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Stefan Haesen · Leopold Verstraelen
Editors

Topics in Modern Differential Geometry

ATLANTIS
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Editors
Stefan Haesen
Department of Teacher Education
Thomas More University College
Vorselaar
Belgium

Leopold Verstraelen
Department of Mathematics
University of Leuven
Leuven
Belgium

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Preface

In 2008 and 2009, the Simon Stevin Institute for Geometry participated in the organization of Ph.D. courses at the universities of Leuven (Belgium), Kragujevac (Serbia), Murcia (Spain) and Brasov (Romania). Besides the main course lectures on "Natural geometrical intrinsic and extrinsic symmetries", there were invited short lectures on a varied selection of topics in differential geometry. Several of the lecturers were able to find the time to prepare their talks for the publication in this book.

Our sincere thanks go to all the students who participated at the various courses, to Prof. Dr. R. Deszcz and Dr. A. Albujer who taught several of the main lectures, to Prof. Dr. L. Alías, Prof. Dr. F. Dillen, Dr. J. Gielis, Prof. Dr. I. Mihai, Prof. Dr. M. Petrović-Torgašev and Prof. Dr. E. Stoica for the local organization of the courses and to all the invited lecturers.

Vorselaar, Belgium — Stefan Haesen
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Contents

The Riemannian and Lorentzian Splitting Theorems

José Luis Flores

Abstract In these notes we are going to briefly review some of the main ideas involved in the formulation and proof of the Riemannian and Lorentzian Splitting Theorems. We will try to emphasize the similarities and differences appeared when passing from the Riemannian to the Lorentzian case, and the way in which these difficulties are overcome by the authors.

1 Introduction

The splitting problem in Riemannian and Lorentzian Geometry is closely related to the idea of "rigidity" in Geometry. So, in order to introduce this problem, first we are going to dedicate some lines to recall this important notion.

Assume that we are interested in studying some Riemannian manifold (M, g). Usually, it is very useful to compare it with some model space M_K, i.e. a complete 1-connected Riemannian manifold of constant sectional curvature K. In fact, there are a series of results which ensure that (M, g) will retain global geometrical properties of M_K under certain strict curvature bounds for (M, g) in terms of K. Even more, under these conditions, it is usually possible to conclude that M will also retain topological properties of M_K. A natural question which arises from this situation is, what happen when one relaxes the condition of "strict" curvature inequality to some "weak" curvature inequality? It is not difficult to realize that, under these new hypotheses, the conclusion may not hold any more. This is clearly illustrated by the following simple observation: there is a crucial difference between the topology of the sphere

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J.L. Flores (✉)
Departamento de Álgebra, Geometría y Topología Facultad de Ciencias,
Universidad de Málaga Campus Teatinos s/n, 29071 Málaga, Spain
e-mail: floresj@agt.cie.uma.es

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($K > 0$) and that of the Euclidean space ($K \equiv 0$), even for spheres of radius very big, and so, with curvature very close to the null curvature of the Euclidean space. However, a relevant property still holds: a conclusion which becomes false when one relaxes the "strict" curvature condition to a "weak" curvature condition usually can be shown to fail only under very special circumstances! This important idea, usually referred as "rigidity" in Geometry, is roughly summarized in the following prototype result:

Prototype Rigidity Theorem: *If M satisfies a "weak" curvature condition, and the geometric restriction derived from the corresponding "strict" curvature condition does not hold any more, then M must be "very special".*

In order to relate the splitting problem to this prototype rigidity theorem, let us recall the following result by Gromoll and Meyer [17]:

Theorem 1.1 (Gromoll, Meyer) *A complete Riemannian manifold (M, g) of dimension $n \geq 2$ such that $Ric(v, v) > 0$ for all $v \in TM$ is connected at infinity.*

This is a typical result where a strict curvature inequality ($Ric(v, v) > 0$) implies a topological restriction on the manifold (connectedness at infinity). Now, suppose that we replace the strict curvature condition $Ric(v, v) > 0$ by the weak curvature condition $Ric(v, v) \geq 0$ and, consequently, we assume that (M, g) fails to be connected at infinity. Since M is complete, now one can ensure the existence of a line joining any two different ends of M. Under these new hypotheses, Cheeger and Gromoll proved that (M, g) must be isometric to a product manifold. So, this is a typical rigidity theorem in the sense described above. This result and its Lorentzian version constitute the central subject of these notes.

In the next section we will establish with precision the Riemannian Splitting Theorem. We will also provide some brief comments about the initial motivation and the precedents of the theorem. Finally, we will introduce some basic notions and results which will be used later in the proof. In Sect. 3 we will outline the main ideas involved in the original proof by Cheeger and Gromoll. This proof strongly uses the theory of elliptic operators, and, indeed, it is stronger than actually needed. So, in Sect. 4 we will describe an alternative proof of the same result, given by Eschenburg and Heintze, which minimizes the use of the elliptic theory. This second approach will be relevant for us because it introduces a new viewpoint useful for the proof of the Lorentzian version of the theorem. In Sect. 5, we will recall some basic notions and results from Lorentzian Geometry. After that, we will establish the Lorentzian Splitting Theorem in Sect. 6, providing some brief comments about the main hits in the history of its solution. The proof of this result will be studied in Sect. 7. We will essentially follow the arguments given by Galloway in [14]: after some previous technical lemmas in Subsects. 7.1–7.3, the proof will be delivered in six steps in Subsect. 7.4. Finally, in Sect. 8 we will recall a related open problem with a physical significance, the Bartnik's Conjecture.

