

Power Electronics and Power Systems

Arun G. Phadke
James S. Thorp

Synchronized Phasor Measurements and Their Applications

Second Edition

 Springer

Power Electronics and Power Systems

Series editors

Joe H. Chow, Rensselaer Polytechnic Institute, Troy, New York, USA

Alex M. Stankovic, Tufts University, Medford, Massachusetts, USA

David Hill, The University of Hong Kong, Sydney, New South Wales, Australia

The Power Electronics and Power Systems Series encompasses power electronics, electric power restructuring, and holistic coverage of power systems. The Series comprises advanced textbooks, state-of-the-art titles, research monographs, professional books, and reference works related to the areas of electric power transmission and distribution, energy markets and regulation, electronic devices, electric machines and drives, computational techniques, and power converters and inverters. The Series features leading international scholars and researchers within authored books and edited compilations. All titles are peer reviewed prior to publication to ensure the highest quality content. To inquire about contributing to the Power Electronics and Power Systems Series, please contact Dr. Joe Chow, Administrative Dean of the College of Engineering and Professor of Electrical, Computer and Systems Engineering, Rensselaer Polytechnic Institute, Jonsson Engineering Center, Office 7012, 110 8th Street, Troy, NY USA, 518-276-6374, chowj@rpi.edu.

More information about this series at <http://www.springer.com/series/6403>

Arun G. Phadke · James S. Thorp

Synchronized Phasor Measurements and Their Applications

Second Edition

 Springer

Arun G. Phadke
Department of Electrical and Computer
Engineering
Virginia Tech
Blacksburg, VA
USA

James S. Thorp
Department of Electrical and Computer
Engineering
Virginia Tech
Blacksburg, VA
USA

ISSN 2196-3185 ISSN 2196-3193 (electronic)
Power Electronics and Power Systems
ISBN 978-3-319-50582-4 ISBN 978-3-319-50584-8 (eBook)
DOI 10.1007/978-3-319-50584-8

Library of Congress Control Number: 2016959259

© Springer International Publishing AG 2008, 2017

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made.

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface to the Second Edition

The first edition of this book was published in the beginning of 2008. These intervening years have seen a steadily growing adaption of this technology to improve the monitoring, protection, and control of the power systems around the world. Many countries are busily installing phasor measurement units (PMUs) and wide area measurement systems (WAMS) on their networks. In the USA, activities related to this technology continue to be supported by various electric utilities, government agencies—particularly the US Department of Energy—and many other industry groups. Although the number of substations equipped by PMUs and WAMS remains small compared to the size of the US power grid, an excellent start has been made, and in general, individual US utilities continue their efforts to improve the coverage provided by WAMS.

In our view, the new frontier in this area is the development of new applications of PMUs and WAMS. Work on new applications continues in many organizations around the world. Postmortem analysis after major system disturbances is a very useful application which assists in determining the causes and timelines of events which contributed to the system disturbance. To this end, obtaining and keeping records of such events are important, requiring selection of important data from WAMS which will validate system models used in simulations of the postmortem analysis. At present, there is no convenient tool which will examine all the saved data and determine which data segments are of most interest. We expect applications to achieve such a data selection task to be one of the outcomes of the present research.

State estimation using PMU measurements alone or in conjunction with SCADA is being developed in many organizations. A new development in this regard is to perform state estimation in phase coordinates, rather than using positive sequence estimation. Protection and control using WAMS is also being developed by many organizations, and some of those research results are reported in this book.

We have clarified many sections in this book which we hope will make our ideas clearer.

A particularly beneficial feature of this revision is that we have been able to draw upon the latest work done by our colleagues. Instead of paraphrasing their work

ourselves, we invited them to make direct contributions to this book. Three colleagues and friends have made such contributions, which have been attributed to them at appropriate places in this book. Dr. Anamitra Pal, a former graduate student at Virginia Tech, has contributed to several sections in this book, and in addition, he contributed Chap. 9, which deals with the advanced developments in the area of control using wide area measurement systems. Kenneth E. Martin has been leading the effort to create PMUs and WAMS standards in the Power System Relaying Committee of the Power and Energy Society, as well as the parallel effort in IEC to make the standards more precise and relevant to the evolving technology. He has contributed the new Sect. 5.6, which deals with the present state of these standards. We recognize that standards are a moving target, and they will surely be modified in the coming years as newer developments in this technology emerge. The work on the calibration of instrument transformers and estimating the transmission network parameters was originally developed by Dr. Zhongyu Wu while she was a Ph.D. student at Virginia Tech. She has continued to work in this area and has contributed to Sect. 7.6 bringing together the latest developments in this field.

We are grateful to Dr. Pal, Mr. Martin, and Dr. Wu for their important contributions to our book. We believe this has added greatly to the overall coverage of the subject.

Finally, we wish to thank many colleagues throughout the world who have used our book and have corresponded with us on questions of mutual interest. We hope this new edition of this book will continue to be of interest to students, researchers, and industry practitioners.

Wilsonville, Oregon
Blacksburg, Virginia
January 2017

Arun G. Phadke
James S. Thorp

Preface to the First Edition

Synchronized phasor measurements have become the measurement technique of choice for electric power systems. They provide positive sequence voltage and current measurements synchronized to within a microsecond. This has been made possible by the availability of Global Positioning System and the sampled data processing techniques developed for computer-relaying applications. In addition to positive sequence voltages and currents, these systems also measure local frequency and rate of change of frequency and may be customized to measure harmonics, negative and zero-sequence quantities, as well as individual phase voltages and currents. At present, there are about two dozen commercial manufacturers of phasor measurement units (PMUs), and industry standards developed in the Power System Relaying Committee of IEEE have made possible the interoperability of units from different manufacturers.

Recent spate of spectacular blackouts on power systems throughout the world has provided an added impetus to wide-scale deployment of PMUs. Positive sequence measurements provide the most direct access to the state of the power system at any given instant. Many applications of these measurements have been discussed in the technical literature, and no doubt many more applications will be developed in the coming years.

The authors have been associated with this technology since its birth, and they and their colleagues and students have produced a rich body of literature on the subject of phasor measurement technology and its applications. Other researchers around the world have also made significant contributions to the field. Our aim in writing this book is to present to the interested reader a coherent account of the development of the technology and of the emerging applications of these measurements. It is our hope that this book will help power system engineers understand the basics of synchronized phasor measurement systems. This technology is bound to inaugurate an era of improved monitoring, protection and control of power systems.

