DANIEL J. INMAN

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WITH CONTROL

SECOND EDITION

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Vibration with Control

Vibration with Control

Second Edition

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This book is dedicated to our grand children

Griffin Amherst Pitre
Benjamin Lafe Scamacca
Lauren Cathwren Scamacca
Conor James Scamacca
Jacob Carlin Scamacca

And our children

Daniel John Inman, II
Angela Wynne Pitre
Jennifer Wren Scamacca

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Preface

Advance level vibration topics are presented here, including lumped mass and distributed mass systems in the context of the appropriate mathematics along with topics from control that are useful in vibration analysis, testing and design. This text is intended for use in a second course in vibration, or a combined course in vibration and control. It is also intended as a reference for the field of structural control and could be used as a text in structural control. The control topics are introduced at the beginners' level with no prerequisite knowledge in controls needed to read the book.

The text is an attempt to place vibration and control on a firm mathematical basis and connect the disciplines of vibration, linear algebra, matrix computations, control and applied functional analysis. Each chapter ends with notes on further references and suggests where more detailed accounts can be found. In this way I hope to capture a "bigger picture" approach without producing an overly large book. The first chapter presents a quick introduction using single degree of freedom systems (second-order ordinary differential equations) to the following chapters, which extend these concepts to multiple degree of freedom systems (matrix theory and systems of ordinary differential equations) and distributed parameter systems (partial differential equations and boundary value problems). Numerical simulations and matrix computations are also presented through the use of MATLAB™.

New In This Edition – The book chapters have been reorganized (there are now 12 instead of 13 chapters) with the former chapter on design removed and combined with the former chapter on control to form a new chapter titled *Vibration Suppression*. Some older, no longer used material, has been deleted in an attempt to keep the book limited in size as new material has been added.

The new material consists of adding several modeling sections to the text, including corresponding problems and examples. Many figures have been redrawn throughout to add clarity with more descriptive captions. In addition, a number of new figures have been added. New problems and examples have been added and some old ones removed. In total, seven new sections have been added to introduce modeling, coupled systems, the use of piezoelectric materials, metastructures, and validation and verification.

Instructor Support – Power Point slides are available for presentation of the material, along with a complete solutions manual. These materials are available from the publisher for those who have adapted the book. The author is pleased to answer questions via the email listed below.

Student Support – The best place to get help is your instructor and others in your peer group through discussion of the material. There are also many excellent texts as referenced throughout the book and of course Internet searches can provide lots of help. In addition, feel free to email the author at the address below (but don't ask me to do your homework!).

Acknowledgements – I would like to thank two of my current PhD students, Katie Reichl and Brittany Essink, for checking some of the homework and providing some plots. I would like to thank all of my former and current PhD students for 36 years of wonderful research and discussions. Thanks are owed to the instructors and students of the previous edition who have sent suggestions and comments. Last, thanks to my lovely wife Catherine Ann Little for putting up with me.

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About the Companion Website

Vibration with Control, Second Edition is accompanied by a companion website:

www.wiley.com/go/inmanvibrationcontrol2e

The website includes:

- Powerpoint slides
- Solutions manual

1

Single Degree of Freedom Systems

1.1 Introduction

In this chapter, the vibration of a single degree of freedom system (SDOF) will be analyzed and reviewed. Analysis, measurement, design and control of SDOF systems are discussed. The concepts developed in this chapter constitute a review of introductory vibrations and serve as an introduction for extending these concepts to more complex systems in later chapters. In addition, basic ideas relating to measurement and control of vibrations are introduced that will later be extended to multiple degree of freedom systems and distributed parameter systems. This chapter is intended to be a review of vibration basics and an introduction to a more formal and general analysis for more complicated models in the following chapters.

Vibration technology has grown and taken on a more interdisciplinary nature. This has been caused by more demanding performance criteria and design specifications of all types of machines and structures. Hence, in addition to the standard material usually found in introductory chapters of vibration and structural dynamics texts, several topics from control theory are presented. This material is included not to train the reader in control methods (the interested student should study control and system theory texts), but rather to point out some useful connections between vibration and control as related disciplines. In addition, structural control has become an important discipline requiring the coalescence of vibration and control topics. A brief introduction to nonlinear SDOF systems and numerical simulation is also presented.

