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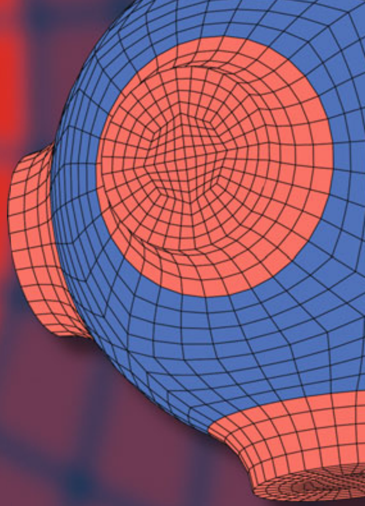

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Variational Continuum Multiphase Poroelasticity

Theory and Applications

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Theory and Applications

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To our beloved wives Ilaria and Pamela,
to our children Lorenza, Alessandro and Dylan,
and to our parents

Foreword

It is a great pleasure to present this nice book in a subject which occupied my studies for a long while. The authors have enthusiastically started a cultural and scientific endeavour which surely deserves a great investment in intelligence and study. Indeed to base the continuum mechanics of porous media on variational principles presents some relevant difficulties of mathematical nature as well as very important issues related to applicability to real-world problems. It has been a pleasure to see how the younger (compared with me) authors have approached the increasingly difficult problems which they met in pursuing their scientific objective and I hope that the discussions which we had were of some use. Actually the mathematical description of the flow of a compressible fluid in a deformable medium presents relevant difficulties and we cannot expect that these are solved quickly and naturally without an important change of the conceptual paradigm used to confront them. This monograph presents some ideas of the authors which are framed nicely in the logic started by the papers by Biot and, more recently, by Coussy and it seems to me that many of the presented methods are well grounded. The state of the art is examined from the point of view of the authors and seems rather complete, while the formulation of the mathematical models follows the standards commonly accepted in continuum mechanics. One can expect therefore that this work will have some beneficial effects in the scientific community interested in such a kind of problems. Indeed: (1) mathematicians will find a wealth of interesting problems to be studied and formalised, (2) engineers may find interesting methods for forecasting the behaviour of applicable mechanical systems, (3) theoretical mechanicians may find a further evidence about the importance of Lagrangian methods.

The enthusiasm of the authors may have led them to underestimate the relevance of some of their simplifying assumptions: however, this is the needed approach for attacking problems which resisted too much to the efforts of scientists. It is very good that they did not want to follow my invitation to prudence and to circumspection and they finally wanted to dare to formulate models in which the microstructure of the deformable matrix was explicitly taken into account: maybe it could have been done in a better way. However, it is better to start an investigation instead of postponing it, while waiting for the moment in which the logical tools are ripened. Indeed the logical tools will ripen under the push of the conjectures which are presented, for instance, in this monograph. I wish to the authors a long scientific career, which seems to me has started under the best auspices.

Rome, Italy
October 2016

Francesco dell'Isola

Preface

The main objective of this monograph is to provide a comprehensive picture of the Variational Macroscopic Theory of Porous Media (VMTPM), a general two-phase variational continuum theory with microstructure which we have been developing since 2013, based on a previous theory originally proposed in 2011. Therefore, this book contains a detailed derivation of VMTPM based on canonical arguments of variational continuum mechanics, followed by the presentation of several applications to consolidation problems we believe to be of relevance in both geomechanics and biomechanics. The intent is to show the variational consistency of this theory and to exemplify its capability to describe a large class of linear and non-linear mechanical behaviors observed in two-phase saturated materials.

During these years, VMTPM was consolidated in the theoretical fundamentals and corroborated with studies showing its capability of predicting established experimental evidences as well as of encompassing paradigms of widespread use in multiphase poroelasticity applied to geomechanics and biomechanics, such as Terzaghi's stress partitioning principle and Biot's equations. Most of the results produced by this research have been published on specialized journals and presented at international meetings in the field. Nevertheless, we believe that the monograph format provides the ideal ground to report a revisited exposition of this variational theory keeping uniformity of treatment and of notation.