2 Riemannian Splitting Theorem

The Riemannian Splitting Theorem can be stated in the following way [7]:

Riemannian Splitting Theorem (Cheeger, Gromoll) *Suppose that the Riemannian manifold (M, g), of dimension $n \geq 2$, satisfies the following conditions:*

(1) *(M,g) is geodesically complete,*
(2) $Ric(v, v) \geq 0$ *for all* $v \in TM$,
(3) *M has a line (i.e. a complete unitary geodesic* $\gamma : \mathbb{R} \to (M, g)$ *realizing the distance between any two of its points).*

Then M is isometric to the product $(M, g) \cong (\mathbb{R}^k \times M_1, g_0 \oplus g_1)$, $k > 0$, *where* (M_1, g_1) *contains no lines and* g_0 *is the standard metric on* $\mathbb{R}^k$.

This is a very important result which has been extensively used in Riemannian Geometry in the last decades. An important precedent of this result is due to Topogonov [23], who obtained the same thesis under the more restrictive curvature assumption of nonnegative sectional curvature. The proof of the Topogonov's result lies on the Triangle Comparison Theorem by the same author. The original motivation for the Cheeger and Gromoll's result was the necessity to extend the existing results concerning the fundamental group of manifolds of nonnegative sectional curvature [8] to the case of nonnegative Ricci curvature. In particular, they needed a splitting theorem under the weaker hypothesis of nonnegative Ricci curvature, for which the Topogonov's Triangle Comparison Theorem does not work. A first splitting result of this type were obtained by Cohn-Vossen in [9]. However, the general result required totally new arguments, which were not developed till the publication of the remarkable paper [7].

In order to describe the proof of the Cheeger–Gromoll Splitting Theorem, firstly we need to introduce some previous notions, which are of interest by itself:

By a *ray* γ we will understand an unitary geodesic defined on $[0, \infty)$ which realizes the distance between any of its points. Then, the *Busemann function (associated to* γ) is defined as the function $b_\gamma : M \to \mathbb{R}^3$ obtained from the limit

$$b_\gamma(\cdot) := \lim_{r\to\infty} (r - d(\cdot, \gamma(r))), \tag{1}$$

where d is the distance associated to the Riemannian metric g. It is not difficult to prove that previous limit always exists (is finite) and the resulting function is continuous. In fact, the limit (1) exists and is finite because, from the triangle inequality, the map

$$r \mapsto b_r(p) = r - d(p, \gamma(r))$$

is nondecreasing

$$r_2 - r_1 = d(\gamma(r_1), \gamma(r_2)) \geq d(p, \gamma(r_2)) - d(p, \gamma(r_1)) \quad \text{if} \quad r_1 \leq r_2$$

and bounded above

$$r - d(p, \gamma(r)) \leq r + d(p, \gamma(0)) - d(\gamma(0), \gamma(r)) = r + d(p, \gamma(0)) - r = d(p, \gamma(0)).$$

On the other hand, the Busemann function b_γ is continuous because $b_\gamma(\cdot) = \lim_{r\to\infty} b_r(\cdot)$, being $\{b_r(\cdot)\}_r$ a family of uniformly equicontinuous functions:

$$|b_r(p) - b_r(q)| = |r - d(p, \gamma(r)) - r + d(q, \gamma(r))| \leq d(p, q).$$

In particular:

$$|b_\gamma(p) - b_\gamma(q)| = \lim_{r\to\infty} |b_r(p) - b_r(q)| \leq d(p, q). \tag{2}$$

Given a ray γ, we say that $\alpha : [0, \infty) \to M$ is an *asymptote from p to γ* if it is a ray which arises as limit of minimal geodesic segments α_n from p to $\gamma(r_n)$, $r_n \to \infty$. A simple limit argument on the initial velocities of α_n shows that any ray γ in a complete Riemannian manifold (M, g) admits some asymptote from any point p, even though it is not necessarily unique.

In general, Busemann functions b_γ are not necessarily differentiable. However, they admit very simple expressions when evaluated on any asymptote α with respect to γ. In fact, from the uniform convergence of α_n to γ over compact subintervals $[0, t]$ for every $t \in (0, \infty)$, it easily follows:

$$b_\gamma(\alpha(t)) = t + b_\gamma(\alpha(0)) \quad \forall t \in [0, \infty). \tag{3}$$

Given a line γ, there are two natural rays associated to γ: the restriction $\gamma_+ := \gamma \mid_{[0,\infty)}$ and the curve $\gamma_-(t) := \gamma(-t)$, $t \in [0, \infty)$. We will denote by $b_\pm$ the corresponding Busemann functions associated to $\gamma_\pm$.