1.2 Spring-Mass System

Simple harmonic motion, or oscillation, is exhibited by structures that have elastic restoring forces. Such systems can be modeled, in some situations, by a spring-mass schematic (Figure 1.1). This constitutes the most basic vibration model of a structure and can be used successfully to describe a surprising number of devices, machines and structures. The methods presented here for solving such a simple mathematical model may seem to be more sophisticated than the problem

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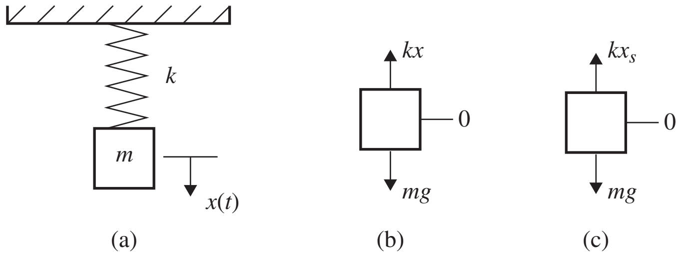

Figure 1.1 (a) A spring-mass schematic, (b) a free body diagram, and (c) a free body diagram of the static spring mass system.

requires. However, the purpose of this analysis is to lay the groundwork for solving more complex systems discussed in the following chapters.

If $x = x(t)$ denotes the displacement (in meters) of the mass m (in kg) from its equilibrium position as a function of time, t (in sec), the equation of motion for this system becomes (upon summing the forces in Figure 1.1b)

$$m\ddot{x} + k(x + x_s) - mg = 0$$

where k is the stiffness of the spring (N/m), x_s is the static deflection (m) of the spring under gravity load, g is the acceleration due to gravity (m/s^2) and the over dots denote differentiation with respect to time. A discussion of dimensions appears in Appendix A and it is assumed here that the reader understands the importance of using consistent units. From summing forces in the free body diagram for the static deflection of the spring (Figure 1.1c), $mg = kx_s$ and the above equation of motion becomes

$$m\ddot{x}(t) + kx(t) = 0 \tag{1.1}$$

This last expression is the equation of motion of an SDOF system and is a linear, second-order, ordinary differential equation with constant coefficients.

Figure 1.2 indicates a simple experiment for determining the spring stiffness by adding known amounts of mass to a spring and measuring the resulting static deflection, x_s. The results of this static experiment can be plotted as force (mass times acceleration) versus x_s, the slope yielding the value of k for the linear portion of the plot. This is illustrated in Figure 1.3.

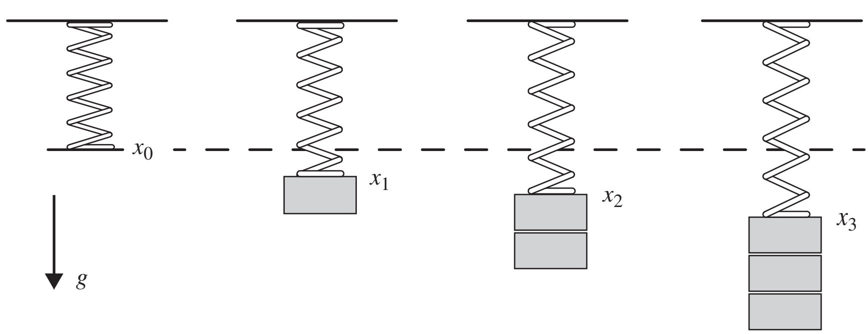

Figure 1.2 Measurement of spring constant using static deflection caused by added mass.

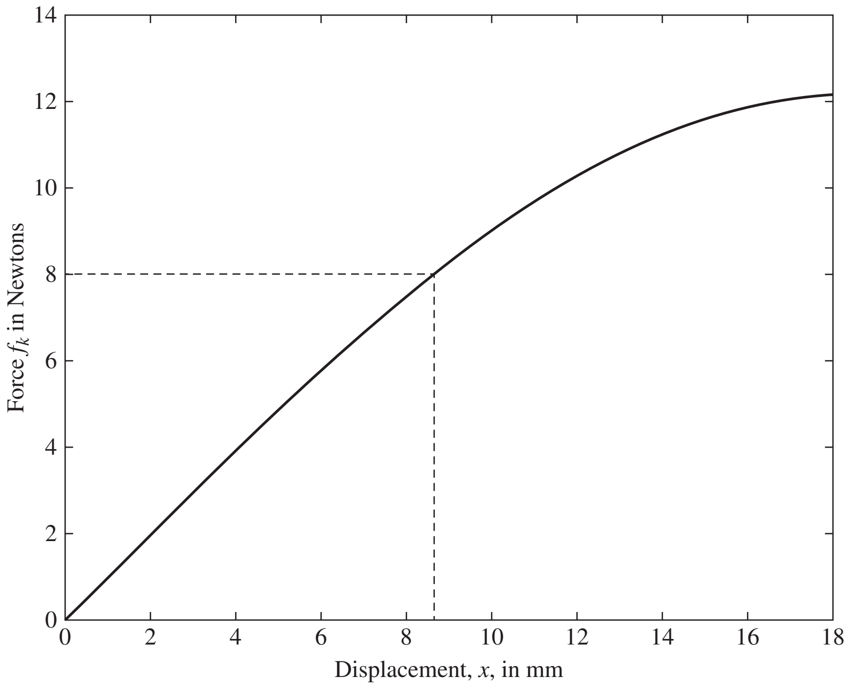

Figure 1.3 Determination of the spring constant. The dashed box indicates the linear range of the spring.