Blacksburg, Virginia
January 2008

Arun G. Phadke
James S. Thorp

Contents

Part I Phasor Measurement Techniques

1	Introduction	3
1.1	Historical Overview	3
1.2	Phasor Representation of Sinusoids	5
1.3	Fourier Series and Fourier Transform	6
1.3.1	Fourier Series	6
1.3.2	Fourier Transform	8
1.4	Sampled Data and Aliasing	14
1.5	Discrete Fourier Transform (DFT)	16
1.5.1	DFT and Fourier Series	20
1.5.2	DFT and Phasor Representation	22
1.6	Leakage Phenomena	24
	References	27
2	Phasor Estimation of Nominal Frequency Inputs	29
2.1	Phasors of Nominal Frequency Signals	29
2.2	Formulas for Updating Phasors	30
2.2.1	Non-recursive Updates	30
2.2.2	Recursive Updates	31
2.3	Effect of Signal, Noise, and Window Length	34
2.3.1	Errors in Sampling Times	37
2.4	Phasor Estimation with Fractional Cycle Data Window	37
2.5	Quality of Phasor Estimate and Transient Monitor	39
2.6	DC Offset in Input Signals	42
2.7	Non-DFT Estimators	45
	References	45

3	Phasor Estimation at Off-Nominal Frequency Inputs	47
3.1	Types of Frequency Excursions Found in Power Systems	47
3.2	DFT Estimate at Off-Nominal Frequency with a Nominal Frequency Clock.	48
3.2.1	Input Signal at Off-Nominal Frequency.	48
3.3	Post Processing for Off-Nominal Frequency Estimates	56
3.3.1	A Simple Averaging Digital Filter for $2f_0$	56
3.3.2	A Re-sampling Filter	57
3.4	Phasor Estimates of Pure Positive Sequence Signals.	58
3.4.1	Symmetrical Components	58
3.5	Estimates of Unbalanced Input Signals.	62
3.5.1	Unbalanced Inputs at Off-Nominal Frequency	62
3.5.2	A Nomogram	66
3.6	Sampling Clocks Locked to the Power Frequency	69
3.7	Non-DFT Type Phasor Estimators	71
	References.	72
4	Frequency Estimation	73
4.1	Historical Overview of Frequency Measurement	73
4.2	Frequency Estimates from Balanced Three-Phase Inputs	74
4.3	Frequency Estimates from Unbalanced Inputs	78
4.4	Nonlinear Frequency Estimators	78
4.5	Other Techniques for Frequency Measurements	81
	References.	82
5	Phasor Measurement Units and Phasor Data Concentrators	83
5.1	Introduction	83
5.2	A Generic PMU	84
5.3	The Global Positioning System.	85
5.4	Hierarchy for Phasor Measurement Systems.	86
5.5	Communication Options for PMUs.	88
5.6	Standards	91
5.6.1	History	91
5.6.2	Synchrophasor Measurement	94
5.6.3	Synchrophasor Communication	103
5.6.4	PDC Files	108
	References.	109
6	Transient Response of Phasor Measurement Units	111
6.1	Introduction	111
6.2	Nature of Transients in Power Systems	112
6.2.1	Electromagnetic Transients	112
6.2.2	Electromechanical Transients.	113
6.3	Transient Response of Instrument Transformers	117
6.3.1	Voltage Transformers	117
6.3.2	Current Transformers.	118

6.4	Transient Response of Filters	119
6.4.1	Surge Suppression Filters	119
6.4.2	Anti-aliasing Filters	119
6.5	Transient Response During Electromagnetic Transients	121
6.6	Transient Response During Power Swings	122
6.6.1	Amplitude Modulation.	123
6.6.2	Frequency Modulation.	125
6.6.3	Simultaneous Amplitude and Frequency Modulation	126
6.6.4	Aliasing Considerations in Phasor Reporting Rates	128
	References.	130

Part II Phasor Measurement Applications

7	State Estimation	133
7.1	History-Operator's Load Flow	133
7.2	Weighted Least Square.	134
7.2.1	Least Square	134
7.2.2	Linear Weighted Least Squares	135
7.2.3	Condition Numbers, Leverage, and LAV in Linear Least Squares	137
7.2.4	Nonlinear Weighted Least Squares	140
7.3	Static State Estimation	141
7.4	Bad Data Detection.	145
7.5	State Estimation with Phasor Measurements.	147
7.5.1	Linear State Estimation	150
7.5.2	An Alternative for Including Phasor Measurements.	153
7.5.3	Incomplete Observability Estimators	153
7.5.4	Partitioned State Estimation.	159
7.6	Calibration	164
7.6.1	Calibration with Positive Sequence Measurements	165
7.6.2	Calibration with Phase Measurements	168
7.6.3	Simultaneous Calibration of Line Parameters and Transducers.	174
7.7	Dynamic Estimators	182
	References.	183
8	Control with Phasor Feedback	185
8.1	Introduction	185
8.2	Linear Optimal Control.	185
8.3	Linear Optimal Control Applied to the Nonlinear Problem.	187
8.4	Coordinated Control of Oscillations	193

8.5	Polytopic Control Using LMIs	197
8.5.1	Phasor Measurement-Based Adaptive Control	205
8.5.2	Future Research on Phasor-Based Controls	207
	References.	208
9	Phasor Measurement-Enabled Decision Making	211
	Anamitra Pal	
9.1	Discrete Event Control	211
9.2	Decision Trees	213
9.2.1	Classification and Regression Tree (CART)	214
9.2.2	Fisher’s Linear Discriminant Applied to Synchrophasor Data (FLDSD) Technique	215
9.2.3	Applications of FLDSD in Power Systems	218
9.3	Synchrophasor Data Conditioning and Validation.	222
9.3.1	Three Sample-Based Quadratic Prediction Algorithm	223
9.3.2	A Methodology for Performing Synchrophasor Data Conditioning and Validation	227
9.3.3	Alternate Approaches for Addressing Data Quality Issues in a LSE	238
	References.	241
10	Protection Systems with Phasor Inputs	245
10.1	Introduction	245
10.2	Differential Protection of Transmission Lines	245
10.3	Distance Relaying of Multiterminal Transmission Lines	248
10.4	Adaptive Protection	250
10.4.1	Adaptive Out-of-Step Protection	250
10.4.2	Security Versus Dependability.	253
10.4.3	Transformer.	255
10.4.4	Adaptive System Restoration.	255
10.5	Control of Backup Relay Performance	257
10.5.1	Hidden Failures	259
10.6	Intelligent Islanding	261
10.7	Supervisory Load Shedding	262
	References.	263
11	Electromechanical Wave Propagation	265
11.1	Introduction	265
11.2	The Model	267
11.3	Electromechanical Telegrapher’s Equation	271
11.4	Continuum Voltage Magnitude.	273
11.5	Effects on Protection Systems.	275
11.5.1	Overcurrent Relays	276
11.5.2	Impedance Relays	276