3 Cheeger and Gromoll's Proof

In this section we are going to review the Cheeger and Gromoll's proof of the Riemannian Splitting Theorem (see [7] for details). The main step in the argument will be the proof that Busemann functions $b_\pm$ are sub-harmonic. This remarkable property joined to some basic relations for $b_\pm$ will imply that $b_\pm$ are, indeed, harmonic. Then, the integral curves of $\operatorname{grad} b_+$ will be shown to be geodesics, and $\operatorname{grad} b_+$ parallel. From here, the de Rham Decomposition Theorem will provide the global splitting of M in terms of the level surfaces and the integral curves of b_+.

Let γ be the line ensured by the hypotheses of the theorem. One easily deduces the following relations for the Busemann functions:

$$b_+ + b_- \leq 0 \;\text{ on } M \qquad \text{and} \qquad b_+ + b_- \equiv 0 \;\text{ on } \gamma. \tag{4}$$

In fact, from the triangle inequality it is

$$\begin{aligned} b_+(p) + b_-(p) &= \lim_{r\to\infty}(r - d(p, \gamma_+(r)) + r - d(p, \gamma_-(r))) \\ &= \lim_{r\to\infty}(r - d(p, \gamma(r)) + r - d(p, \gamma(-r))) \\ &\leq \lim_{r\to\infty}(2r - d(\gamma(-r), \gamma(r))) \\ &= \lim_{r\to\infty}(2r - 2r) \\ &= 0 \end{aligned}$$

for any $p \in M$, being the inequality "$\leq$" an equality "$=$" if $p \in \gamma$ (recall that γ is a line).

As commented above, the main ingredient in the Cheeger and Gromoll's proof is the sub-harmonic character of the Busemann functions $b_\pm$. This property requires the nonnegative character of the Ricci curvature:

Theorem 3.1 *If the Ricci curvature is nonnegative then functions $b_\pm$ are sub-harmonic.*

Recall that, in principle, Busemann functions are not necessarily differentiable on M. So, in Theorem 3.1 we are implicitly assuming the following well-known notion of sub-harmonicity for continuous functions: *a continuous function $f : M \to \mathbb{R}$ is* sub-harmonic *if, given any connected compact region D in M with smooth boundary ∂D, one has $f \leq h$ on D, being h the continuous function on D which is harmonic on int D and satisfies $h|_{\partial D} \equiv f|_{\partial D}$.*

Sketch of proof of Theorem 3.1. Denote by $d_p(\cdot) := d(\cdot, p)$ the distance function on M with respect to p. By using the nonnegative character of the Ricci curvature and the fundamental inequality for the index form, one deduces the following upper bound estimate for the Laplacian of $d_p(\cdot)$:

$$\Delta d_p(q) \leq (n-1)/d_p(q) \quad \text{for any } q \text{ outside the cut locus of } p. \tag{5}$$

In particular:

$$\Delta d_{\gamma(r)}(q) \leq (n-1)/d_{\gamma(r)}(q) \quad \text{for any } q \text{ outside the cut locus of } \gamma(r).$$

From here, a simple limit argument suggests that $b_\gamma(\cdot) = \lim_r (r - d_{\gamma(r)}(\cdot))$ has non-negative Laplacian whenever it is differentiable. In particular, one is tempted to deduce that b_γ is sub-harmonic. But notice that $d_{\gamma(r)}$ is not differentiable on the cut locus of $\gamma(r)$, and so, b_γ may not be differentiable anywhere. Moreover, even though b_γ were differentiable almost everywhere with $\Delta b_\gamma \geq 0$, the conclusion is not clear at all, as illustrated by the simple example $f(x) = -x^{2/3}$ (which is differentiable everywhere up to $x = 0$, with nonnegative Laplacian, but it is not sub-harmonic according to the definition above). Therefore, the conclusion follows after a sophisticated analysis of the behavior of the gradient near the points of non-differentiability, in order to avoid pathological behaviors as that showed by function $f(x)$ at $x = 0$ (see [7] for details). □

The next step consists of showing that relations (4) joined to the sub-harmonic character of $b_\pm$ imply that $b_\pm$ are differentiable and harmonic on M. To this aim, consider any point $q \in \gamma$ and any connected region D with $q \in \text{int}D$. Let $h_\pm$ be the continuous functions on D which are harmonic on int$D$ and satisfy $h_\pm \mid_{\partial D} = b_\pm \mid_{\partial D}$. In particular, $h_+ + h_- = b_+ + b_- \leq 0$ on ∂D. From the Maximum Principle applied to $h_+ + h_-$ on D, we deduce $h_+ + h_- \leq 0$ on D, and thus,

$$h_+(q) + h_-(q) \leq 0 = b_+(q) + b_-(q). \tag{6}$$

Since $b_\pm$ are sub-harmonic, it is also $b_\pm \leq h_\pm$ on D. Hence $b_\pm(q) \leq h_\pm(q)$. This joined to (6) implies $b_\pm(q) = h_\pm(q)$, or equivalently, $(b_\pm - h_\pm)(q) = 0$. But recall that $b_\pm - h_\pm$ is sub-harmonic and $b_\pm - h_\pm \leq 0$ on ∂D, hence $b_\pm = h_\pm$ on D. Since D is arbitrary, we have proved that $b_\pm$ are differentiable and harmonic on M.

