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The Nikolai Vasilevskii Anniversary Volume

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The Life and Work of Nikolai Vasilevski

Sergei Grudsky, Yuri Latushkin and Michael Shapiro

Nikolai Leonidovich Vasilevski was born on January 21, 1948 in Odessa, Ukraine. His father, Leonid Semenovich Vasilevski, was a lecturer at Odessa Institute of Civil Engineering, his mother, Maria Nikolaevna Krivtsova, was a docent at the Department of Mathematics and Mechanics of Odessa State University.

In 1966 Nikolai graduated from Odessa High School Number 116, a school with special emphasis in mathematics and physics, that made a big impact at his creative and active attitude not only to mathematics, but to life in general. It was a very selective high school accepting talented children from all over the city, and famous for a high quality selection of teachers. A creative, nonstandard, and at the same time highly personal approach to teaching was combined at the school with a demanding attitude towards students. His mathematics instructor at the high school was Tatjana Aleksandrovna Shevchenko, a talented and dedicated teacher. The school was also famous because of its quite unusual by Soviet standards system of self-government by the students. Quite a few graduates of the school later became well-known scientists, and really creative researchers.

In 1966 Nikolai became a student at the Department of Mathematics and Mechanics of Odessa State University. Already at the third year of studies, he began his serious mathematical work under the supervision of the well-known Soviet mathematician Georgiy Semenovich Litvinchuk. Litvinchuk was a gifted teacher and scientific adviser. He, as anyone else, was capable of fascinating his students by new problems which have been always interesting and up-to-date. The weekly Odessa seminar on boundary value problems, chaired by Prof. Litvinchuk for more than 25 years, very much influenced Nikolai Vasilevski as well as others students of G.S. Litvinchuk.

N. Vasilevski started to work on the problem of developing the Fredholm theory for a class of integral operators with nonintegrable integral kernels. In essence, the integral kernel was the Cauchy kernel multiplied by a logarithmic factor. The integral operators of this type lie between the singular integral operators and the integral operators whose kernels have weak (integrable) singularities. A famous Soviet mathematician F.D. Gakhov posted this problem in early 1950ies, and it remained open for more than 20 years. Nikolai managed to provide a complete solution in the setting which was much more general than the original. Working on this problem, Nikolai has demonstrated one of the main traits of his mathematical talent: his ability to achieve a deep penetration in the core of the problem, and to see rather unexpected connections between different theories. For instance, in order to solve Gakhov's Problem, Nikolai utilized the theory of singular integral operators with coefficients having discontinuities of first kind, and the theory of operators whose integral kernels have fixed singularities – both theories just appeared at that time. The success of the young mathematician was well recognized by a broad circle of experts working in the area of boundary value problems and operator theory. In 1971 Nikolai was awarded the prestigious M. Ostrovskii Prize, given to the young Ukrainian scientists for the best research work. Due to his solution of the famous problem, Nikolai quickly entered the mathematical community, and became known to many prominent mathematicians of that time. In particular, he was very much influenced by the his regular interactions with such outstanding mathematicians as M.G. Krein and S.G. Mikhlin.

In 1973 N. Vasilevski defended his PhD thesis entitled "To the Noether theory of a class of integral operators with polar-logarithmic kernels". In the same year he became an Assistant Professor at the Department of Mathematica and Mechanics of Odessa State University, where he was later promoted to the rank of Associate Professor, and, in 1989, to the rank of Full Professor.

Having received the degree, Nikolai continued his active mathematical work. Soon, he displayed yet another side of his talent in approaching mathematical problems: his vision and ability to use general algebraic structures in operator theory, which, on one side, simplify the problem, and, on another, can be used in many other problems. We will briefly describe two examples of this.