11.5.3	Out-of-Step Relays	277
11.5.4	Load Shedding	278
11.6	Dispersion.	279
11.7	Parameter Distribution	279
	References.	281
Index		283

Part I

Phasor Measurement Techniques

Chapter 1

Introduction

1.1 Historical Overview

Phase angles of voltage phasors of power network buses have always been of special interest to power system engineers. It is well known that active (real) power flow in a power line is very nearly proportional to the sine of the angle difference between voltages at the two terminals of the line. As many of the planning and operational considerations in a power network are directly concerned with the flow of real power, measuring angle differences across transmission have been of concern for many years. The earliest modern application involving direct measurement of phase angle differences was reported in three papers in the early 1980s [1–3]. These systems used LORAN-C, GOES satellite transmissions, and the HBG radio transmissions (in Europe) in order to obtain synchronization of reference time at different locations in a power system. The next available positive-going zero-crossing of a phase voltage was used to estimate the local phase angle with respect to the time reference. Using the difference of measured angles on a common reference at two locations, the phase angle difference between voltages at two buses was established. Measurement accuracies achieved in these systems were of the order of 40 μ s. Single-phase voltage angles were measured, and of course, no attempt was made to measure the prevailing voltage phasor magnitude. Neither was any account taken of the harmonics contained in the voltage waveform. These methods of measuring phase angle differences are not suitable for generalization for wide-area phasor measurement systems and remain one-of-a-kind systems which are no longer in use.

The modern era of phasor measurement technology has its genesis in research conducted on computer relaying of transmission lines. Early work on transmission line relaying with microprocessor-based relays showed that the available computer power in those days (1970s) was barely sufficient to manage the calculations needed to perform all the transmission line relaying functions.

A significant portion of the computations was dedicated to solving six fault loop equations at each sample time in order to determine whether any one of the ten types of faults possible on a three-phase transmission line is present. The search for methods which would eliminate the need to solve the six equations finally yielded a new relaying technique which was based on symmetrical component analysis of line voltages and currents. Using symmetrical components, and certain quantities derived from them, it was possible to perform all fault calculations with a single equation. In a paper published in 1977 [4], this new symmetrical component-based algorithm for protecting a transmission line was described. As a part of this theory, efficient algorithms for computing symmetrical components of three-phase voltages and currents were described, and the calculation of positive sequence voltages and currents using the algorithms of that paper gave an impetus for the development of modern phasor measurement systems. It was soon recognized that the positive sequence measurement (a part of the symmetrical component calculation) is of great value in its own right. Positive sequence voltages of a network constitute the state vector of a power system, and it is of fundamental importance in all of the power system analysis. The first paper to identify the importance of positive sequence voltage and current phasor measurements and some of the uses of these measurements was published in 1983 [5], and this last paper can be viewed as the starting point of modern synchronized phasor measurement technology. The Global Positioning System (GPS) [6] was beginning to be fully deployed around that time. It became clear that this system offered the most effective way of synchronizing power system measurements over great distances. The first prototypes of the modern phasor measurement units (PMUs) using GPS were built at Virginia Tech in the early 1980s, and two of these prototypes are shown in Fig. 1.1. The prototype PMU built at Virginia Tech was deployed at a few substations of the Bonneville Power Administration, the American Electric Power Service Corporation, and the New York Power Authority. The first commercial manufacture of PMUs with Virginia Tech collaboration was started by Macrodyne in 1991 [7]. At present, a number of manufacturers offer PMUs as a commercial product, and deployment of PMUs on power systems is being carried out in earnest in many countries around the world. IEEE published a standard in 1991 [8, 9], governing the format of data files created and transmitted by the PMU. A revised version of the standard was issued in 2005.

Concurrently with the development of PMUs as measurement tools, research was ongoing on the applications of the measurements provided by the PMUs. These applications will be discussed in greater detail in later chapters of this book. It can be said now that finally the technology of synchronized phasor measurements has come of age, and most modern power systems around the world are in the process of installing wide-area measurement systems consisting of the phasor measurement units.

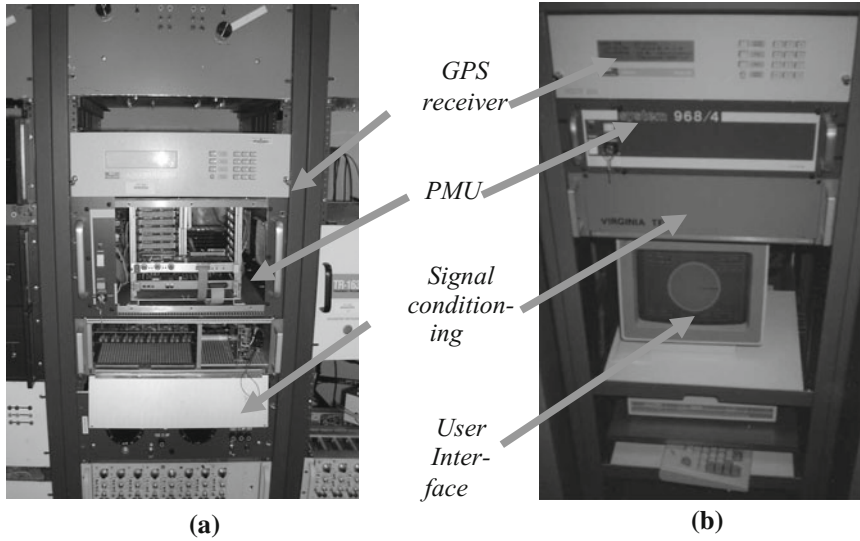


Fig. 1.1 The first phasor measurement units built at the power systems research laboratory at Virginia Tech. The GPS receiver clock was external to the PMU, and with the small number of GPS satellites deployed at that time, the clock had to be equipped with a precision internal oscillator which maintained accurate time in the absence of visible satellites

1.2 Phasor Representation of Sinusoids

Consider a pure sinusoidal quantity given by

$$x(t) = X_m \cos(\omega t + \phi) \quad (1.1)$$

ω being the frequency of the signal in radians per second, and ϕ being the phase angle in radians. X_m is the peak amplitude of the signal. The root-mean-square (RMS) value of the input signal is $(X_m/\sqrt{2})$. Recall that RMS quantities are particularly useful in calculating active power and reactive power in an AC circuit.

