$

where first relation immediately follows from the equation of motion, whereas the second one is obtained from the continuity of the tangent derivative of l on the weak discontinuity line. Therefore, along the characteristic under consideration, we have $[l_t] = 0$, $[l_x] = 0$, and, as a consequence, $[u_x] = [r_x]/2$. Differentiating the first equation of the original system with respect to x, considering the jump, and taking into account the above properties $l_x(x - 0, t) = l_x(x + 0, t) = 0$ for $x = c_0 t + a$, we