The first example is the method of orthogonal projections. In 1979, studying the algebra of operators generated by the Bergman projection, and by the operators of multiplication by piece-wise continuous functions, N. Vasilevski gave a description of the C^*-algebra generated by two self-adjoint elements s and n satisfying the properties $s^2 + n^2 = \mathrm{e}$ and $sn + ns = 0$. A simple substitution $p = (\mathrm{e} + s - n)/2$ and $q = (\mathrm{e} - s - n)/2$ shows that this algebra is also generated by two self-adjoint idempotents (orthogonal projections) p and q (and the identity element e). During the last quarter of the past century, the latter algebra has been rediscovered by many authors all over the world. Among all algebras generated by orthogonal projections, the algebra generated by two projections is the only tame algebra (excluding the trivial case of the algebra with identity generated by one orthogonal projection). All algebras generated by three or more orthogonal projections are known to be wild, even when the projections satisfy some additional constrains. Many model algebras arising in operator theory are generated by orthogonal projections, and thus any information of their structure essentially broadens the set of operator algebras admitting a reasonable description. In particular, two and more orthogonal projections naturally appear in the study of various algebras generated by the Bergman projection and by piece-wise continuous functions having two or more different limiting values at a point. Although these projections, say, P, $Q_1, \dots, Q_n$, satisfy an extra condition $Q_1 + \cdots + Q_n = I$, they still generate, in general, a wild C^*-algebra. At the same time, it was shown that the structure of the algebra just mentioned is determined by the joint properties of certain positive injective contractions C_k, $k = 1, \ldots, n$, satisfying the identity $\sum_{k=1}^{n} C_k = I$, and, therefore, the structure is determined by the structure of the C^*-algebra generated by the contractions. The principal difference between the case of two projections and the general case of a finite set of projections is now completely clear: for $n = 2$ (and the projections P and $Q + (I - Q) = I$) we have only one contraction, and the spectral theorem directly leads to the desired description of the algebra. For $n \geq 2$ we have to deal with the C^*-algebra generated by a finite set of noncommuting positive injective contractions, which is a wild problem. Fortunately, for many important cases related to concrete operator algebras, these projections have yet another special property: the operators $PQ_1P, \ldots, PQ_nP$ mutually commute. This property makes the respective algebra tame, and thus it has a nice and simple description as the algebra of all $n \times n$ matrix-valued functions that are continuous on the joint spectrum Δ of the operators $PQ_1P, \ldots, PQ_nP$, and have certain degeneration on the boundary of Δ.

Another notable example of the algebraic structures used and developed by N. Vasilevski is his version of the Local Principle. The notion of locally equivalent operators, and localization theory were introduced and developed by I. Simonenko in mid-sixtieth. According to the tradition of that time, the theory was focused on the study of individual operators, and on the reduction of the Fredholm properties of an operator to local invertibility. Later, different versions of the local principle have been elaborated by many authors, including, among others, G.R. Allan, R. Douglas, I.Ts. Gohberg and N.Ia. Krupnik, A. Kozak, B. Silbermann. In spite of the fact that many of these versions are formulated in terms of Banach- or C^*-algebras, the main result, as before, reduces invertibility (or the Fredholm property) to local invertibility. On the other hand, at about the same time, several papers on the description of algebras and rings in terms of continuous sections were published by J. Dauns and K.H. Hofmann, M.J. Dupré, J.M.G. Fell, M. Takesaki and J. Tomiyama. These two directions have been developed independently, with no known links between the two series of papers. N. Vasilevski was the one who proposed a local principle which gives the global description of the algebra under study in terms of continuous sections of a certain canonically defined C^*-bundle. This approach is based on general constructions of J. Dauns and K.H. Hofmann, and results of J. Varela. The main contribution consists of a deep re-comprehension of the traditional approach to the local principles unifying the ideas coming from both directions mentioned above, which results in a canonical procedure that provides the global description of the algebra under consideration in terms of continuous sections of a C^*-bundle constructed by means of local algebras.

In the eighties and even later, the main direction of the work of Nikolai Vasilevski has been the study of multi-dimensional singular integral operators with discontinuous coefficients. The main philosophy here was to study first algebras containing these operators, thus providing a solid foundation for the study of various properties (in particular, the Fredholm property) of concrete operators. The main tool has been the described above version of the local principle. This principle was not merely used to reduce the Fredholm property to local invertibility but also for a global description of the algebra as a whole based on the description of the local algebras. Using this methodology, Nikolai Vasilevski obtained deep results in the theory of operators with Bergman's kernel and piece-wise continuous coefficients, in the theory of multi-dimensional Toeplitz operators with pseudo-differential presymbols, in the theory of multi-dimensional Bitsadze operators, in the theory of multi-dimensional operators with shift, etc. In 1988 N. Vasilevski defended the Doctor of Sciences dissertation, based on these results, and entitled "Multi-dimensional singular integral operators with discontinuous classical symbols".