Once m and k are determined from static experiments, Equation (1.1) can be solved to yield the time history of the position of the mass m, given the initial position and velocity of the mass. The form of the solution of Equation (1.1) is found from substitution of an assumed periodic motion (from experience watching vibrating systems) of the form

$$x(t) = A\sin(\omega_n t + \phi) \tag{1.2}$$

where $\omega_n = \sqrt{k/m}$ is called the *natural frequency* in radians per second (rad/s). Here A, the *amplitude*, and ϕ, the *phase shift*, are constants of integration determined by the initial conditions.

The existence of a *unique* solution for Equation (1.1) with two specific initial conditions is well known and is given in Boyce and DiPrima (2012). Hence, if a solution of the form of Equation (1.2) is guessed and it works, then it is *the* solution. Fortunately, in this case, the mathematics, physics and observation all agree.

To proceed, if x_0 is the specified initial displacement from equilibrium of mass m, and v_0 is its specified initial velocity, simple substitution allows the constants of integration A and ϕ to be evaluated. The unique solution is

$$x(t) = \sqrt{\frac{\omega_n^2 x_0^2 + v_0^2}{\omega_n^2}} \sin\left[\omega_n t + \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right)\right] \tag{1.3}$$

Alternately, $x(t)$ can be written as

$$x(t) = \frac{v_0}{\omega_n} \sin \omega_n t + x_0 \cos \omega_n t \tag{1.4}$$

by using a simple trigonometric identity or by direct substitution of the initial conditions (Example 1.2.1).

A purely mathematical approach to the solution of Equation (1.1) is to assume a solution of the form $x(t) = Ae^{\lambda t}$ and solve for λ, i.e.

$$m\lambda^2 e^{\lambda t} + ke^{\lambda t} = 0$$

This implies that (because $e^{\lambda t} \neq 0$ and $A \neq 0$)

$$\lambda^2 + \left(\frac{k}{m}\right) = 0$$

or that

$$\lambda = \pm j \left(\frac{k}{m}\right)^{1/2} = \pm\omega_n j$$

where $j = (-1)^{1/2}$. Then the general solution becomes

$$x(t) = A_1 e^{-\omega_n jt} + A_2 e^{\omega_n jt} \tag{1.5}$$

where A_1 and A_2 are arbitrary complex conjugate constants of integration to be determined by the initial conditions. Use of Euler's formulas then yields Equations (1.2) and (1.4) (Inman, 2014). For more complicated systems, the exponential approach is often more appropriate than first guessing the form (sinusoid) of the solution from watching the motion.

Another mathematical comment is in order. Equation (1.1) and its solution are valid only as long as the spring is linear. If the spring is stretched too far or too much force is applied to it, the curve in Figure 1.3 will no longer be linear. Then Equation (1.1) will be nonlinear (Section 1.10). For now, it suffices to point out that initial conditions and springs should always be checked to make sure that they fall into the linear region, if linear analysis methods are going to be used.

Example 1.2.1

Assume a solution of Equation (1.1) of the form

$$x(t) = A_1 \sin \omega_n t + A_2 \cos \omega_n t$$

and calculate the values of the constants of integration A_1 and A_2 given arbitrary initial conditions x_0 and v_0, thus verifying Equation (1.4).

Solution: The displacement at time $t = 0$ is

$$x(0) = x_0 = A_1 \sin(0) + A_2 \cos(0)$$

or $A_2 = x_0$. The velocity at time $t = 0$ is

$$\dot{x}(0) = v_0 = \omega_n A_1 \cos(0) - \omega_n x_0 \sin(0)$$

Solving this last expression for A_1 yields $A_1 = v_0/x_0$, so that Equation (1.4) results in

$$x(t) = \frac{v_0}{x_0}\sin\omega_n t + x_0\cos\omega_n t$$

Example 1.2.2
Compute and plot the time response of a linear spring-mass system to initial conditions of $x_0 = 0.5$ mm and $v_0 = 2\sqrt{2}$ mm/s, if the mass is 100 kg and the stiffness is 400 N/m.

Solution: The frequency is

$$\omega_n = \sqrt{k/m} = \sqrt{400/100} = 2 \text{ rad/s}$$

Next compute the amplitude from Equation (1.3):

$$A = \sqrt{\frac{\omega_n^2 x_0^2 + v_0^2}{\omega_n^2}} = \sqrt{\frac{2^2(0.5)^2 + (2\sqrt{2})^2}{2^2}} = 1.5 \text{ mm}$$

From Equation (1.3) the phase is

$$\phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right) = \tan^{-1}\left(\frac{2(0.5)}{2\sqrt{2}}\right) \approx 10 \text{ rad}$$

Thus the response has the form

$$x(t) = 1.5\sin(2t + 10)$$

and this is plotted in Figure 1.4.