Equation (1.1) can also be written as

$$x(t) = \text{Re}\{X_m e^{j(\omega t + \phi)}\} = \text{Re}\left[\left\{e^{j(\omega t)}\right\} X_m e^{j\phi}\right]$$

It is customary to suppress the term $e^{j(\omega t)}$ in the expression above, with the understanding that the frequency is ω . The sinusoid of Eq. (1.1) is represented by a complex number $\mathbf{X}$ known as its phasor representation:

$$x(t) \leftrightarrow \mathbf{X} = \left(X_m/\sqrt{2}\right)e^{j\phi} = (X_m/\sqrt{2})[\cos \phi + j \sin \phi] \quad (1.2)$$

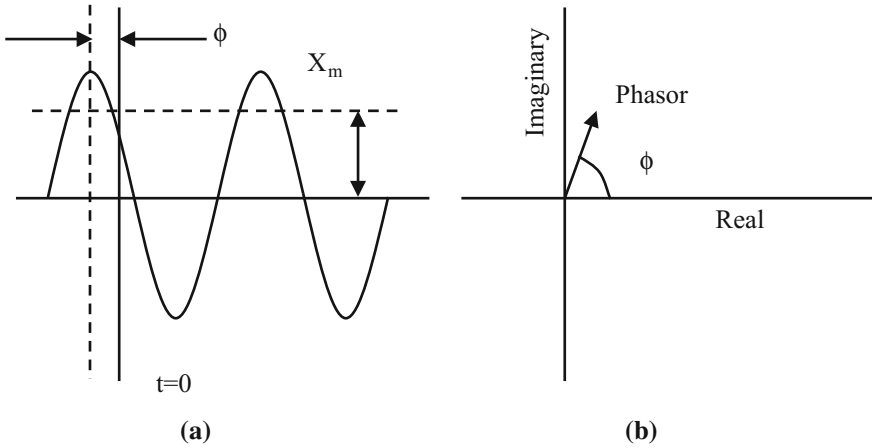


Fig. 1.2 A sinusoid (a) and its representation as a phasor (b). The phase angle of the phasor is arbitrary, as it depends upon the choice of the axis $t = 0$. Note that the length of the phasor is equal to the RMS value of the sinusoid

A sinusoid and its phasor representation are shown in Fig. 1.2.

It was stated earlier that the phasor representation is only possible for a pure sinusoid. In practice, a waveform is often corrupted with other signals of different frequencies. It then becomes necessary to extract a single frequency component of the signal (usually the principal frequency of interest in an analysis) and then represent it by a phasor. Extracting a single frequency component is often done with a Fourier transform calculation. In sampled data systems, this becomes the discrete Fourier transform (DFT) or the fast Fourier transform (FFT). These transforms are reviewed in the next section. The phasor definition also implies that the signal is unchanging for all time. However, in all practical cases, it is only possible to consider a portion of time span over which the phasor representation is considered. This time span, also known as the ‘data window’ is very important in phasor estimation of practical waveforms. It will be considered in greater detail in later sections.

1.3 Fourier Series and Fourier Transform

1.3.1 Fourier Series

Let $x(t)$ be a periodic function of t , with a period equal to T . Then, $x(t + kT) = x(t)$ for all integer values of k . A periodic function can be expressed as a Fourier series:

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi kt}{T}\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2\pi kt}{T}\right) \quad (1.3)$$

where the constants a_k and b_k are given by

$$\begin{aligned} a_k &= \frac{2}{T} \int_{-T/2}^{+T/2} x(t) \cos\left(\frac{2\pi kt}{T}\right) dt, \quad k = 0, 1, 2, \dots \\ b_k &= \frac{2}{T} \int_{-T/2}^{+T/2} x(t) \sin\left(\frac{2\pi kt}{T}\right) dt, \quad k = 1, 2, \dots \end{aligned} \quad (1.4)$$

The Fourier series can also be written in the exponential form

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\frac{j2\pi kt}{T}} \quad (1.5)$$

With

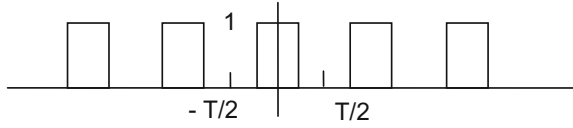
$$\alpha_k = \frac{1}{T} \int_{-T/2}^{+T/2} x(t) e^{-\frac{j2\pi kt}{T}} dt, \quad k = 0, \pm 1, \pm 2, \dots \quad (1.6)$$

Note that the summation in Eq. (1.5) goes from $-\infty$ to $+\infty$, while the summations in Eq. (1.3) go from 1 to $+\infty$. The change in summation limits is accomplished by noting that the cosine and sine functions are even and odd functions of k , and thus, expanding the summation limits to $(-\infty$ to $+\infty)$ and removing the factor 2 in front of the integrals for a_k and b_k lead to the desired exponential form of the Fourier series.

Example 1.1 Consider a periodic square wave signal with a period T as shown in Fig. 1.3. This is an even function of time. The Fourier coefficients (in exponential form) are given by

$$\begin{aligned} \alpha_k &= \frac{1}{T} \int_{-T/4}^{+T/4} e^{-\frac{j2\pi kt}{T}} dt, \quad k = 0, \pm 1, \pm 2, \dots \\ &= \frac{1}{\pi k} \sin\left(\frac{k\pi}{2}\right) \end{aligned}$$

Fig. 1.3 A square wave function with a period T , with duty cycle equal to half, with the $t = 0$ axis so chosen that the function is an even function



Hence,

$$\alpha_0 = 1/2,$$

$$\alpha_1 = 1/\pi, \alpha_{-1} = 1/\pi,$$

$$\alpha_3 = -1/3\pi, \alpha_{-3} = -1/3\pi, \text{ and}$$

$$\alpha_5 = 1/5\pi, \alpha_{-5} = 1/5\pi, \text{ etc., and all even coefficients are zero.}$$

Thus, the Fourier series of the square wave signal is

$$x(t) = \frac{1}{2} + \frac{2}{\pi} \left[\cos\left(\frac{2\pi t}{T}\right) - \frac{1}{3} \cos\left(\frac{6\pi t}{T}\right) + \frac{1}{5} \cos\left(\frac{10\pi t}{T}\right) - \dots \right]$$

The sum of first seven terms of the series is shown in Fig. 1.4.