Besides being a very active mathematician, N. Vasilevski has been an excellent lecturer. His lectures are always clear, and sparkling, and full of humor, which so natural for someone who grew up in Odessa, a city with a longstanding tradition of humor and fun. He was the first at Odessa State University who designed and started to teach a class in general topology. Students happily attended his lectures in Calculus, Real Analysis, Complex Analysis, Functional Analysis. He has been one of the most popular professor at the Department of Mathematics and Mechanics of Odessa State University. Nikolai is a master of presentations, and his colleagues always enjoy his talks at conferences and seminars.

In 1992 Nikolai Vasilevski moved to Mexico. He started his career there as an Investigator (Full Professor) at the Mathematics Department of CINVESTAV (Centro de Investagacion y de Estudios Avansados). His appointment significantly strengthen the department which is one of the leading mathematical centers in Mexico. His relocation also visibly revitalized mathematical activity in the country in the field of operator theory. Actively pursuing his own research agenda, Nikolai also served as the organizer of several important conferences. For instance, let us mention the (regular since 1998) annual workshop "Análisis Norte-Sur", and the well-known international conference IWOTA-2009. He initiated the relocation to Mexico a number of active experts in operator theory such as Yu. Karlovich and S. Grudsky, among others.

During his tenure in Mexico, Nikolai Vasilevski produced a sizable group of students and younger colleagues; five of young mathematicians received PhD under his supervision.

The contribution of N. Vasilevski in the theory of multi-dimensional singular integral operators found its rather unexpected development in his work on quaternionic and Clifford analysis, published mainly with M. Shapiro in 1985–1995, starting still in the Soviet Union, with the subsequent continuation during the Mexican period of his life. Among others, the following topics have been considered: The settings for the Riemann boundary value problem for quaternionic functions that are taking into account both the noncommutative nature of quaternionic multiplication and the presence of a family of classes of hyperholomorphic functions, which adequately generalize the notion of holomorphic functions of one complex variable; algebras, generated by the singular integral operators with quaternionic Cauchy kernel and piece-wise continuous coefficients; operators with quaternion and Clifford Bergman kernels. The Toeplitz operators in quaternion and Clifford setting have been introduced and studied in the first time. This work found the most favorable response and initiated dozens of citations.

During his life in Mexico, the scientific interests of Nikolai Vasilevski mainly concentrated around the theory of Toeplitz operators on Bergman and Fock spaces. In the end of 1990ies, N. Vasilevski discovered a quite surprising phenomenon in the theory of Toeplitz operators on the Bergman space. Unexpectedly, there exists a rich family of commutative C^*-algebras generated by Toeplitz operators with non-trivial defining symbols. In 1995 B. Korenblum and K. Zhu proved that the Toeplitz operators with radial defining symbols acting on the Bergman space over the unit disk can be diagonalized with respect to the standard monomial basis in the Bergman space. The C^*-algebra generated by such Toeplitz operators is therefore obviously commutative. Four years later N. Vasilevski also showed the commutativity of the C^*-algebra generated by the Toeplitz operators acting on the Bergman space over the upper half-plane and with defining symbols depending only on $\operatorname{Im} z$. Furthermore, he discovered the existence of a rich family of commutative C^*-algebras of Toeplitz operators. Moreover, it turned out that the smoothness properties of the symbols do not play any role in commutativity: the symbols can be merely measurable. Surprisingly, everything is governed by the geometry of the underlying manifold, the unit disk equipped with the hyperbolic metric. The precise description of this phenomenon is as follows. Each pencil of hyperbolic geodesics determines the set of symbols which are constant on the corresponding cycles, the orthogonal trajectories to geodesics forming the pencil. The C^*-algebra generated by the Toeplitz operators with such defining symbols is commutative. An important feature of such algebras is that they remain commutative for the Toeplitz operators acting on each of the commonly considered weighted Bergman spaces. Moreover, assuming some natural conditions on "richness" of the classes of symbols, the following complete characterization has been obtained: A C^*-algebra generated by the Toeplitz operators is commutative on each weighted Bergman space if and only if the corresponding defining symbols are constant on cycles of some pencil of hyperbolic geodesics. Apart from its own beauty, this result reveals an extremely deep influence of the geometry of the underlying manifold on the properties of the Toeplitz operators over the manifold. In each of the mentioned above cases, when the algebra is commutative, a certain unitary operator has been constructed. It reduces the corresponding Toeplitz operators to certain multiplication operators, which also allows one to describe their representations of spectral type. This gives a powerful research tool for the subject, in particular, yielding direct access to the majority of the important properties such as boundedness, compactness, spectral properties, invariant subspaces, of the Toeplitz operators under study.