Now that we have the differentiability of b_+, we are going to show that $|\text{grad}\, b_+| \equiv 1$. From (2) we have $|\text{grad}\, b_+| \leq 1$. From (3) we also have $|b_+(p) - b_+(q)| = d(p, q)$ for all p, q over an asymptote σ to γ. Hence, $|\text{grad}\, b_+| = 1$. In particular, the integral curves of grad b_+ must coincide with the asymptotes σ to γ, and so, they are geodesics.

Finally, denote by N the gradient of b_+. Since the integral curves of N are geodesics, it is $\nabla_N N = 0$. On the other hand, recall that b_+ is harmonic. Let $\{N, E_1, \ldots, E_{n-1}\}$ be a parallel orthonormal base along the asymptote. Then, a direct computation gives:

$$\begin{aligned} Ric(N) &= \textstyle\sum_{i=1}^{n-1} \langle R(E_i, N)N, E_i \rangle \\ &= \textstyle\sum_{i=1}^{n-1} \langle \nabla_{E_i} \nabla_N N - \nabla_N \nabla_{E_i} N - \nabla_{[E_i, N]} N, E_i \rangle \\ &= -N(\Delta b_+) - |\nabla N|^2 \\ &= -|\nabla N|^2. \end{aligned}$$

Since $Ric(N) \geq 0$, it is $\nabla N \equiv 0$. Whence N is parallel, and the de Rham Decomposition Theorem ensures that the map

$$I : (b_+)^{-1}(0) \times \mathbb{R} \to M, \quad (p, t) \mapsto \exp(t \cdot N(p))$$

is an isometry. The conclusion follows after a finite induction on the lines of (M, g). □

Remark 3.2 As we have seen, the approach followed by Cheeger and Gromoll in his proof strongly uses the existence and regularity theory of elliptic equations. In particular, it is far from being adaptable to the Lorentzian case, since the d'Alambertian (i.e. the Lorentzian Laplacian) operator is hyperbolic, not elliptic.

4 Eschenburg and Heintze's Proof

In this section we are going to review the Eschenburg and Heintze's alternative proof of the Riemannian Splitting Theorem. The key point consists of minimizing the use of the elliptic theory of equations by using a Calabi's version of the Hopf Maximum

Principle. As we will see later, this accurate approach introduces an useful viewpoint for spacetimes.

First, the same relations for $b_\pm$ as in the Cheeger and Gromoll's proof are deduced. Concretely:

$$b_+ + b_- \leq 0 \ \text{ on } \ M \qquad \text{and} \qquad b_+ + b_- \equiv 0 \ \text{ on } \ \gamma. \tag{7}$$

For any $p \in M$, $r \in \mathbb{R}$, define the functions

$$b^\pm_{p,r} : M \to \mathbb{R}, \qquad b^\pm_{p,r}(x) := b_\pm(p) - r + d(x, \exp(rv)),$$

where v is the direction of some asymptote to $\gamma_\pm$ from p. It can be proved that $b^\pm_{p,r}$ are *lower support functions* of $b_\pm$ at p, i.e.

$$b^\pm_{p,r}(q) \leq b_\pm(q) \ \text{ for all } \ q \in M \qquad \text{and} \qquad b^\pm_{p,r}(p) = b_\pm(p), \tag{8}$$

which are C^∞ around p (where the points x remain out of the cut locus of $\exp(rv)$) and satisfies $|\text{grad}\, b^\pm_{p,r}| = 1$. From the nonnegative Ricci curvature hypothesis, one can estimate the following lower bound for the Laplacian of the sum of these functions:

$$\Delta(b^+_{p,r} + b^-_{p,r}) \geq -2(n-1)/r \qquad \text{for all } \ p \in M, \quad r \in \mathbb{R}.$$

Under this inequality, a Calabi's version of the Hopf Maximum Principle ([12][Sect. 6]; see also [6, 19]) ensures that $b_+ + b_-$ attains no maximum unless it is constant. But, according to (7), the function $b_+ + b_-$ does attain a maximum on γ. Hence:

$$b_+ + b_- \equiv 0 \quad \text{on} \ \ M. \tag{9}$$

From (8) and (9), one can write the following sandwich expression for $b_\pm$ in terms of the lower support functions $b^\pm_{p,r}$:

$$b^+_{p,r} \leq b_+ = -b_- \leq -b^-_{p,r} \quad \text{on} \ \ M, \qquad \text{with “=” at } p.$$

From this sandwich expression and the differentiability of $b^\pm_{p,r}$, one deduces that Busemann functions $b_\pm$ must be once differentiable at p, and $\text{grad}\, b_\pm(p) = \text{grad}\, b^\pm_{p,r}(p)$. In particular,

$$|\text{grad}\, b_\pm| = 1.$$

Therefore, the asymptotes to $\gamma_\pm$ at any p are uniquely determined and fit together to a line.