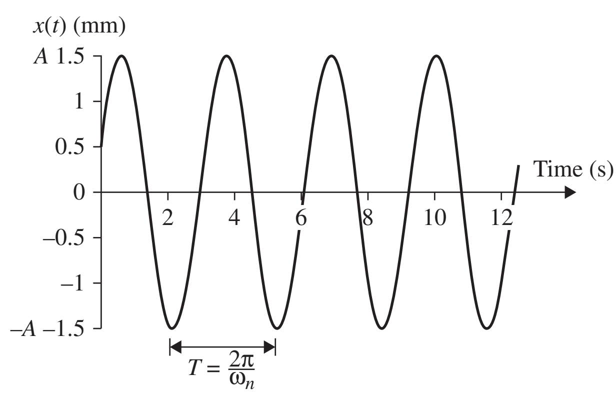

Figure 1.4 The response of a simple spring-mass system to an initial displacement of $x_0 = 0.5$ mm and an initial velocity of $v_0 = 2\sqrt{2}$ mm/s. The period, defined as the time it takes to complete one cycle off oscillation, $T = 2\pi/\omega_n$, becomes $T = 2\pi/2 = \pi$s.

1.3 Spring-Mass-Damper System

Most systems will not oscillate indefinitely when disturbed, as indicated by the solution in Equation (1.3). Typically, the periodic motion dies down after some time. The easiest way to treat this mathematically is to introduce a velocity term, $c\dot{x}$, into Equation (1.1) and examine the equation

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{1.6}$$

This also happens physically with the addition of a *dashpot* or *damper* to dissipate energy, as illustrated in Figure 1.5.

Equation (1.6) agrees with summing forces in Figure 1.5 if the dashpot exerts a dissipative force proportional to velocity on the mass m. Unfortunately, the constant of proportionality, c, cannot be measured by static methods as m and k are. In addition, many structures dissipate energy in forms not proportional to velocity. The constant of proportionality c is given in Newton-second per meter (Ns/m) or kilograms per second (kg/s) in terms of fundamental units.

Again, the unique solution of Equation (1.6) can be found for specified initial conditions by assuming that $x(t)$ is of the form

$$x(t) = Ae^{\lambda t}$$

and substituting this into Equation (1.6) to yield

$$A\left(\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m}\right)e^{\lambda t} = 0 \tag{1.7}$$

Since a trivial solution is not desired, $A \neq 0$, and since $e^{\lambda t}$ is never zero, Equation (1.7) yields

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0 \tag{1.8}$$

Equation (1.8) is called the *characteristic equation* of Equation (1.6). Using simple algebra, the two solutions for λ are

$$\lambda_{1,2} = -\frac{c}{2m} \pm \frac{1}{2}\sqrt{\frac{c^2}{m^2} - 4\frac{k}{m}} \tag{1.9}$$

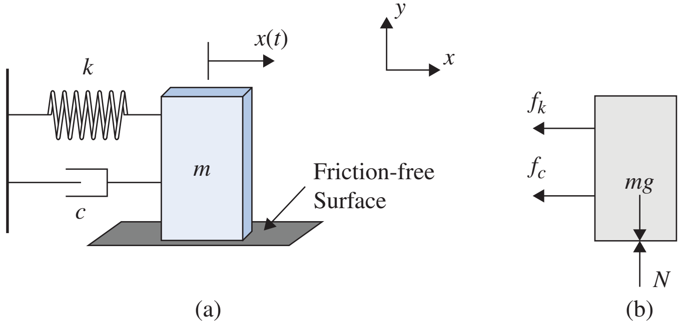

Figure 1.5 (a) Schematic of spring-mass-damper system. (b) A free-body diagram of the system in part (a).

The quantity under the radical is called the *discriminant* and together with the sign of m, c and k determines whether or not the roots are complex or real. Physically, m, c and k are all positive in this case, so the value of the discriminant determines the nature of the roots of Equation (1.8).

It is convenient to define the dimensionless *damping ratio*, ζ, as

$$\zeta = \frac{c}{2\sqrt{km}}$$

In addition, let the *damped natural frequency*, ω_d, be defined by (for $0 < \zeta < 1$)

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} \tag{1.10}$$

Then Equation (1.6) becomes

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0 \tag{1.11}$$

and Equation (1.9) becomes

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm \omega_d j, \quad 0 < \zeta < 1 \tag{1.12}$$

Clearly the value of the damping ratio, ζ, determines the nature of the solution of Equation (1.6). There are three cases of interest. The derivation of each case is left as an exercise and can be found in almost any introductory text on vibrations (Inman, 2014; Meirovitch, 1986).