The results of the research in this directions became a part of the monograph “Commutative Algebras of Toeplitz Operators on the Bergman Space” published by N. Vasilevski in Birkhäuser in 2008.

Nikolai Leonidovich Vasilevski passed his sixties birthday on full speed, and being in excellent shape. We, his friends, students, and colleagues, wish him further success and, above all, many new interesting and successfully solved problems.

Principal publications of Nikolai Vasilevski

Book

1. N.L. Vasilevski. *Commutative Algebras of Toeplitz Operators on the Bergman Space*, Operator Theory: Advances and Applications, Vol. 183, Birkhäuser Verlag, Basel-Boston-Berlin, 2008, XXIX, 417 p.

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On the Structure of the Eigenvectors of Large Hermitian Toeplitz Band Matrices

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For Nikolai Vasilevski on His 60th Birthday

Abstract. The paper is devoted to the asymptotic behavior of the eigenvectors of banded Hermitian Toeplitz matrices as the dimension of the matrices increases to infinity. The main result, which is based on certain assumptions, describes the structure of the eigenvectors in terms of the Laurent polynomial that generates the matrices up to an error term that decays exponentially fast. This result is applicable to both extreme and inner eigenvectors.

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1. Introduction and main results

Given a function a in L^1 on the complex unit circle $\mathbf{T}$, we denote by a_ℓ the ℓth Fourier coefficient,

$$a_\ell = \frac{1}{2\pi}\int_0^{2\pi} a(e^{ix})e^{-i\ell x}dx \quad (\ell \in \mathbf{Z}),$$

and by $T_n(a)$ the $n \times n$ Toeplitz matrix $(a_{j-k})_{j,k=1}^n$. We assume that a is real-valued, in which case the matrices $T_n(a)$ are all Hermitian. Let

$$\lambda_1^{(n)} \le \lambda_2^{(n)} \le \cdots \le \lambda_n^{(n)}$$

be the eigenvalues of $T_n(a)$ and let

$$\{v_1^{(n)}, v_2^{(n)}, \ldots, v_n^{(n)}\}$$

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be an orthonormal basis of eigenvectors such that $T_n(a)v_j^{(n)} = \lambda_j^{(n)} v_j^{(n)}$. The present paper is dedicated to the asymptotic behavior of the eigenvectors $v_j^{(n)}$ as $n \to \infty$.

To get an idea of the kind of results we will establish, consider the function $a(e^{ix}) = 2 - 2\cos x$. The range $a(\mathbf{T})$ is the segment $[0, 4]$. It is well known that the eigenvalues and eigenvectors of $T_n(a)$ are given by

$$\lambda_j^{(n)} = 2 - 2\cos\frac{\pi j}{n+1}, \quad x_j^{(n)} = \sqrt{\frac{2}{n+1}} \left(\sin \frac{m\pi j}{n+1} \right)_{m=1}^{n}. \tag{1.1}$$

(We denote the eigenvectors in this reference case by $x_j^{(n)}$ and reserve the notation $v_j^{(n)}$ for the general case.) Let φ be the function

$$\varphi : [0, 4] \to [0, \pi], \quad \varphi(\lambda) = \arccos \frac{2 - \lambda}{2}.$$

We have $\varphi(\lambda_j^{(n)}) = \pi j/(n + 1)$ and hence, apart from the normalization factor $\sqrt{2/(n+1)}$, $x_{j,m}^{(n)}$ is the value of $\sin(m\varphi(\lambda))$ at $\lambda = \lambda_j^{(n)}$. In other words, an eigenvector for λ is given by $(\sin(m\varphi(\lambda)))_{m=1}^n$. A speculative question is whether in the general case we can also find functions Ω_m such that, at least asymptotically, $(\Omega_m(\lambda))_{m=1}^n$ is an eigenvector for λ. It turns out that this is in general impossible but that after a slight modification the answer to the question is in the affirmative. Namely, we will prove that, under certain assumptions, there are functions Ω_m, Φ_m and real-valued functions σ, η such that an eigenvector for $\lambda = \lambda_j^{(n)}$ is always of the form

$$\left(\Omega_m(\lambda) + \Phi_m(\lambda) + (-1)^{j+1} e^{-i(n+1)\sigma(\lambda)} e^{-i\eta(\lambda)} \overline{\Phi_{n+1-m}(\lambda)} + \text{error term} \right)_{m=1}^{n}. \tag{1.2}$$