In this contribution, we strove to provide a theoretical approach capable of attaining a *medium-independent* framework, presenting to the poroelasticity community a set of equations which *any other continuum theory of poroelasticity should be downward compatible to*. This is indeed rather an ambitious plan, since it requires a general enough statement of the variational model, as well as a due discussion of a number of limit cases which should be consistently embraced by any candidate general medium-independent theory of this alleged kind. Accordingly, to achieve generality, the variational theory is developed in this work proceeding from a finite kinematic description. Just to mention a few of the limit cases specifically addressed here, it is shown that VMTPM is downward compatible to single-continuum elasticity when porosity achieves zero or unity limit conditions; special care was also taken in showing that the kinematics and the mechanics of VMTPM consistently include the description of fluid flow outside of a porous body, and consistently address the presence of free solid-fluid surfaces. A discussion is also included on the extent to which the equations of this theory apply, beyond the purely mechanical context, to media with inelastic dissipative behavior, such as in elastoplasticity. Hence, the monograph format provided a wider editorial template suitable to accommodate this more extended treatment.

This work was written for an intended audience including investigators in the fields of continuum mechanics, geomechanics and biomechanics, as they will find in this contribution not only a thorough presentation of VMTPM as a theoretical framework for porous media, but also several of its applications of relevance for their research.

The authors would like to acknowledge Prof. Luciano Rosati from University of Naples Federico II, Dr. Alessandro della Corte from University of Rome La Sapienza, and Dr. Shihab Asfour from University of Miami for their scientific contribution to the material presented in Chaps. 1 and 5.

Finally, the authors wish to thank Prof. Francesco dell'Isola for the encouragement to undertake the task of writing this monographic work, for his support, and for the many insightful discussions on the roots of continuum mechanics.

Benevento, Italy Roberto Serpieri
Miami, USA Francesco Travascio

Contents

Chapter 1
Variational Multi-phase Continuum Theories of Poroelasticity: A Short Retrospective

Abstract This chapter aims at offering a comprehensive overview on the family of two-phase continuum poroelasticity theories whose formulations are based on the application of classical variational methods, or on variants of Hamilton's Least Action Principle. The reader will be walked through several theoretical approaches to poroelasticity, starting from the early use of variational concepts by Biot, then covering the variational frameworks which employ porosity-enriched kinematics, such as those proposed by Cowin and co-workers and by Bedford and Drumheller, to conclude with the most recent variational theories of multiphase poroelasticity. Arguments are provided to show that, as a widespread opinion in the poroelasticity community, even the formulation of a simplest two-phase purely-mechanical poroelastic continuum theory remains, under several respects, a still-open problem of applied continuum mechanics, with the *closure problem* representing a crucial issue where important divergencies are found among the several proposed frameworks. Concluding remarks are finally drawn, pointing out the existence of delicate open issues even in the subclass of variational two-phase theories of poroelasticity.

1.1 Introduction

Continuum poroelastic frameworks are employed in a wide range of applications. Aside from their classical use in the field of soil mechanics [14, 38, 83], poroelastic models have been garnering increasing popularity for describing the complex phenomenology of biological tissue mechanics and its remodeling processes (see, for instance, [3, 4, 22, 35, 42, 66, 70]). Several problems encountered in geomechanics or biomechanics require, however, a multiphase description if one aims at capturing many concurring deformation-related phenomena. For instance, in geomechanics, soils might be partially saturated and should be consequently treated as systems composed of three phases (i.e., solid phase, water and air) [64, 76]. Also, in the field of biomechanics, cartilaginous tissues have been represented as mixtures of a solid and electrically charged network of structural macromolecules embedded in an interstitial solution of water and solutes [48, 52, 54, 84].

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Despite this broad range of applications, by comprehensively scoping the past and recent literature related to continuum poroelasticity, one would be led to conclude that, to date, the achievement of a unified theory of multiphase continuum poroelasticity, capable of addressing multiphase systems with any range of compressibility of the constituents, still represents a challenge of theoretical and applied continuum mechanics, even for the simpler two-phase problem. Actually, if by the term *standard* poroelasticity one refers to a generally agreed minimal closed set of mathematically consistent and physically plausible governing equations of two-phase poroelasticity, deducible from the classical principles of physics and with assessed predictive capabilities, insightful and comprehensive survey works have highlighted that the development of such a *standard* theory has been complex and controversial since the first conception of multiphase continuum theories.