Underdamping occurs if the system's parameters are such that

$$0 < \zeta < 1$$

so that the discriminant in Equation (1.12) is negative and the roots form a complex conjugate pair of values. The solution of Equation (1.11) then becomes

$$x(t) = e^{-\zeta\omega_n t}(A \cos \omega_d t + B \sin \omega_d t) \tag{1.13}$$

or

$$x(t) = Ce^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$$

where A, B, C and ϕ are constants determined by the specified initial velocity, v_0 and position, x_0

$$\begin{aligned} A &= x_0 & C &= \frac{\sqrt{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}}{\omega_d} \\ B &= \frac{(v_0 + \zeta\omega_n x_0)}{\omega_d} & \phi &= \tan^{-1}\left[\frac{x_0\omega_d}{(v_0 + \zeta\omega_n x_0)}\right] \end{aligned} \tag{1.14}$$

The underdamped response has the form given in Figure 1.6 and consists of a decaying oscillation of frequency ω_d.

Overdamping occurs if the system's parameters are such that

$$\zeta > 1$$

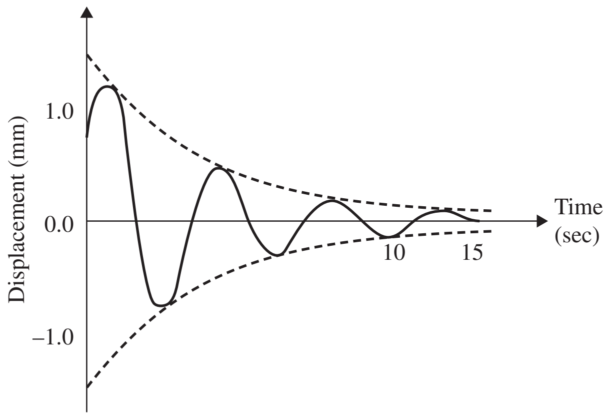

Figure 1.6 Response of an underdamped system illustrating oscillation with exponential decay.

so that the discriminant in Equation (1.12) is positive and the roots are a pair of negative real numbers. The solution of Equation (1.11) then becomes

$$x(t) = Ae^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + Be^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} \tag{1.15}$$

where A and B are again constants determined by v_0 and x_0. They are

$$A = \frac{v_0 + \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}} \text{ and } B = -\frac{v_0 + \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}} \tag{1.16}$$

The overdamped response has the form given in Figure 1.7. An overdamped system does not oscillate, but rather returns to its rest position exponentially.

Critical Damping occurs if the system's parameters are such that $\zeta = 1$, so that the discriminant in Equation (1.12) is zero and the roots are a pair of negative real repeated numbers. The solution of Equation (1.11) then becomes

$$x(t) = e^{-\omega_n t}[(v_0 + \omega_n x_0)t + x_0] \tag{1.17}$$

The critically damped response is plotted in Figure 1.8 for values of the initial velocity v_0 of different signs and $x_0 = 0.25$ mm.

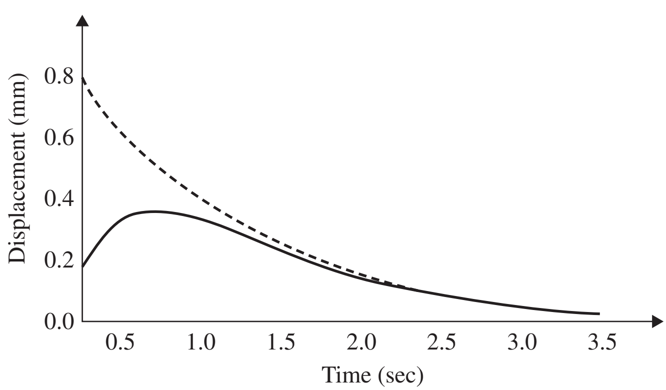

Figure 1.7 Response of an overdamped system illustrating exponential decay without oscillation.

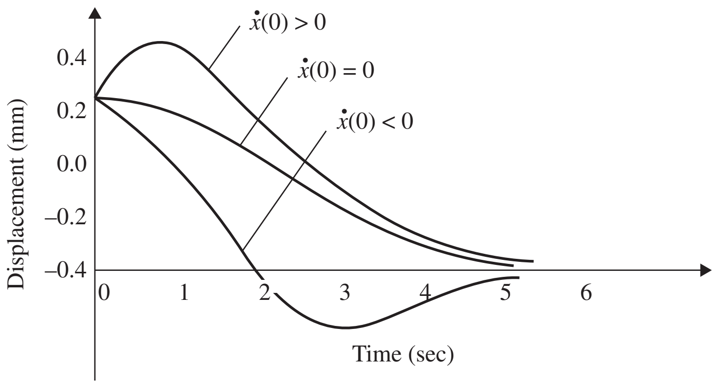

Figure 1.8 Response of critically damped system to an initial displacement and three different initial velocities indicating no oscillation.

It should be noted that critically damped systems can be thought of in several ways. First, they represent systems with the minimum value of damping rate that yields a non-oscillating system (Exercise 1.5). Critical damping can also be thought of as the case that separates non-oscillation from oscillation.