The error term will be shown to decrease to zero exponentially fast and uniformly in j and m as $n \to \infty$. Moreover, we will show that $\Omega_m(\lambda)$ is an oscillating function of m for each fixed λ and that $\Phi_m(\lambda)$ decays exponentially fast to zero as $m \to \infty$ for each λ (which means that $\Phi_{n+1-m}(\lambda)$ is an exponentially increasing function of m for each λ). Finally, it will turn out that

$$\sum_{m=1}^{n} |\Phi_m(\lambda)|^2 \Big/ \sum_{m=1}^{n} |\Omega_m(\lambda)|^2 = O\left(\frac{1}{n}\right)$$

as $n \to \infty$, uniformly in λ. Thus, the dominant term in (1.2) is $\Omega_m(\lambda)$, while the terms containing $\Phi_m(\lambda)$ and $\Phi_{n+1-m}(\lambda)$ may be viewed as twin babies.

If a is also an even function, $a(e^{ix}) = a(e^{-ix})$ for all x, then all the matrices $T_n(a)$ are real and symmetric. In [4], we conjectured that then, again under additional but reasonable assumptions, the appropriately rotated extreme eigenvectors $v_j^{(n)}$ are all close to the vectors $x_j^{(n)}$. To be more precise, we conjectured that if $n \to \infty$ and j (or $n - j$) remains fixed, then there are complex numbers $\tau_j^{(n)}$ of modulus 1 such that

$$\left\| \tau_j^{(n)} v_j^{(n)} - x_j^{(n)} \right\|_2 = o(1), \tag{1.3}$$

where $\|\cdot\|_2$ is the ℓ^2 norm. Several results related to this conjecture were established in [3] and [4]. We here prove this conjecture under assumptions that will be specified in the following paragraph. We will even be able to show that the $o(1)$ in (1.3) is $O(j/n)$ if $j/n \to 0$ and $O(1-j/n)$ if $j/n \to 1$.

Throughout what follows we assume that a is a Laurent polynomial

$$a(t) = \sum_{k=-r}^{r} a_k t^k \quad (t = e^{ix} \in \mathbf{T})$$

with $r \geq 2$, $a_r \neq 0$, and $\overline{a_k} = a_{-k}$ for all k. The last condition means that a is real-valued on $\mathbf{T}$. We assume without loss of generality that $a(\mathbf{T}) = [0, M]$ with $M > 0$ and that $a(1) = 0$ and $a(e^{i\varphi_0}) = M$ for some $\varphi_0 \in (0, 2\pi)$. We require that the function $g(x) := a(e^{ix})$ is strictly increasing on $(0, \varphi_0)$ and strictly decreasing on $(\varphi_0, 2\pi)$ and that the second derivatives of g at $x = 0$ and $x = \varphi_0$ are nonzero. Finally, we denote by $[\alpha, \beta] \subset [0, M]$ a segment such that if $\lambda \in [\alpha, \beta]$, then the $2r - 2$ zeros of the Laurent polynomial $a(z) - \lambda$ that lie in $\mathbf{C} \setminus \mathbf{T}$ are pairwise distinct.

Note that we exclude the case $r = 1$, because in this case the eigenvalues and eigenvectors of $T_n(a)$ are explicitly available. Also notice that if $r = 2$, which is the case of pentadiagonal matrices, then for every $\lambda \in [0, M]$ the polynomial $a(z) - \lambda$ has two zeros on $\mathbf{T}$, one zero outside $\mathbf{T}$, and one zero inside $\mathbf{T}$. Thus, in this situation the last requirement of the previous paragraph is automatically satisfied for $[\alpha, \beta] = [0, M]$.