A quite complete picture of the researches conducted from the beginning of the last century to the eighties can be gained from the historical review by De Boer [24] and from the comprehensive survey by Bedford and Drumheller on the group of theories frequently gathered under the term *Theories of Immiscible Mixtures* (TIM) [9]. The review [24] covers, in particular, most of the continuum poroelastic approaches proposed since the early Terzaghi-Fillunger dispute [38, 83], including the fundamental theoretical contributions by Biot [14] and the compelling experimental evidences from geomechanics [73, 81]. The review on TIMs encompasses classical theories such as the so-called *Continuum theory of mixtures* contributed by Truesdell [85] as well as poroelasticity theories deriving from generalized continuum formulations which employ enhanced microstructural descriptions of the solid phase deformation (such as Mindlin's theory of linear elasticity with microstructure [67] and the micromorphic theory by Eringen [36]), up to multiphase theories, contemporary to the end of the seventies and the beginning of the eighties, [7–9, 21, 43].

From the eighties onwards, theoretical research efforts have been aimed at developing general and comprehensive multiphase flow theories, driven by the increase of advanced applications of multiphase poroelasticity in geomechanics, biomechanics, environmental engineering and material engineering. On the one hand, research in this area has accordingly experienced a proliferation of porous media frameworks which have proceeded quite independently by stressing different arguments in order to achieve the formulation of a standard macroscopic governing set of continuum equations. On the other hand, theoretical research has kept steadily searching for a fundamental set of governing equations achieving general consensus.

To find a possible logical organization of the several research efforts driven by such a *multiplication of languages* from the eighties until current times, a classification of the mainstream approaches might be attempted, without any claim of completeness and of clean-cut separation.

A first classification criterion can be considered according to the conceptual scheme followed for introducing enhanced mechanical features into the theory, whereby two approaches can be identified: (1) *Purely Macroscale Theories* (PMT), which are based on the introduction of kinematic descriptors or constitutive features expressly at the macroscale level, and (2) *upscaling/Averaging Theories* (AT), which proceed by considering a detailed representation of the geometry and flow processes at the microscale to subsequently apply space averaging techniques.

Both PMT and AT approaches are not exempt to criticisms (see for intance the debates in [5, 46, 47]). In any case, PMT approaches are exposed to the criticism of lacking a strong connection with the pore scale physics and of performing implicit approximations. On the other hand, AT frameworks have been criticized for being frequently employed in combination with assumptions justifiable only on a heuristic basis, and also for lacking sometimes in their application a clear link with the macroscopic measurement processes.

As observed in [40], purely macro-scale theories can be further classified according to the setting employed for the definition of energy potentials of the constituent phases. In particular, a first group of PMT approaches adopts a single macroscopic energy potential of the whole saturated mixture from which stresses of individual phases are derived (see for instance [20]). A second group of PMT includes approaches where the two phases are treated as superposed continua, each one endowed with a separate energy potential. As observed by several authors, among them Svendsen and Hutter [82], the use of individual strain energy potentials for each phase requires, alongside of linear momentum and mass balances, an additional governing equation to match the number of unknown fields with the number of equations. This lacking equation is referred to by most authors as the *closure equation*.

It should be remarked that, in the specialized literature, very different solutions to the problem of the proper identification of such missing closure equation (or of the set of closure equations) have been proposed to construct a minimal set of governing balance equations achieving a general consistent formulation of compressible poroelasticity. For instance, in several works, in agreement with an early indication by Truesdell (requoted after Bedford and Drumheller) according to whom "*the 'missing principle', surely, is a proper generalization of the Clausius-Duhem inequality*", the closure of the poroelastic problem has been attempted by supplementing momentum and mass balance equations with the second law of thermodynamics [17, 49, 76]. In [17], the closure of the problem has been also attempted by supplying the governing equations with the momentum of momentum balance (tracing back to Cosserat's theory [19, 37]) and by further including evolution equations of volume fractions similar to [32]. Further approaches deployed to achieve the closure of the biphasic problem have been proposed by incorporating a saturation constraint in the entropy inequality, together with the use of an incompressibility hypothesis and of a Lagrangian multiplier [24, 82]. Also, in [31], a multiplicative decomposition of the strain tensor has been considered in combination with moment of momentum balance. Among several other solutions proposed for the closure problem, Albers and Wilmański, upon adding porosity as an additional independent kinematic descriptor field, have investigated, as candidate closure equations, a postulated porosity balance equation and an equation representing an integrability condition for the deformation of the solid skeleton [2, 88]. More recently, a geometric saturation constraint has also been considered, combined with a multiplicative decomposition of the deformation gradient, by de Boer [16].