Example 1.3.1

Derive the constants A and B of integration for the overdamped case of Equation (1.15).

Solution: Substitution of $x(0) = x_0$ into Equation (1.15) yields

$$x(0) = Ae^0 + Be^0 \text{ or } x_0 = A + B \tag{1.18}$$

Differentiating Equation (1.15) and setting $t = 0$ in the result yields

$$\dot{x}(0) = A\lambda_1 e^0 + B\lambda_2 e^0 \text{ or } v_0 = \lambda_1 A + \lambda_1 B \tag{1.19}$$

where λ_1 and λ_2 are defined in Equation (1.12). These two initial conditions result in two independent equations in two unknowns, A and B, which can be solved in many ways. Writing Equations (1.17) and (1.18) as a single matrix equation yields

$$\begin{bmatrix} x_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \text{ or } \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}^{-1} \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}$$

Solving by computing matrix inverse (see Appendix B for details on computing a matrix inverse) yields

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}$$

Expanding, substituting in the values for λ_1 and λ_2, recalling that they are real numbers (i.e. $\zeta^2 > 1$) and writing as two separate equations results in

$$A = \frac{-v_0 + (-\zeta - \sqrt{\zeta^2 - 1})\omega_n}{-2\omega_n\sqrt{\zeta^2 - 1}} \text{ and } B = \frac{v_0 + (\zeta - \sqrt{\zeta^2 - 1})\omega_n}{-2\omega_n\sqrt{\zeta^2 - 1}}$$

Factoring out the minus sign in the denominator results in Equations (1.16).

1.4 Forced Response

The preceding analysis considers the vibration of a device or structure due to some initial disturbance (nonzero v_0 and x_0). In this section, the vibration of a spring-mass-damper system subjected to an external force is considered. In particular, the response to harmonic excitations, impulses and step forcing functions is examined.

In many environments, rotating machinery, motors, etc., cause periodic motions of structures to induce vibrations into other mechanical devices and structures nearby. It is common to approximate the driving forces, $F(t)$, as periodic of the form

$$F(t) = F_0 \sin \omega t \tag{1.20}$$

where F_0 represents the amplitude of the applied force and ω denotes the frequency of the applied force, or the driving frequency, in rad/s. On summing forces, the equation for the forced vibration of the system in Figure 1.9 becomes

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t \tag{1.21}$$

Recall from the discipline of differential equations (Boyce and DiPrima, 2012), that the solution of Equation (1.21) consists of the sum of the homogeneous solution Equation (1.5) and a particular solution. These are usually referred to as the *transient response* and the *steady-state response*, respectively. Physically, there is motivation to assume that the steady state response will follow the forcing function. Hence, it is tempting to assume that the particular solution has the form

$$x_p(t) = X \sin(\omega t - \theta) \tag{1.22}$$

where X is the steady-state amplitude and θ is the phase shift at steady state. Mathematically, the method is referred to as the method of undetermined coefficients. Substitution of Equation (1.22) into Equation (1.21) yields

$$X = \frac{F_0/k}{\sqrt{(1 - m\omega^2/k)^2 + (c\omega/k)^2}}$$

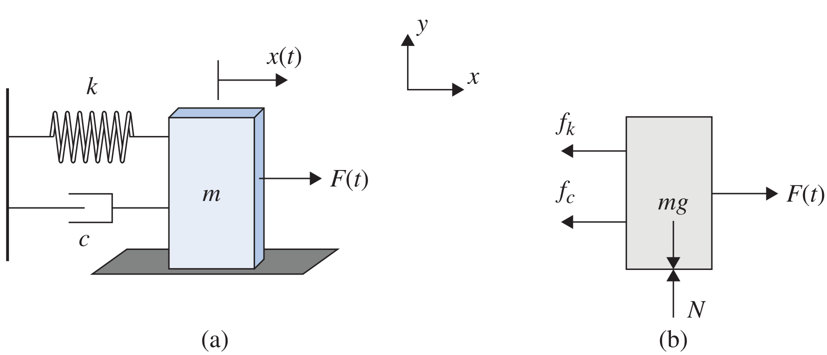

Figure 1.9 **(a)** The schematic of the forced spring-mass-damper system, assuming no friction on the surface. **(b)** The free-body diagram of the system of part (a).

or

$$\frac{Xk}{F_0} = \frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + [2\zeta(\omega/\omega_n)]^2}} \tag{1.23}$$

and

$$\tan\theta = \frac{(c\omega/k)}{1 - m\omega^2/k} = \frac{2\zeta(\omega/\omega_n)}{1 - (\omega/\omega_n)^2} \tag{1.24}$$

where $\omega_n = \sqrt{k/m}$ as before. Since the system is linear, the sum of two solutions is a solution, and the total time response for the system in Figure 1.9 for the case $0 < \zeta < 1$ becomes

$$x(t) = e^{-\zeta\omega_n t}(A\sin\omega_d t + B\cos\omega_d t) + X\sin(\omega t - \theta) \tag{1.25}$$

Here A and B are constants of integration determined by the initial conditions and the forcing function (and in general will be different than the values of A and B determined for the free response). See Examples 1.4.2 and 1.5.1 for the case where the driving force is a cosine function.