The asymptotic behavior of the extreme eigenvalues and eigenvectors of $T_n(a)$, that is, of $\lambda_j^{(n)}$ and $v_j^{(n)}$ when j or $n - j$ remain fixed, has been studied by several authors. As for extreme eigenvalues, the pioneering works are [7], [9], [11], [12], [18], while recent papers on the subject include [3], [6], [8], [10], [13], [14], [15], [19], [20]. See also the books [1] and [5]. Much less is known about the asymptotics of the eigenvectors. Part of the results of [4] and [19] may be interpreted as results on the behavior of the eigenvectors "in the mean" on the one hand and as insights into what happens if eigenvectors are replaced by pseudomodes on the other. In [3], we investigated the asymptotics of the extreme eigenvectors of certain Hermitian (and not necessarily banded) Toeplitz matrices. Our paper [2] may be considered as a first step to the understanding of the asymptotic behavior of individual inner eigenvalues of Toeplitz matrices. In the same vein, this paper intends to understand the nature of individual eigenvectors as part of the whole, independently of whether they are extreme or inner ones.

To state our main results, we need some notation. Let $\lambda \in [0, M]$. Then there are uniquely defined $\varphi_1(\lambda) \in [0, \varphi_0]$ and $\varphi_2(\lambda) \in [\varphi_0 - 2\pi, 0]$ such that

$$g(\varphi_1(\lambda)) = g(\varphi_2(\lambda)) = \lambda;$$

recall that $g(x) := a(e^{ix})$. We put

$$\varphi(\lambda) = \frac{\varphi_1(\lambda) - \varphi_2(\lambda)}{2}, \quad \sigma(\lambda) = \frac{\varphi_1(\lambda) + \varphi_2(\lambda)}{2}.$$

We have

$$\begin{aligned} a(z) - \lambda &= z^{-r}\Big(a_r z^{2r} + \cdots + (a_0 - \lambda)z^r + \cdots + a_{-r}\Big) \\ &= a_r z^{-r} \prod_{k=1}^{2r} (z - z_k(\lambda)), \end{aligned}$$

and our assumptions imply that we can label the zeros $z_k(\lambda)$ so that the collection $\mathcal{Z}(\lambda)$ of the zeros may be written as

$$\begin{aligned} &\{z_1(\lambda), \dots, z_{r-1}(\lambda), z_r(\lambda), z_{r+1}(\lambda), z_{r+2}(\lambda), \dots, z_{2r}(\lambda)\} \\ &= \{u_1(\lambda), \dots, u_{r-1}(\lambda), e^{i\varphi_1(\lambda)}, e^{i\varphi_2(\lambda)}, 1/\overline{u}_1(\lambda), \dots, 1/\overline{u}_{r-1}(\lambda)\} \end{aligned} \tag{1.4}$$

where $|u_\nu(\lambda)| > 1$ for $1 \le \nu \le r-1$ and each $u_\nu(\lambda)$ depends continuously on $\lambda \in [0, M]$. Here and in similar places below we write $\overline{u}_k(\lambda) := \overline{u_k(\lambda)}$. We define $\delta_0 > 0$ by

$$e^{\delta_0} = \min_{\lambda \in [0,M]} \min_{1 \le \nu \le r-1} |u_\nu(\lambda)|.$$

Throughout the following, δ stands for any number in $(0, \delta_0)$. Further, we denote by h_λ the function

$$h_\lambda(z) = \prod_{\nu=1}^{r-1} \left(1 - \frac{z}{u_\nu(\lambda)}\right).$$

The function $\Theta(\lambda) = h_\lambda(e^{i\varphi_1(\lambda)})/h_\lambda(e^{i\varphi_2(\lambda)})$ is continuous and nonzero on $[0, M]$ and we have $\Theta(0) = \Theta(M) = 1$. In [2], it was shown that the closed curve

$$[0, M] \to \mathbf{C} \setminus \{0\}, \quad \lambda \mapsto \Theta(\lambda)$$

has winding number zero. Let $\theta(\lambda)$ be the continuous argument of $\Theta(\lambda)$ for which $\theta(0) = \theta(M) = 0$.