1.3.2 Fourier Transform

There are several excellent textbooks devoted to the subject of Fourier transforms [10, 11]. The reader should consult those books for a more complete account of the Fourier transform theory. Here, we present only those topics which are of direct interest for phasor estimation in power system applications.

The Fourier transform of a continuous time function $x(t)$ satisfying certain integrability conditions [10] is given by

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt \quad (1.7)$$

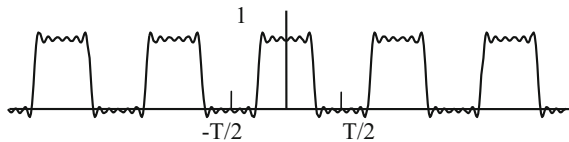


Fig. 1.4 A square wave approximated by 7 terms of the Fourier series. With more terms, the waveform approaches the square shape. The oscillations are known as the Gibbs phenomenon and are inescapable when step functions are approximated by the Fourier series

and the inverse Fourier transform recovers the time function from its Fourier transform:

$$x(t) = \int_{-\infty}^{+\infty} X(f) e^{j2\pi ft} df \quad (1.8)$$

An important function frequently used in calculations using sampled data is the impulse function $\delta(t)$ defined by

$$x(t_0) = \int_{-\infty}^{+\infty} \delta(t - t_0) x(t) dt \quad (1.9)$$

The impulse function (also known as a distribution or a Dirac delta function) is a sampling function in the sense that when the integration in Eq. (1.9) is performed, the result is the sampled value of the function $x(t)$ at $t = t_0$. The integrals of the type shown in Eq. (1.9) are known as convolutions. Thus, the sampling process at uniform intervals ΔT apart can be considered to be a convolution of the input signal and a string of impulse functions $\delta(t - k\Delta T)$ where k ranges from $-\infty$ to $+\infty$.

The convolutions of two time functions and their Fourier transforms have a convenient relationship. Consider the convolution $z(t)$ of two time functions $x(t)$ and $y(t)$

$$z(t) = \int_{-\infty}^{+\infty} x(\tau) y(\tau - t) d\tau \equiv x(t) * y(t) \quad (1.10)$$

The important result regarding convolutions is the following property:

Property 1 *The Fourier transform of a convolution is equal to the product of the Fourier transform of the functions being convolved, or:*

$$\text{If } s(t) = x(t) * y(t), \text{ then } S(f) = X(f) \cdot Y(f)$$

and similarly, the inverse Fourier transform of a convolution of two Fourier transforms is a product of the corresponding inverse Fourier transforms:

$$\text{If } Z(f) = X(f) * Y(f), \text{ then } z(t) = x(t) \cdot y(t)$$

Next, we illustrate the second of the above two statements. Consider the functions $x(t) = \cos(\omega_0 t)$ and $y(t) = \sin(\omega_0 t)$, with $\omega_0 = 2\pi f_0$. The Fourier transforms of $x(t)$ and $y(t)$ are as follows:

$$\begin{aligned}
X(f) &= \int_{-\infty}^{+\infty} \cos(2\pi f_0 t) e^{-j2\pi f t} dt = \int_{-\infty}^{+\infty} \frac{e^{-j2\pi(f-f_0)t} + e^{-j2\pi(f+f_0)t}}{2} dt \\
&= \frac{1}{2} [\delta(f-f_0) + \delta(f+f_0)]
\end{aligned}$$

and similarly

$$Y(f) = \frac{j}{2} [\delta(f+f_0) - \delta(f-f_0)]$$

The Fourier transforms of a pure cosine wave of unit amplitude are a pair of real impulse functions in frequency domain located at $\pm f_0$ and that of a pure sine wave of unit amplitude is a pair of imaginary impulse functions of opposite signs at $\pm f_0$.

The convolution of the two Fourier transforms determined above in the frequency domain is as follows:

$$\begin{aligned}
S(f) &= \int_{-\infty}^{+\infty} \frac{1}{2} [\delta(\phi-f_0) + \delta(\phi+f_0)] \frac{j}{2} [\delta(f+f_0-\phi) - \delta(f-f_0-\phi)] d\phi \\
&= \frac{j}{4} \int_{-\infty}^{+\infty} [\delta(\phi-f_0)\delta(f+f_0-\phi) + \delta(\phi+f_0)\delta(f-f_0-\phi) \\
&\quad - \delta(\phi-f_0)\delta(f-f_0-\phi) - \delta(\phi+f_0)\delta(f+f_0-\phi)] d\phi
\end{aligned}$$

Using the sampling property of the integrals involving impulse functions

$$S(f) = \frac{j}{4} [\delta(f-2f_0) - \delta(f+2f_0)]$$

The inverse Fourier transform of $S(f)$ is clearly

$$S(t) = \frac{1}{2} \sin(4\pi f t) = \sin(2\pi f t) \cos(2\pi f t) = x(t) \cdot y(t)$$

This property of convolutions will be used in discussing the sampling process and the DFT. Some other properties of the Fourier transform which are particularly useful in our development are stated next with accompanying examples.

Property 2 *The Fourier transform of an even function is an even function of frequency. If the even function is real, the Fourier transform is also real and even.*

Consider an even function of time, $x(t)$, so that $x(-t) = x(t)$. Let $x(t)$ be complex, $x(t) = r(t) + js(t)$. The Fourier transform $X(f)$ of this function is given by

$$\begin{aligned}
X(f) &= \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft} dt = \int_{-\infty}^{+\infty} r(t)e^{-j2\pi ft} dt + j \int_{-\infty}^{+\infty} s(t)e^{-j2\pi ft} dt \\
&= \int_{-\infty}^{+\infty} r(t) \cos(2\pi ft) dt + j \int_{-\infty}^{+\infty} r(t) \sin(2\pi ft) dt \\
&\quad + j \int_{-\infty}^{+\infty} s(t) \cos(2\pi ft) dt - \int_{-\infty}^{+\infty} s(t) \sin(2\pi ft) dt
\end{aligned}$$

The second and fourth integrals are zero, since the integrands are odd functions of time. Thus,

$$X(f) = \int_{-\infty}^{+\infty} r(t) \cos(2\pi ft) dt + j \int_{-\infty}^{+\infty} s(t) \cos(2\pi ft) dt$$

Since $\cos(2\pi ft) = \cos(-2\pi ft)$, it follows that $X(f) = X(-f)$.