On the other hand, by using the estimate (5) for the Laplacian of the distance function out of the cut locus, one can deduce the following limit for the Hessian of the lower support functions:

$$\lim_{r\to\infty} \text{Hess}\, b^{\pm}_{p,r}(p) = 0.$$

Thus, for any geodesic c, the composition of $b^{\pm}_{p,r} \circ c$ provides lower support functions at any $t \in \mathbb{R}$ for $b_{\pm} \circ c$ with arbitrarily small 2^{nd} derivative at t. Observe that this remains true for $b_{\pm} \circ c - l$, being l any affine function. Therefore, by the (trivial 1-dimensional) Maximum Principle, the functions $b_{\pm} \circ c$ are convex. Since $b_+ = -b_-$, they are also concave. Hence, for any geodesic c with initial velocity tangent to a level surface of b_+, the composition $b_+ \circ c$ is constant. This means that any such c remains contained in the level surface, and so, b_+ has totally geodesic level sets. Consequently, $N = \text{grad}\, b_+$ is a parallel vector field, and, by the de Rham Decomposition Theorem, the map

$$I : (b_+)^{-1}(0) \times \mathbb{R} \to M, \quad (p,t) \mapsto \exp(t \cdot N(p))$$

is an isometry. The conclusion follows after a finite induction on the lines of (M, g). □

Next, we are going to study the Lorentzian version of the Splitting Theorem. But, first, let us recall some basic notions and results from Lorentzian Geometry. Our notation and conventions follow the standard ones (see, for example, [3, 21]).

5 Preliminaries on Lorentzian Geometry

By a spacetime (M, g) we understand a (connected) oriented smooth manifold M endowed with a metric tensor g of signature $(-, +, \ldots, +)$. A tangent vector $v \in T_pM$, $p \in M$ is named *timelike* (resp. *lightlike*; *causal*) if $g(v, v) < 0$ (resp. $g(v, v) = 0$, $v \neq 0$; v is either timelike or lightlike). Accordingly, a smooth curve $\gamma : I \to M$ (I real interval) is called *timelike* (resp. *lightlike*; *causal*) if $\dot\gamma(s)$ is timelike (resp. lightlike; causal) for all s. Spacetimes are assumed to be *time-oriented*, i.e. they are endowed with a continuous, globally defined, timelike vector field X. Fixed a *time-orientation* X, causal tangent vectors $v \in T_pM$ are distributed in two cones, each one containing future $g(v, X(p)) < 0$ or past-directed $g(v, X(p)) > 0$ causal vectors. So, a causal curve $\gamma(s)$ is said *future-directed* (resp. *past-directed*) if $g(\dot\gamma(s), X(\gamma(s))) < 0$ (resp. $g(\dot\gamma(s), X(\gamma(s))) > 0$) for all s. Future-directed causal curves represent all the physically admissible trajectories for material particles and light rays in the universe.

A *(smooth) spacelike hypersurface* is a smooth codimension one submanifold with everywhere timelike normal. A spacelike hypersurface is said *maximal* if the mean (extrinsic) curvature vanishes identically.

Two events $p, q \in M$ are *chronologically related* $p \ll q$ (resp. *causally related* $p \leq q$) if there exists some future-directed timelike (resp. causal) curve from p to q (the case $p = q$ is also allowed in $p \leq q$). The *chronological past* (resp. *future*) of p, $I^-(p)$ ($I^+(p)$) is defined as:

$$I^-(p) = \{q \in M : q \ll p\} \qquad (\text{resp. } I^+(p) = \{q \in M : p \ll q\}).$$

On the other hand, the *causal past* (resp. *future*) of p, $J^-(p)$ (resp. $J^+(p)$) is defined as:

$$J^-(p) = \{q \in M : q \leq p\} \qquad (\text{resp. } J^+(p) = \{q \in M : p \leq q\}).$$

We will denote by $I^{\pm}(p, U)$ the chronological past and future of p defined with respect to an open set $U \subset M$.

A subset $A \subset M$ is said *achronal* (resp. *acausal*) if it does not contain points chronologically (causally) related between them. The *edge* of an achronal set $A \subset M$ is the set of points $p \in \overline{A}$ satisfying that every neighborhood U of p contains a timelike curve from $I^-(p, U)$ to $I^+(p, U)$ which does not meet A.

A spacetime is *chronological* if it does not admit closed timelike curves, *causal* if it does not admit closed causal curves, and *strongly causal* if it does not admit neither closed nor "almost closed" causal curves. A spacetime is *globally hyperbolic* if it is strongly causal and $J^+(p) \cap J^-(q)$ is compact for any $p, q \in M$. Here, global hyperbolicity is the most restrictive causality condition, while chronological is the most general one. Global hyperbolicity is equivalent to the following condition: the spacetime admits a *Cauchy hypersurface*, i.e. a topological hypersurface that is met exactly once by every inextensible timelike curve.