In [2], we proved that if n is large enough, then the function

$$f_n : [0, M] \to [0, (n+1)\pi], \quad f_n(\lambda) = (n+1)\varphi(\lambda) + \theta(\lambda)$$

is bijective and increasing and that if $\lambda_{j,*}^{(n)}$ is the unique solution of the equation $f_n(\lambda_{j,*}^{(n)}) = \pi j$, then the eigenvalues $\lambda_j^{(n)}$ satisfy

$$|\lambda_j - \lambda_{j,*}^{(n)}| \le K\, e^{-\delta n}$$

for all $j \in \{1, \dots, n\}$, where K is a finite constant depending only on a. Thus, we have

$$(n+1)\varphi(\lambda_j^{(n)}) + \theta(\lambda_j^{(n)}) = \pi j + O(e^{-\delta n}), \tag{1.5}$$

uniformly in $j \in \{1, \dots, n\}$.

Now take λ from (α, β). For $j \in \{1, \dots, n\}$ and $\nu \in \{1, \dots, r-1\}$, we put

$$A(\lambda) = \frac{e^{i\sigma(\lambda)}}{2i\, h_\lambda(e^{i\varphi_1(\lambda)})}, \quad B(\lambda) = \frac{e^{i\sigma(\lambda)}}{2i\, h_\lambda(e^{i\varphi_2(\lambda)})},$$

$$D_\nu(\lambda) = \frac{e^{2i\sigma(\lambda)} \sin\varphi(\lambda)}{(u_\nu(\lambda) - e^{i\varphi_1(\lambda)})(u_\nu(\lambda) - e^{i\varphi_2(\lambda)}) h'_\lambda(u_\nu(\lambda))},$$

$$F_\nu(\lambda) = \frac{\sin\varphi(\lambda)}{(\overline{u}_\nu(\lambda) - e^{-i\varphi_1(\lambda)})(\overline{u}_\nu(\lambda) - e^{-i\varphi_2(\lambda)})\overline{h'_\lambda(u_\nu(\lambda))}} \times \frac{|h_\lambda(e^{i\varphi_1(\lambda)}) h_\lambda(e^{i\varphi_2(\lambda)})|}{h_\lambda(e^{i\varphi_1(\lambda)}) h_\lambda(e^{i\varphi_2(\lambda)})}$$

and define the vector $w_j^{(n)}(\lambda) = (w_{j,m}^{(n)}(\lambda))_{m=1}^n$ by

$$w_{j,m}^{(n)}(\lambda) = A(\lambda) e^{-im\varphi_1(\lambda)} - B(\lambda) e^{-im\varphi_2(\lambda)} + \sum_{\nu=1}^{r-1} \left(D_\nu(\lambda) \frac{1}{u_\nu(\lambda)^m} + F_\nu(\lambda) \frac{(-1)^{j+1} e^{-i(n+1)\sigma(\lambda)}}{\overline{u}_\nu(\lambda)^{n+1-m}} \right).$$

The assumption that zeros $u_\nu(\lambda)$ are all simple guarantees that $h'(u_\nu) \neq 0$. We denote by $\|\cdot\|_2$ and $\|\cdot\|_\infty$ the ℓ^2 and ℓ^∞ norms on $\mathbf{C}^n$, respectively.

Here are our main results.

Theorem 1.1. *As $n \to \infty$ and if $\lambda_j^{(n)} \in (\alpha, \beta)$,*

$$\|w_j^{(n)}(\lambda_j^{(n)})\|_2^2 = \frac{n}{4} \left(\frac{1}{|h_\lambda(e^{i\varphi_1(\lambda)})|^2} + \frac{1}{|h_\lambda(e^{i\varphi_2(\lambda)})|^2} \right)\Bigg|_{\lambda=\lambda_j^{(n)}} + O(1),$$

uniformly in j.

Theorem 1.2. *Let $n \to \infty$ and suppose $\lambda_j^{(n)} \in (\alpha, \beta)$. Then the eigenvectors $v_j^{(n)}$ are of the form*

$$v_j^{(n)} = \tau_j^{(n)} \left(\frac{w_j^{(n)}(\lambda_j^{(n)})}{\|w_j^{(n)}(\lambda_j^{(n)})\|_2} + O_\infty(e^{-\delta n}) \right)$$

where $\tau_j^{(n)} \in \mathbf{T}$ and $O_\infty(e^{-\delta n})$ denotes vectors $\xi_j^{(n)} \in \mathbf{C}^n$ such that $\|\xi_j^{(n)}\|_\infty \leq Ke^{-\delta n}$ for all j and n with some finite constant K independent of j and n.