Also, if $x(t)$ is real $= r(t)$, the Fourier transform of $x(t)$ is $R(f)$, which is real and even.

Property 3 *The Fourier transform of an odd function is an odd function of frequency. If the odd function is real, the Fourier transform is imaginary and odd.*

Consider an odd function of time, $x(t)$, so that $x(-t) = -x(t)$. Let $x(t)$ be complex, $x(t) = r(t) + js(t)$. The Fourier transform $X(f)$ of this function is given by

$$\begin{aligned}
X(f) &= \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft} dt = \int_{-\infty}^{+\infty} r(t)e^{-j2\pi ft} dt + j \int_{-\infty}^{+\infty} s(t)e^{-j2\pi ft} dt \\
&= \int_{-\infty}^{+\infty} r(t) \cos(2\pi ft) dt + j \int_{-\infty}^{+\infty} r(t) \sin(2\pi ft) dt + j \int_{-\infty}^{+\infty} s(t) \cos(2\pi ft) dt \\
&\quad - \int_{-\infty}^{+\infty} s(t) \sin(2\pi ft) dt
\end{aligned}$$

The first and third integrals are zero, since the integrands are odd functions of time. Thus,

$$X(f) = j \int_{-\infty}^{+\infty} r(t) \sin(2\pi ft) dt - \int_{-\infty}^{+\infty} s(t) \sin(2\pi ft) dt$$

Since $\sin(2\pi ft) = -\sin(-2\pi ft)$, it follows that $X(f) = -X(-f)$.

Also, if $x(t)$ is real $= r(t)$, the Fourier transform of $x(t)$ is $jR(f)$, which is imaginary and odd.

Property 4 *The Fourier transform of a real function has an even real part and an odd imaginary part.*

Consider a real function of time $x(t) = r(t) + j0$. The Fourier transform is given by

$$X(f) = \int_{-\infty}^{+\infty} r(t) \cos(2\pi ft) dt + j \int_{-\infty}^{+\infty} r(t) \sin(2\pi ft) dt = R_1(f) + jR_2(f)$$

Since the cosine and sine functions are, respectively, even and odd functions of frequency, it is clear that $R_1(f)$ is an even function, and $R_2(f)$ is an odd function of frequency.

Property 5 *The Fourier transform of a periodic function is a series of impulse functions of frequency.*

If $x(t)$ is a periodic function of t , with a period equal to T , it can be expressed as an exponential Fourier series given by Eqs. (1.5) and (1.6):

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j\frac{2\pi kt}{T}}$$

with

$$\alpha_k = \frac{1}{T} \int_{-T/2}^{+T/2} x(t) e^{-j\frac{2\pi kt}{T}} dt, \quad k = 0, \pm 1, \pm 2, \dots$$

The Fourier transform of $x(t)$ expressed in the exponential form is given by

$$\begin{aligned} X(f) &= \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt = \int_{-\infty}^{+\infty} \left[\sum_{k=-\infty}^{\infty} \alpha_k e^{j\frac{2\pi kt}{T}} \right] e^{-j2\pi ft} dt = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{+\infty} \alpha_k e^{j\frac{2\pi kt}{T}} e^{-j2\pi ft} dt \\ &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{+\infty} \alpha_k e^{-j2\pi t \{ \frac{k}{T} - f \}} dt \end{aligned}$$

where the order of the summation and integration has been reversed (assuming that this is permissible). Setting $f_0 = 1/T$, the fundamental frequency of the periodic signal, the integral of the exponential term in the last form is the impulse function $\delta(kf_0 - f)$, and thus, the Fourier transform of periodic $x(t)$ is

$$X(f) = \sum_{k=-\infty}^{\infty} \alpha_k \delta\left(f - \frac{k}{T}\right), \quad \text{with}$$

$$\alpha_k = \frac{1}{T} \int_{-T/2}^{+T/2} x(t) e^{-\frac{j2\pi kt}{T}} dt, \quad k = 0, \pm 1, \pm 2, \dots$$

These are a series of impulses located at multiples of the fundamental frequency f_0 of the periodic signal with impulse magnitudes being equal to amplitude of each frequency component in the input signal.

Property 6 *The Fourier transform of a series of impulses is a series of impulse functions in the frequency domain.*

Consider the function

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

This is a periodic function with period T . Hence, its Fourier transform (by Property 5 above) is

$$X(f) = \sum_{k=-\infty}^{\infty} \alpha_k \delta\left(f - \frac{k}{T}\right), \quad \text{with}$$

$$\alpha_k = \frac{1}{T} \int_{-T/2}^{+T/2} \delta(t) e^{-\frac{j2\pi kt}{T}} dt, \quad k = 0, \pm 1, \pm 2, \dots$$

Since the delta function in the integrand produces a sample of the exponent at $t = 0$, α_k is equal to $1/T$ for all k , and the Fourier transform of $x(t)$ becomes

$$X(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right)$$

which is a pulse train in the frequency domain at intervals kf_0 and a magnitude of $1/T$.

Example 1.2 Consider a rectangular input signal as shown in Fig. 1.5. This is an even function of time.

The Fourier transform of this time function is given by

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt = \int_{t_1}^{t_1 + T_0} e^{-j2\pi ft} dt = e^{j2\pi f(t_1 + \frac{T_0}{2})} T_0 \frac{\sin(2\pi f \frac{T_0}{2})}{(2\pi f \frac{T_0}{2})}$$

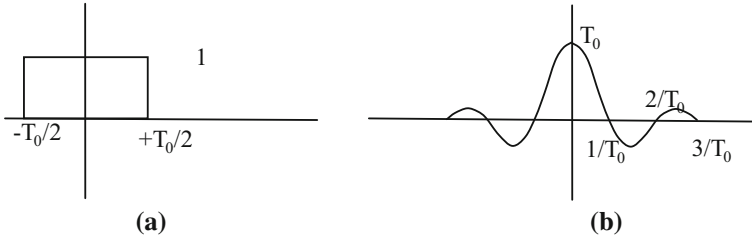


Fig. 1.5 **a** A rectangular function of time with the $t = 0$ axis so chosen that the function is an even function. The duration of the signal is $2T_0$. **b** Fourier transform of the function

The first term in the Fourier transform is a phase shift factor and has been omitted from the plot in Fig. 1.5b for convenience. If the rectangular wave is centered at the origin, $t_1 = -T_0/2$, and the phase shift factor vanishes. This is also in keeping with property 1.2 of the Fourier transform given above, which states that the Fourier transform of a real even function must be real and even function of frequency.