Examining Equation (1.25), two features are important and immediately obvious. First, as t gets larger, the transient response (the first term) becomes very small – hence the term steady-state response is assigned to the particular solution (the second term). The second observation is that the coefficient of the steady state response, or particular solution, becomes large when the excitation frequency is close to the undamped natural frequency, i.e. $\omega \approx \omega_n$. This phenomenon is known as *resonance* and is extremely important in design, vibration analysis and testing.

Example 1.4.1

Compute the response of the following system (assuming consistent units)

$$\ddot{x}(t) + 0.4\dot{x}(t) + 4x(t) = \frac{1}{\sqrt{2}}\sin 3t, \quad x(0) = \frac{-3}{\sqrt{2}}, \quad \dot{x}(0) = 0$$

Solution: First solve for the particular solution by using the more convenient form of

$$x_p(t) = X_1\sin 3t + X_2\cos 3t$$

rather than the magnitude and phase form, where X_1 and X_2 are the constants to be determined. Differentiating x_p yields

$$\dot{x}_p(t) = 3X_1\cos 3t - 3X_2\sin 3t$$
$$\ddot{x}_p(t) = -9X_1\sin 3t - 9X_2\cos 3t$$

Substitution of x_p and its derivatives into the equation of motion and collecting like terms yields

$$\left(-9X_1 - 1.2X_2 + 4X_1 - \frac{1}{\sqrt{2}}\right)\sin 3t + (-9X_2 + 1.2X_1 + 4X_2)\cos 3t = 0$$

Since the sine and cosine are independent, the two coefficients in parenthesis must vanish, resulting in two equations in the two unknowns, X_1 and X_2. This solution yields

$$x_p(t) = -0.134 \sin 3t - 0.032 \cos 3t$$

Next consider adding the free response to this. From the problem statement

$$\omega_n = 2 \text{ rad/s}, \quad \zeta = \frac{0.4}{2\omega_n} = 0.1 < 1, \quad \omega_d = \omega_n \sqrt{1-\zeta^2} = 1.99 \text{ rad/s}$$

Thus, the system is underdamped, and the total solution is of the form

$$x(t) = e^{-\zeta\omega_n t}(A \sin \omega_d t + B \cos \omega_d t) + X_1 \sin \omega t + X_2 \cos \omega t$$

Applying the initial conditions requires the derivative

$$\dot{x}(t) = e^{-\zeta\omega_n t}(\omega_d A \cos \omega_d t - \omega_d B \sin \omega_d t) + \omega X_1 \cos \omega t \\ - \omega X_2 \sin \omega t - \zeta\omega_n e^{-\zeta\omega_n t}(A \sin \omega_d t + B \cos \omega_d t)$$

The initial conditions yield the constants A and B

$$x(0) = B + X_2 = \frac{-3}{\sqrt{2}} \Rightarrow B = -X_2 - \frac{3}{\sqrt{2}} = -2.089$$

$$\dot{x}(0) = \omega_d A + \omega X_1 - \zeta\omega_n B = 0 \Rightarrow A = \frac{1}{\omega_d}(\zeta\omega_n B - \omega X_1) = -0.008$$

Thus the total solution is

$$x(t) = -e^{-0.2t}(0.008 \sin 1.99t + 2.089 \cos 1.99t) - 0.134 \sin 3t - 0.032 \cos 3t$$

Example 1.4.2

Calculate the form of the forced response if, instead of a sinusoidal driving force, the applied force is given by

$$F(t) = F_0 \cos \omega t.$$

Solution: In this case, assume that the response is also a cosine function out of phase or

$$x_p(t) = X \cos(\omega t - \theta)$$

To make the computations easy to follow, this is written in the equivalent form using a basic trig identity

$$x_p(t) = A_s \cos \omega t + B_s \sin \omega t$$

where the constants $A_s = X \cos \theta$ and $B_s = X \sin \theta$ satisfying

$$X = \sqrt{A_s^2 + B_s^2} \quad \text{and} \quad \theta = \tan^{-1} \frac{B_s}{A_s}$$

are undetermined constant coefficients. Taking derivatives of the assumed form of the solution and substitution of these into the equation of motion yields

$$\left(-\omega^2 A_s + 2\zeta\omega_n\omega B_s + \omega_n^2 A_s - f_0\right) \cos\omega t + \left(-\omega^2 B_s - 2\zeta\omega_n\omega A_s + \omega_n^2 B_s\right) \sin\omega t = 0$$