The above highlighted diverging solutions given to the closure problem by different research groups motivates the opinion, widely spread indeed in the multiphase poroelasticity community, that even the simpler two-phase purely-mechanical problem of poroelasticity remains to date an open problem of continuum mechanics. This opinion is well represented by the words of De Boer: "*the necessity to attack the problem of developing a consistent general poroelasticity theory is still existent*" [16], and it has been more recently remarked also by Lopatnikov and Gillespie [61] "... *in spite of a tremendous number of publications in this field, the discussion continues about physical background of the poroelastic theory. Even the form of basic governing equations are sufficiently different [...] in frame of different approaches that one can find in literature. It seems that there is no final agreement about consistency of proposed different approaches*".

The objective of the present chapter is to provide an updated survey on the family of two-phase continuum poroelasticity theories which can be identified to be of *variational* type and to be based on a purely macroscale formulation (PMT). The survey reported in this chapter retraces the review on variational multiphase theories reported in [80] and is mainly intended to provide a scientific background for the subsequent chapters where a variational multiphase theory is proposed.

The reason for paying special attention to variational theories is that we share the opinion that variational statements are privileged means for the continuum description of physical phenomena ensuring "*a natural and rigorously correct way to think of [...] continuum physics*" [74]. Far from being only a matter of formal elegance and consistency, the minimization principle built in variational approaches is also very convenient as a natural ground for the development of advanced numerical integration schemes for multiphase problems. Actually, as the minimization automatically provides theories in weak form whose discretization naturally leads to Finite Element (FE) formulations, variational methods appear to be suited for developing equations prone to robust numerical integration. This is especially true in presence of high-order differential terms, such as those characteristically stemming from generalized continuum theories and multiphase theories, especially when high-regularity interpolations are invoked. This has been recently shown in different contexts, in particular, when one desires to deal with complex geometries by invoking isogeometric analysis (see e.g., [18, 23, 51, 58, 69]).

The survey herein reported, refraining from pursuing a comprehensive updated review of the whole class and variants of the currently available porous media multiphase frameworks (what would indeed represent a major bibliographical review effort) restricts the attention to the subclass of those two-phase and multi-phase continuum poroelasticity theories whose formulations are based on the application of differently named declinations and variants of classical variational methods (Principle of Virtual Powers, Principle of Virtual Works, extended Hamilton-Rayleigh principle, etc.), all ultimately stemming from Hamilton's Least Action Principle [10, 27, 56, 68]. Attention is primarily payed on those theoretical works which share a specific focus on solid mechanics. This survey recalls also few works which, although not properly variational, employ, at least in part, some variational concepts to multiphase continuum solid mechanics.

To keep the number of pages of this monograph limited, this chapter does not include further important works where more elaborated concepts have been applied to the multiphase poroelastic problem, such as the dependency of the energy functionals upon higher-order deformation and density gradients, [25, 44, 79]. Further important theoretical contributions which, although having a variational content, are primarily focused on fluid mechanics, e.g., [41, 45], are also not included in the present review.

A final remark concerns the notation employed in the mathematical expressions reported in this chapter. We deliberately decided to leave the number of recalled equations and symbols to a minimum. Also, in reviewing some equations, we preferred to preserve a coherent and uniform notation, even though this required, in some cases, the modification of the original format of some equations. All the relations in the present chapter were however intended to be a faithful rendering of the original ones in the cited papers. These choices were made necessary to deal with the fact that, especially among pioneering works, substantially different notations and conventions are employed.

1.2 Variational Theories from the 70s to the 80s

An early use of variational concepts in the derivation of a mixture theory has been claimed by Truesdell [87] (p. 567) to trace back to Duhem [33].

It should also be recalled that even the seminal and very popular theory of poroelasticity by Biot [13–15], although partly developed also with the aid of definition of stress measures and elastic relations based on intuitive (and sometimes heuristic) mechanical considerations [6], was subsequently framed into a quasi-static isothermal variational framework [11] whose equilibrium equations are obtained proceeding from the statement of a principle of virtual works. Later, this framework was further extended to account for nonisothermal deformations and to incorporate dynamics [12].

Specifically, in [11], variational concepts are applied proceeding from the consideration of an 'isothermal free energy density function'. This function is defined to be dependent upon the finite strain tensor of the solid phase and on a further state descriptor m which is the total mass of fluid added during deformation in the pores of the specimen. It should be remarked that the possibility of defining a proper variational theory combined with the choice of including m among the descriptors has been questioned by several authors. Indeed, it has been observed, for instance by Wilmański [89], that it is not possible to construct a true variational principle since, for open mechanical systems where mass is not fixed, m is a nonequilibrium variable.