1.4 Sampled Data and Aliasing

Sampled data from input signals are the starting point of digital signal processing. The computation of phasors of voltages and currents begins with the samples of the waveform taken at uniform intervals $k\Delta T$ ($k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \dots$). Consider an input signal $x(t)$ which is being sampled, yielding sampled data $x(k\Delta T)$. We may view the sampled data as a time function $x'(t)$ consisting of uniformly spaced impulses, each with a magnitude $x(k\Delta T)$

$$x'(t) = \sum_{k=-\infty}^{\infty} x(k\Delta T) \delta(t - k\Delta T) \quad (1.11)$$

It is interesting to determine the Fourier transform of the sampled data function given by Eq. (1.11). Note that the sampled data function is a product of the function $x(t)$ and the sampling function $\delta(t - k\Delta T)$, the product being interpreted in the sense of Eq. (1.9). Hence, the Fourier transform $X'(f)$ of $x'(t)$ is the convolution of the Fourier transforms of $x(t)$ and of the unit impulse train. By property 1.6 of Sect. 1.3, the Fourier transform of the impulse train is

$$\Delta(f) = \frac{1}{\Delta T} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{\Delta T}\right) \quad (1.12)$$

Hence, the Fourier transform of the sampled data function is the convolution of $\Delta(f)$ and $X(f)$

$$\begin{aligned}
 X'(f) &= \frac{1}{\Delta T} \int_{-\infty}^{+\infty} X(\phi) \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{\Delta T} - \phi\right) d\phi \\
 &= \frac{1}{\Delta T} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{+\infty} X(\phi) \delta\left(f - \frac{k}{\Delta T} - \phi\right) d\phi \\
 &= \frac{1}{\Delta T} \sum_{k=-\infty}^{\infty} X\left(f - \frac{k}{\Delta T}\right)
 \end{aligned} \tag{1.13}$$

Once again, the order of summation and integration has been reversed (it being assumed that this is permissible), and the integral is evaluated by the use of the sampling property of the impulse function.

The relationship between the Fourier transforms of $x(t)$ and $x'(t)$ is as shown in Fig. 1.6. The Fourier transform of $x(t)$ is shown to be band-limited, meaning that it has no components beyond a cutoff frequency f_c . The sampled data have a Fourier transform, which consist of an infinite train of the Fourier transforms of $x(t)$ centered at frequency intervals of $(k/\Delta T)$ for all k . Recall that the sampling interval is ΔT , so that the sampling frequency $f_s = (1/\Delta T)$.

If the cutoff frequency f_c is greater than one-half of the sampling frequency f_s , the Fourier transform of the sampled data will be as shown in Fig. 1.7. In this case, the spectrum of the sampled data is different from that of the input signal in the region where the neighboring spectra overlap as shown by the shaded region in Fig. 1.7. This implies that frequency components estimated from the sampled data in this region will be in error, due to a phenomenon known as **aliasing**.

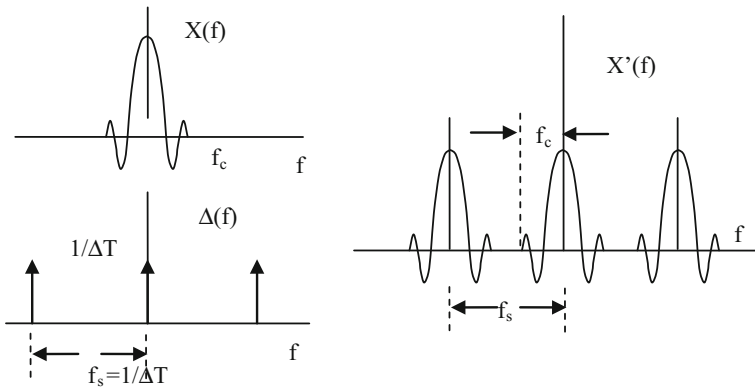


Fig. 1.6 Fourier transform of the sampled data function as a convolution of the transforms $X(f)$ and $\Delta(f)$. The sampling frequency is f_s , and $X(f)$ is band-limited between $\pm f_c$

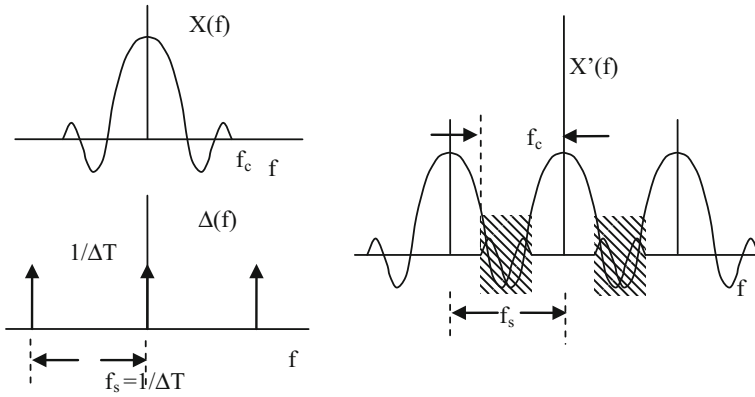


Fig. 1.7 Fourier transform of the sampled data function when the input signal is band-limited to a frequency greater than half of the sampling frequency. The estimate of frequencies from sampled data in the shaded region will be in error because of aliasing

It is clear from the above discussion that in order to avoid errors due to aliasing, the bandwidth of the input signal must be less than half of the sampling frequency utilized in obtaining the sampled data. This requirement is known as the **Nyquist criterion**.