This equation must hold for all time, in particular for $t = \pi/2\omega$, so that the coefficient of sin ωt must vanish. Similarly, for $t = 0$, the coefficient of cos ωt must vanish. This yields the two equations

$$\left(\omega_n^2 - \omega^2\right) A_s + \left(2\zeta\omega_n\omega\right) B_s = f_0$$

and

$$(-2\zeta\omega_n\omega)A_s + \left(\omega_n^2 - \omega^2\right) + B_s = 0$$

in the two undetermined coefficients A_s and B_s. Solving yields

$$A_s = \frac{(\omega_n^2 - \omega^2)f_0}{\left(\omega_n^2 - \omega^2\right)^2 + (2\zeta\omega_n\omega)^2}$$

$$B_s = \frac{2\zeta\omega_n\omega f_0}{\left(\omega_n^2 - \omega^2\right)^2 + (2\zeta\omega_n\omega)^2}$$

Substitution of these expressions into the equations for X and θ yields the particular solution

$$x_p(t) = \overbrace{\frac{f_0}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + (2\zeta\omega_n\omega)^2}}}^{X} \cos\left(\omega t - \overbrace{\tan^{-1}\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}}^{\theta}\right)$$

Resonance is generally to be avoided in designing structures, since it means large amplitude vibrations, which can cause fatigue failure, discomfort, loud noises, etc. Occasionally, the effects of resonance are catastrophic. However, the concept of resonance is also very useful in testing structures and in certain applications such as energy harvesting (Section 7.10). In fact, the process of modal testing (Chapter 12) is based on resonance. Figure 1.10 illustrates how ω_n and ζ affect the amplitude at resonance. The dimensionless quantity Xk/F_0 is called the *magnification factor* and Figure 1.10 is called a *magnification curve* or *magnitude plot*. The maximum value at resonance, called the *peak resonance*, and denoted by M_p, can be shown (Inman, 2014) to be related to the damping ratio by

$$M_p = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \tag{1.26}$$

Also, Figure 1.10 can be used to define the *bandwidth* of the structure, denoted by BW, as the value of the driving frequency at which the magnitude drops below 70.7% of its zero frequency value (also said to be the 3-dB down point from the zero

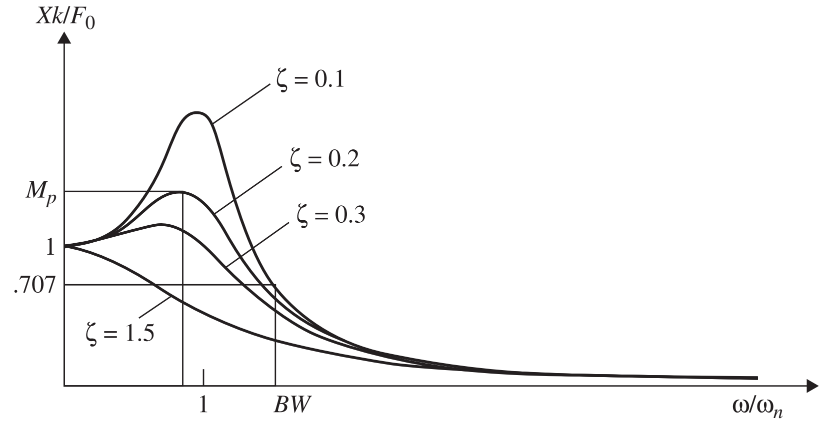

Figure 1.10 Magnification curves (dimensionless) for an SDOF system showing the normalized amplitude of vibration versus the ratio of driving frequency to natural frequency ($r = \omega/\omega_n$).

frequency point). The bandwidth can be calculated (Kuo and Golnaraghi, 2009: p. 359) in terms of the damping ratio by

$$BW = \omega_n \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \tag{1.27}$$

Two other quantities are used in discussing the vibration of underdamped structures. They are the *loss factor* defined at resonance (only) to be

$$\eta = 2\zeta \tag{1.28}$$

and the *Q value*, or *resonance sharpness* factor, given by

$$Q = \frac{1}{2\zeta} = \frac{1}{\eta} \tag{1.29}$$

Another common situation focuses on the transient nature of the response, namely, the response of Equation (1.6) to an impulse, to a step function, or to initial conditions. Many mechanical systems are excited by loads, which act for a very brief time. Such situations are usually modeled by introducing a fictitious function called the *unit impulse function*, or the *Dirac delta function*. This delta function, denoted δ, is defined by the two properties

$$\begin{aligned} &\delta(t - a) = 0 \qquad t \neq a \\ &\int_{-\infty}^{\infty} \delta(t - a)\, dt = 1 \end{aligned} \tag{1.30}$$

where a is the instant of time at which the impulse is applied. Strictly speaking, the quantity $\delta(t)$ is not a function; however, it is very useful in quantifying important physical phenomena of an impulse.