A very useful tool in Lorentzian Geometry is the notion of *temporal separation* or *Lorentzian distance* (even though it is not a distance in a formal way, as we will see in a moment). The *Lorentzian distance* is defined as the map $d : M \times M \to [0, \infty]$ given by:

$$d(p,q) = \begin{cases} 0, & \text{if } C^c_{pq} = \emptyset \\ \sup\{L(\alpha) = \int \sqrt{-g(\dot{\alpha}, \dot{\alpha})},\ \alpha \in C^c_{pq}\}, & \text{if } C^c_{pq} \neq \emptyset, \end{cases}$$

where C^c_{pq} denotes the family of future causal curves (possibly piecewise smooth) which connect p to q. The Lorentzian distance satisfies the following basic properties:

1. $d(p,q) > 0 \Leftrightarrow p \in I^-(q) \Leftrightarrow q \in I^+(p)$. In particular, two different points may have distance equal to zero.
2. The Lorentzian distance from some point to itself may be different from zero. In fact, $d(p, p) = \infty$ if there exists some piecewise smooth timelike curve joining p to itself; otherwise, $d(p, p) = 0$.
3. If $0 < d(p,q) < \infty$ then $d(q, p) = 0$. Therefore, d is not symmetric in general.
4. The Lorentzian distance satisfies a sort of *reverse triangle inequality*:

$$d(p,q) + d(q,r) \leq d(p,r) \quad \text{if} \quad p \leq q \leq r.$$

More sophisticated properties involving the Lorentzian distance are the following ones:

5. In general, d is not continuous, but only *lower semicontinuous*, i.e. if $\{p_n\} \to p$ and $\{q_n\} \to q$ then

$$\liminf_n d(p_n, q_n) \geq d(p, q).$$

If the spacetime is globally hyperbolic, the Lorentzian distance presents a better behavior, as showed by the following two properties:

6. A sort of Hopf-Rinow Theorem holds: *if* (M, g) *is globally hyperbolic and* $p \leq q$ *then there exists a maximal geodesic joining* p to q (Avez-Seifert's result).
7. If (M, g) is globally hyperbolic, d is continuous and finite valued.

The proof of the Lorentzian Splitting Theorem will also require some additional notions analogous to those ones introduced in Sect. 2 for the Riemannian case:

By a *timelike line (resp. timelike ray)* γ we will understand an unitary timelike geodesic defined on $\mathbb{R}$ (resp. $[0, \infty)$) which realizes the Lorentzian distance between any of its points. Then, the *Busemann function (associated to a timelike ray* γ) is defined as the function $b_\gamma : I[\gamma] \subset M \to [-\infty, \infty)$, with $I[\gamma] = I^+(\gamma(0)) \cap I^-[\gamma]$, given by

$$b_\gamma(\cdot) := \lim_{r\to\infty} (r - d(\cdot, \gamma(r))).$$

In fact, the reverse triangle inequality ensures that limit above cannot be ∞. An important difference with respect to the Riemannian case is that now b_γ may take values at $-\infty$ (because the Lorentzian distance d may be ∞; recall property 2. above) and may also be discontinuous (recall that, even, d may be discontinuous – property 5.). Again from the reverse triangle inequality one can deduce the following restriction for the growth of Busemann function:

$$b_\gamma(q) \geq b_\gamma(p) + d(p, q) \qquad \text{for all} \quad p, q \in I[\gamma], \quad p \leq q. \tag{10}$$

From here, one deduces that level sets of Busemann functions are achronal in $I[\gamma]$.

Another concept which can be also defined for spacetimes is the notion of *asymptote*. Given a timelike ray γ, an *asymptote from* $p \in I[\gamma]$ *to* γ is a causal ray $\alpha : [0, \infty) \to M$ which arises as limit of maximal timelike geodesic segments α_n from p to $\gamma(r_n)$, $r_n \to \infty$ (assumed the spacetime is globally hyperbolic). It is not difficult to realize that they need not be timelike, since the limit vector of a sequence of timelike vectors may be lightlike. On the other hand, by using the uniform convergence of α_n to γ over compact subintervals $[0, t]$ for every $t \in (0, \infty)$, one can deduce:

$$b_\gamma(\alpha(t)) = t + b_\gamma(\alpha(0)) \quad \forall t \in [0, \infty). \tag{11}$$

6 Lorentzian Splitting Theorem

In view of the interest of the Riemannian Splitting Theorem, in the early eighties Yau posed the problem of obtaining the Lorentzian analogue of this result. Concretely, he formulated the following conjecture [24]:

Yau's Conjecture *Suppose that the spacetime* (M, g), *of dimension* $n > 2$, *satisfies the following conditions:*

(1) *(M,g) is timelike geodesically complete,*
(2) $Ric(v, v) \geq 0$ *for all timelike* $v \in TM$,
(3) M *has a timelike line.*

Then M *splits isometrically along the line,* $(M, g) \cong (\mathbb{R} \times M_1, -dt^2 \oplus g_1)$, *where* (M_1, g_1) *is a complete Riemannian manifold.*

The timelike character imposed on the hypotheses of this conjecture are required in order to successfully apply Lorentzian tools analogous to the Riemannian case, as the Lorentzian distance (see Sect. 5).