In order to avoid aliasing errors, it is customary in all sampled data systems used in phasor estimation to use anti-aliasing filters which band-limit the input signals to below half of the sampling frequency chosen. Note that the signal input cutoff frequency must be *less than* one-half of the sampling frequency. In practice, the signal is usually band-limited to a value much smaller than the one required for meeting the Nyquist criterion. Anti-aliasing filters are generally passive low-pass R-C filters [12], although active filters may also be used for obtaining a sharp cutoff characteristic. In addition to passive anti-aliasing filters, digital filters may also be used in special cases (e.g., with oversampling and decimation). All anti-aliasing filters introduce frequency-dependent phase shift in the input signal which must be compensated for determining the phasor representation of the input signal. This will be discussed further in Chap. 5 where the ‘synchrophasor’ standard is described.

1.5 Discrete Fourier Transform (DFT)

Discrete Fourier transform (DFT) is a method of calculating the Fourier transform of a small number of samples taken from an input signal $x(t)$. The Fourier transform is calculated at discrete steps in the frequency domain, just as the input signal is sampled at discrete instants in the time domain. Consider the process of selecting N

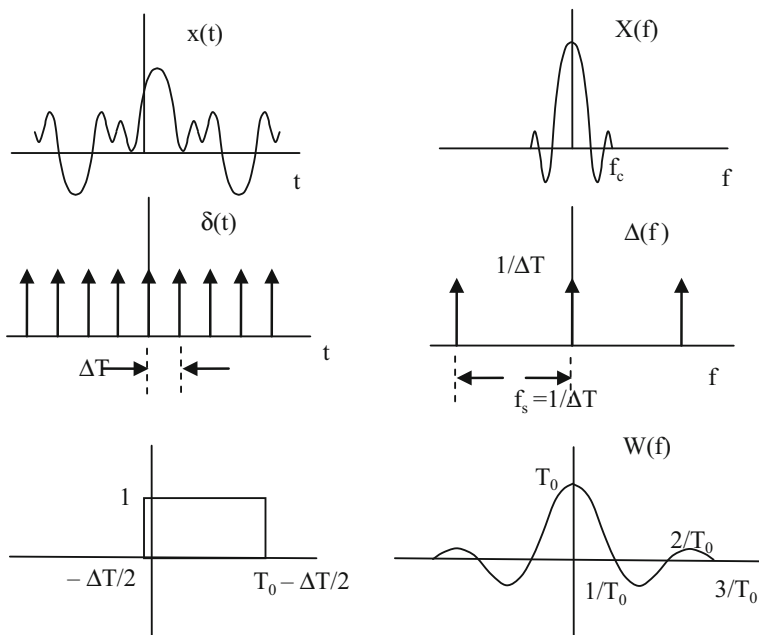


Fig. 1.8 Time functions and Fourier transforms $x(t)$, $\delta(t)$, and $w(t)$. Note that once again, the phase shift factor from $\mathcal{Q}(f)$ has been omitted

samples: $x(k\Delta T)$ with $\{k = 0, 1, 2, \dots, N - 1\}$, ΔT being the sampling interval. This is equivalent to multiplying the sampled data train by a **windowing function** $w(t)$, which is a rectangular function of time with unit magnitude and a span of $N\Delta T$. With the choice of samples ranging from 0 to $N - 1$, it is clear that the windowing function can be viewed as starting at $-\Delta T/2$ and ending at $(N - 1/2)\Delta T$. The function $x(t)$, the sampling function $\delta(t)$, and the windowing function $w(t)$ along with their Fourier transforms are shown in Fig. 1.8.

Consider the collection of signal samples which fall in the data window: $x(k\Delta T)$ with $\{k = 0, 1, 2, \dots, N - 1\}$. These samples can be viewed as being obtained by the multiplication of the signal $x(t)$, the sampling function $\delta(t)$, and the windowing function $w(t)$:

$$y(t) = x(t)\delta(t)w(t) = \sum_{k=0}^{N-1} x(k\Delta T)\delta(t - k\Delta T) \quad (1.14)$$

where once again the multiplication with the delta function is to be understood in the sense of the integral in Eq. (1.9). The Fourier transform of the sampled windowed function $y(t)$ is then the convolution of Fourier transforms of the three functions.

The Fourier transform of $y(t)$ is to be sampled in the frequency domain in order to obtain the DFT of $y(t)$. The discrete steps in the frequency domain are multiples of $1/T_0$, where T_0 is the span of the windowing function. The frequency sampling function $\Phi(f)$ is given by

$$\Phi(f) = \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right) \quad (1.15)$$

and its inverse Fourier transform (by property 1.6 of Fourier transforms) is

$$\phi(t) = T_0 \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \quad (1.16)$$

In order to obtain the samples in the frequency domain, we must multiply the Fourier transform $Y(f)$ with $F(f)$. To obtain the corresponding time-domain function $x'(t)$, we will require a convolution in the time domain of $y(t)$ and $\phi(t)$:

$$\begin{aligned} x'(t) &= y(t) * \phi(t) \\ x'(t) &= y(t) * \phi(t) = \left[\sum_{k=0}^{N-1} x(k\Delta T) \delta(t - k\Delta T) \right] * \left[T_0 \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \right] \\ &= T_0 \sum_{n=-\infty}^{\infty} \left[\sum_{k=0}^{N-1} x(k\Delta T) \delta(t - k\Delta T - nT_0) \right] \end{aligned} \quad (1.17)$$

This function is periodic with a period T_0 . The functions $x(t)$, $y(t)$, and $x'(t)$ are shown in Fig. 1.9. The windowing function limits the data to samples 0 through $N - 1$, and the sampling in frequency domain transforms the original N samples in time domain to an infinite train of N samples with a period T_0 as shown in Fig. 1.9c. Note that although the original function $x(t)$ was not periodic, the function $x'(t)$ is, and we may consider this function to be an approximation of $x(t)$.

The Fourier transform of the periodic function $x'(t)$ is a sequence of impulse functions in frequency domain by property 1.5 of the Fourier transform. Thus,

$$\begin{aligned} X'(f) &= \sum_{n=-\infty}^{\infty} \alpha_n \delta\left(f - \frac{n}{T_0}\right), \quad \text{with} \\ \alpha_n &= \frac{1}{T_0} \int_{-T_0/2}^{T_0-T_0/2} x'(t) e^{-\frac{j2\pi nt}{T_0}} dt, \quad n = 0, \pm 1, \pm 2, \dots \end{aligned} \quad (1.18)$$

Substituting for $x'(t)$ in the above expression for α_n ,