Other formulations appeared in the seventies which combine some variational ideas with postulated momentum balance equations for deriving multiphase porous media theories are those in [1, 53]. In particular, in [53], the employed kinematic descriptors are the densities and the deformation gradients of each phase, and a variational postulate is proposed to obtain the linear momentum balance equations introduced by Truesdell [86].

1.2.1 Cowin's Theory

The multiphase theories of mixtures by Nunziato and Walsh [72] and by Passman [75] can be also stated to have a partly variational character. Such frameworks turn out to be extensions of the continuum theory for granular materials by Goodman and Cowin [43], and exploit the idea of adding the volume fraction of the solid phase as an additional kinematic continuum scalar descriptor. In the above mentioned works an additional balance scalar equation, proposed by Goodman and Cowin [43] and termed *equation of balance of equilibrated force*, is considered to pair the number of unknown fields with the number of governing PDEs, incremented by one due to the addition of this scalar descriptor.

While the formulation by Goodman and Cowin had not been originarily presented, in [43], in a variational guise as it contained ad-hoc modified forms of mass balance and ad-hoc modified momentum and energy balances, the same authors in a subsequent study, [21], were able show that the equation denominated 'balance of equilibrated forces' can be actually derived by a postulated variational principle encompassing the dependence of a stored energy density function upon the solid volume fraction, the true density of the solid porous phase $\hat{\rho}^{(s)}$, and upon the solid volume fraction $\phi^{(s)}$ and its space gradient $\phi^{(s)}\nabla$.

However, it might still be remarked that such restated variational theory does not appear to be a standard one. In particular, Eq. (13) therein presents a postulated condition, directly expressed in the form of first-variations, which contains two postulated generalized stress quantities, H, l, termed *self-equilibrated stress system* and *self-equilibrated body force*, respectively. Also, in this theory the stress tensor of the solid phase is not defined as a quantity work-associated with the symmetric part of the displacement gradient, being instead introduced as a quantity work-associated with the solid true density $\hat{\rho}^{(s)}$.

1.2.2 Mindlin's Variational Single-Phase Theory

Mindlin's single-phase continuum theory of materials with microstructure [67], although not directly applied by Mindlin to multiphase problems, has a prominent importance in the formulation of multiphase theories, as it has laid the ground for several subsequent consistent developments, on a variational basis, of multiphase poroelastic continuum frameworks which, similar to the works of the previous section, have made use of porosity, and volume fractions, as additional kinematic descriptors.

In [67], the equations of motion are derived via Hamilton's principle:

$$\delta \int_{t_1}^{t_2} (T - V) d t + \int_{t_1}^{t_2} \delta W d t = 0, \tag{1.1}$$

where t_1 and t_2 are two arbitrarily assigned time instants, T is the kinetic energy, V is the internal potential energy. The term δW in (1.1) synthetically comprehends the virtual work of external body forces, external traction vectors, generalized body forces and generalized surface forces (referred to as *double forces*).

Mindlin's theory is quite general and very well known, indeed, since it has set the basis for several important formulations of generalized continua. In Mindlin's framework a macroscopic second-order tensor field, the *microdeformation*, is added as a further kinematic descriptor complementing the displacement field. Next, a vectorial linear momentum balance is derived expressing the stationarity of (1.1) with respect to the displacement field, and additional scalar equations are derived expressing stationarity with respect to the independent components of the microdeformation.

The strain measures of this theory are the standard strain tensor, defined as the symmetric part of the gradient of the displacement field, together with two additional strain measure fields related to the microdeformation: the *microdeformation gradient*, and the so-called *relative deformation field*, defined as the difference between the gradient of the displacement and the microdeformation. On this basis, the strain energy turns out to be a homogeneous quadratic function of the strain tensor, of the gradient of the microdeformation, and of the relative deformation field.

1.2.3 The Variational Theory of Immiscible and Structured Mixtures by Bedford and Drumheller

Advances in the derivation of variational theories of multiphase porous media and structured mixtures were provided by Bedford and Drumheller. In their works [7–9], these authors extended the original Mindlin's ideas of single-continuum framework of microstructured continua and the approaches for the variational treatment of a single continuum in solid and fluid mechanics found in [34, 39, 50, 55, 57, 74] to derive the balance equations for porous multiphase problems by means of the Hamilton's principle.