The proof of this conjecture constitutes one of most important hits in the history of Lorentzian Geometry. It has involved multiple leading authors, sometimes in a joint effort, during more than one decade: Beem, Ehrlich, Markovsen, Galloway, Eschenburg, Heintze and Newman. The main contributions to the proof of the Yau's Conjecture can be outlined as follows:

The first relevant progress in the solution of this conjecture appeared two years later. In 1984, Galloway solved the problem under the assumption that M admits a smooth function whose level sets are compact spacelike Cauchy hypersurfaces [13]. Afterwards, Beem, Ehrlich, Markovsen and Galloway solved the problem by assuming global hyperbolicity instead of timelike completeness, and the sectional curvature inequality $K \leq 0$ instead of the Ricci curvature inequality $Ric \geq 0$ [4, 5]. The assumption of global hyperbolicity is not a big restriction, and, in certain sense, can be considered a more natural condition than timelike geodesic completeness. However, the sectional curvature inequality is significatively more restrictive than that for Ricci curvature, and does not admit a clear interpretation from a physical point of view. Actually, this strong curvature hypothesis is assumed in order to apply a Lorentzian adaptation of the Topogonov's argument [23], via the Harris' Lorentzian Triangle Comparison Theorem (see [1, Appendix A], [18]). By the same year Eschenburg and Heintze gave their proof of the Riemannian Splitting Theorem (see Sect. 4), which provided an alternative viewpoint helpful for the Lorentzian case.

In 1988, Eschenburg solved the problem for $Ric \geq 0$, by assuming both, global hyperbolicity and timelike completeness [10]. The key point was the observation that the geometry of a neighborhood of the timelike line is so well-behaved that allows certain arguments to be successfully modified from $K \leq 0$ to $Ric \geq 0$.

In 1989, Galloway removed the assumption of timelike completeness from Eschenburg's work [14]. The main ingredient was the use of a result by Bartnik on the existence of maximal spacelike hypersurfaces [2].

In 1990 Newman obtained a proof assuming timelike completeness instead of global hyperbolicity, and thus, solved the Yau's Conjecture [20].

Finally, in 1996 Galloway and Horta revisited the whole problem making important simplifications in the proof [16]. The new idea was to replace the use of causal geodesic connectedness ensured by global hyperbolicity by certain limiting arguments within a tubular neighborhood of the given ray.

These results can be summarized in the following general statement of the Lorentzian Splitting Theorem, which, in particular, contains Yau's Conjecture:
Lorentzian Splitting Theorem: *Suppose that the spacetime* (M, g), *of dimension* $n > 2$, *satisfies the following conditions:*

(1) *(M,g) is either timelike geodesically complete or globally hyperbolic,*
(2) $Ric(v, v) \geq 0$ *for all timelike* $v \in TM$,
(3) M *has a timelike line.*

Then M *splits isometrically along the line* $(M, g) \cong (\mathbb{R} \times M_1, -dt^2 \oplus g_1)$, *where* (M_1, g_1) is a *complete Riemannian manifold.*

In order to simplify the exposition, in these notes we are going to restrict our attention to the case where both hypotheses, timelike geodesic completeness and global hyperbolicity, are assumed simultaneously. So, the theorem whose proof we are going to study in the next section is the following one:
(Weak) Lorentzian Splitting Theorem: *Suppose that the spacetime* (M, g), *of dimension* $n > 2$, *satisfies the following conditions:*

(1) *(M,g) is timelike geodesically complete and globally hyperbolic,*
(2) $Ric(v, v) \geq 0$ *for all timelike* $v \in TM$,
(3) M *has a timelike line.*

Then M *splits isometrically along the line* $(M, g) \cong (\mathbb{R} \times M_1, -dt^2 \oplus g_1)$, *where* (M_1, g_1) is a *complete Riemannian manifold.*

7 Proof of the Lorentzian Splitting Theorem

We will essentially follow the arguments given by Galloway in [14]. As we have commented before, the main difficulty in the proof of the Lorentzian Splitting Theorem is the lack of ellipticity of the d'Alambertian operator. The key idea to overcome this problem consists of restricting Busemann functions to a maximal hypersurface Σ having edge(Σ) contained in the level set $b_+ = 0$. The existence of this hypersurface is ensured by a result by Bartnik, and has the remarkable advantage that the induced d'Alambertian becomes elliptic there. Then, a series of maximum principle type arguments ensure that the level sets $b_\pm = 0$ are smooth spacelike hypersurfaces which agree near $\gamma(0)$, providing a posteriori the splitting of a tubular neighborhood of γ. Finally, the global splitting is deduced by applying a continuation type argument which consists of extending flat strips.

We have divided this section in four parts. The first three ones are devoted to study some technical results needed for the proof: the super-harmonicity of the Busemann functions, the nice properties of certain neighborhoods, and a key convexity result. Then, in the forth subsection we outline the proof in six steps.

7.1 Super-Harmonicity of Busemann Functions

We begin with a result which ensures the super-harmonic character of Busemann functions when they are differentiable.

Lemma 7.1 *Assume (M, g) obeys $Ric(v, v) \geq 0$ for all timelike $v \in TM$. Let b_γ be the Busemann function associated to the ray γ. Assume b_γ is smooth on an open set $U \subset I[\gamma]$ with unitary timelike gradient. Then $\Delta b_\gamma \leq 0$ on U.*

In this lemma we have restricted our attention to a domain U where b_γ is differentiable, and so, the Laplacian has sense. Notice also that, here, the nonnegative character of the Ricci curvature implies the super-harmonicity of the Busemann function, in contraposition to the sub-harmonicity ensured in the Riemannian case.