Note that the previous theorem gives (1.2) with

$$\Omega_m(\lambda) = A(\lambda) e^{-im\varphi_1(\lambda)} - B(\lambda) e^{-im\varphi_2(\lambda)}, \quad \Phi_m(\lambda) = \sum_{\nu=1}^{r-1} \frac{D_\nu(\lambda)}{u_\nu(\lambda)^m},$$

$$e^{-i\eta(\lambda)} = \frac{|h_\lambda(e^{i\varphi_1(\lambda)}) h_\lambda(e^{i\varphi_2(\lambda)})|}{h_\lambda(e^{i\varphi_1(\lambda)}) h_\lambda(e^{i\varphi_2(\lambda)})}.$$

Things can be a little simplified for symmetric matrices. Thus, suppose all a_k are real and $a_k = a_{-k}$ for all k. We will show that then $\{u_1(\lambda), \dots, u_{r-1}(\lambda)\} = \{\overline{u}_1(\lambda), \dots, \overline{u}_{r-1}(\lambda)\}$. Put

$$Q_\nu(\lambda) = \frac{|h_\lambda(e^{i\varphi(\lambda)})| \sin \varphi(\lambda)}{(u_\nu(\lambda) - e^{i\varphi(\lambda)})(u_\nu(\lambda) - e^{-i\varphi(\lambda)}) h'_\lambda(u_\nu(\lambda))}$$

and let $y_j^{(n)}(\lambda) = (y_{j,m}^{(n)}(\lambda))_{m=1}^n$ be given by

$$y_{j,m}^{(n)}(\lambda) = \sin\left(m\varphi(\lambda) + \frac{\theta(\lambda)}{2}\right) - \sum_{\nu=1}^{r-1} Q_\nu(\lambda)\left(\frac{1}{u_\nu(\lambda)^m} + \frac{(-1)^{j+1}}{u_\nu(\lambda)^{n+1-m}}\right). \quad (1.6)$$

Theorem 1.3. *Let $n \to \infty$ and suppose $\lambda_j^{(n)} \in (\alpha, \beta)$. If $a_k = a_{-k}$ for all k, then*

$$\|y_j^{(n)}(\lambda_j^{(n)})\|_2^2 = \frac{n}{2} + O(1)$$

uniformly in j, and the eigenvectors $v_j^{(n)}$ are of the form

$$v_j^{(n)} = \tau_j^{(n)}\left(\frac{y_j^{(n)}(\lambda_j^{(n)})}{\|y_j^{(n)}(\lambda_j^{(n)})\|_2} + O_\infty(e^{-\delta n})\right)$$

where $\tau_j^{(n)} \in \mathbf{T}$ and $O_\infty(e^{-\delta n})$ is as in the previous theorem.

Let J be the $n \times n$ matrix with ones on the counterdiagonal and zeros elsewhere. Thus, $(Jv)_m = v_{n+1-m}$. A vector v is called symmetric if $Jv = v$ and skew-symmetric if $Jv = -v$. Trench [17] showed that the eigenvectors $v_1^{(n)}, v_3^{(n)}, \dots$ are all symmetric and that the eigenvectors $v_2^{(n)}, v_4^{(n)}, \dots$ are all skew-symmetric. From (1.5) we infer that

$$\sin\left((n+1-m)\varphi(\lambda_j^{(n)}) + \frac{\theta(\lambda_j^{(n)})}{2}\right)$$
$$= (-1)^{j+1} \sin\left(m\varphi(\lambda_j^{(n)}) + \frac{\theta(\lambda_j^{(n)})}{2}\right) + O(e^{-\delta n})$$

and hence (1.6) implies that

$$(Jy_j^{(n)}(\lambda_j^{(n)}))_m = (-1)^{j+1} y_{j,m}^{(n)}(\lambda_j^{(n)}) + O(e^{-\delta n}).$$

Consequently, apart from the term $O(e^{-\delta n})$, the vectors $y_j^{(n)}(\lambda_j^{(n)})$ are symmetric for $j = 1, 3, \dots$ and skew-symmetric for $j = 2, 4, \dots$. This is in complete accordance with Trench's result.

Due to (1.5), we also have

$$\sin\left(m\varphi(\lambda_j^{(n)}) + \frac{\theta(\lambda_j^{(n)})}{2}\right) = \sin\left(\left(m - \frac{n+1}{2}\right)\varphi(\lambda_j^{(n)})\right) + O(e^{-\delta n}).$$