These authors assume, in particular in [8], that the mechanical response of the generic phase ξ (with $\xi \in \{1, \dots, N\}$ and where N is the number of phases) is defined by a density of strain energy ψ which is only dependent on the *true density* $\hat{\rho}^{(\xi)}$, related to the relevant apparent density $\bar{\rho}^{(\xi)}$ by the usual relation:

$$\hat{\rho}^{(\xi)} = \frac{\bar{\rho}^{(\xi)}}{\phi^{(\xi)}}, \tag{1.2}$$

where $\phi^{(\xi)}$ is the volume fraction of the generic ξ-th phase. In the subsequent work [7] this conceptual scheme is enhanced encompassing a dependence of ψ upon $\hat{\rho}^{(\xi)}$ and upon the (infinitesimal) strain tensor $\boldsymbol{\varepsilon}$.

The primary descriptors of this formulation are $\phi^{(\xi)}$ and $\hat{\rho}^{(\xi)}$, together with the placement field $\boldsymbol{\chi}^{(\xi)}$ which operates the association $\mathbf{x}^{(\xi)} = \boldsymbol{\chi}^{(\xi)}\left(\mathbf{X}^{(\xi)}\right)$ between the current position $\mathbf{x}^{(\xi)}$ of phase ξ and its reference material position $\mathbf{X}^{(\xi)}$. In agreement with Leech [57], the least-Action condition is written integrating over a fixed reference volume domain containing a fixed mass of mixture.

It should be noted that, in this formulation, the primary descriptors are not unconstrained fields. In fact, $\phi^{(\xi)}$, $\hat{\rho}^{(\xi)}$, and $\boldsymbol{\chi}^{(\xi)}$ are constrained by the mass balance:

$$J^{(\xi)}\bar{\rho}^{(\xi)} = \bar{\rho}_0^{(\xi)} \tag{1.3}$$

and by the saturation condition:

$$\sum_{\xi=1}^{N} \phi^{(\xi)} = 1. \tag{1.4}$$

In compliance with (1.3) and (1.4), the variations $\delta\phi^{(\xi)}$, $\delta\hat{\rho}^{(\xi)}$ and $\delta\mathbf{x}^{(\xi)}$ are also constrained to each other. Such constraints are included via the addition of (1.3) and (1.4) into (1.1) with the aid of Lagrange multipliers λ and μ_ξ. The resulting equation has the format:

$$\delta\int_{t_1}^{t_2}(T-V)dt + \int_{t_1}^{t_2}\delta W dt+ \\ +\int_{t_1}^{t_2}\left[\sum_{\xi=1}^{N}\int_{\Omega}\mu_\xi\delta\left(J^{(\xi)} - \frac{\phi_0^{(\xi)}\hat{\rho}_0^{(\xi)}}{\phi^{(\xi)}\hat{\rho}^{(\xi)}}\right)dV_0 - \int_{\Omega}\lambda\delta\left(\sum_{\xi=1}^{N}\phi^{(\xi)}\right)\Bigg|_{\mathbf{x}} dV_0\right]dt = 0. \tag{1.5}$$

The physical interpretation of parameters λ and μ_ξ has been discussed [7]: based on the standard notion of Lagrange multipliers as generalized forces ensuring the constraint to be satisfied, and to some considerations on pressure force balances, the authors interpret λ as an *interface pressure* between constituents, and obtain for μ_ξ the relationship $\mu_\xi = \dfrac{p^{(\xi)}\phi^{(\xi)}}{J^{(\xi)}}$, where $p^{(\xi)}$ indicates the pressure of the ξ-th constituent.

It should be remarked that the mechanical consistency of the choice of incorporating of the effect of constraints in a variational framework has been subjected to debate and objections between researchers. In the review [9], Bedford and Drumheller recall a criticism by Truesdell and Toupin who observed that incorporating the effect of constraints in variational principles "... is a somewhat dubious blessing" [87]. Bedford and Drumheller rebutted that the volume fraction constraint does not entail ill-posedness issues, and remarked that the admissibility and usefulness of the volume fraction constraint in multiphase theories can be standardly accepted as a continuum mechanical analogue to the treatment of connections between rigid bodies in the variational description of the mechanics of these rigid systems.