Sketch of proof of Lemma 7.1. Assume by contradiction $\Delta b_\gamma(p) = H > 0$ for some $p \in U$. Denote $\Sigma = \{b_\gamma = c\} \cap U_0$, where $c = b_\gamma(p)$ and $U_0 = U \cap I^-(\gamma(r_0))$, for some $r_0 > 0$. From the properties of the Busemann function, ∇b_γ is past and unitary. Hence, the mean curvature of Σ becomes $H_\Sigma = \Delta b_\gamma$ along Σ. Choose some $q \in I^+(p) \cap U_0$ close enough to p so that $H_\Sigma(x) \geq H/2$ for all $x \in \Sigma \cap I^-(q)$. Let Σ' be a smooth spacelike hypersurface resulting from a small deformation of Σ around p such that $A \cap I^-(p) \neq \emptyset$. This deformation can be chosen to additionally ensure $H_{\Sigma'}(x) \geq H/3$ for all $x \in A := \Sigma' \setminus \Sigma \subset I^-(q)$. Then, for r sufficiently large, $b_r |_{\Sigma'}$ achieves an inferior minimum $c' < c$ at some point $z \in A$, and hence $\Sigma' \subset \{b_r \geq c'\}$. Define the function $\beta_r : I^-(y_r) \to \mathbb{R}$ by

$$\beta_r(x) = r - ((r - c')/2 + d(x, y_r)),$$

where $y_r = \eta_r(\frac{r-c'}{2})$ and $\eta_r : [0, r - c'] \to M$ is maximal geodesic segment from z to $\gamma(r)$. Near z, the level set $\Sigma_r = \{\beta_r = c'\}$ is a smooth spacelike hypersurface which meets Σ' tangentially at z, and lies at $I^-[\Sigma']$. Therefore, from the Maximum Principle

$$H_{\Sigma_r}(z) \geq H_{\Sigma'}(z) \geq H/3. \tag{12}$$

On the other hand, from the nonnegative Ricci hypothesis the estimate (5) holds. This provides the estimate:

$$H_{\Sigma_r}(z) < 2(n - 1)/(r - c). \tag{13}$$

The contradiction is obtained from (12) and (13) by taking $r \to \infty$. □

7.2 Nice Neighborhoods

A concept which will be crucial in the proof of the Lorentzian Splitting Theorem is the notion of *nice neighborhood*.

Definition 7.2 An open set $U \subset I[\gamma]$ is said to be *nice with respect to* γ if there exist constants $K > 0$ and $T > 0$ such that, for each $q \in U$ and $r > T$, any maximal unit speed geodesic segment σ from q to $\gamma(r)$ satisfies

$$g_0(\sigma'(0), \sigma'(0)) \leq K, \quad g_0 \text{ some fixed Riemannian metric on } M.$$

Properties:

1. For each t, $\gamma(t)$ is contained in a nice neighborhood.
2. Asymptotes to γ from points in nice neighborhoods are always timelike.
3. $\{b_r\}$ converges locally uniformly to b_γ on nice neighborhoods, and hence b_γ is continuous on nice neighborhoods.

The following lemma states that level sets of Busemann functions present a "nice" structure when restricted to a nice neighborhood:

Lemma 7.3 *The level set* $\Sigma_c = \{b_\gamma = c\}$ *of a Busemann function* b_γ *is a* partial Cauchy surface *at any nice neighborhood* U*, i.e.* Σ_c *is closed, edgeless and acausal.*

Proof The closed character of Σ_c in U directly follows from the continuity of b_γ in U. In order to prove that Σ_c is edgeless, assume by contradiction that $p \in \text{edge}(\Sigma_c) \neq \emptyset$. For every neighborhood U of p, there exists a timelike curve in U which goes from $I^-(p, U)$ to $I^+(p, U)$ and does not meet Σ_c. In particular, b_γ does not take the value c along that curve. This contradicts the continuity of b_γ, since b_γ takes values smaller and greater than c at the extremes of the curve (recall (10)).

It remains to show that Σ_c is acausal. We already know that it is achronal. By contradiction, assume that Σ_c is not acausal. Then, there exists $p, q \in \Sigma_c$, $p \leq q$, $p \not\ll q$. From Avez-Seifert's result (property 6. in Sect. 5), there exists a null geodesic η connecting p, q. Let $\{\alpha_n\}_n$ be a sequence of maximal timelike segments connecting q with $\gamma(r_n)$ and let α be a limit timelike geodesic. Let $\widetilde{\eta \cdot \alpha_n}$ be the resulting curve from cutting the corner to the convolution $\eta \cdot \alpha_n$. By making the cuts of the curves appropriately, and after comparing them with the corner of $\eta \cdot \alpha$, we deduce

$$d(p, \gamma(r_n)) \geq \text{length}(\widetilde{\eta \cdot \alpha_n}) \geq \text{length}(\alpha_n) + \epsilon = d(q, \gamma(r_n)) + \epsilon.$$

In particular,

$$b_{r_n}(q) - b_{r_n}(p) = d(p, \gamma(r_n)) - d(q, \gamma(r_n)) \geq \epsilon,$$

in contradiction with $b_\gamma(p) = b_\gamma(q) = c$. □

7.3 Key Convexity Result

The proof of the Lorentzian Splitting Theorem requires the following convexity result.

Lemma 7.4 *Assume (M, g) obeys $Ric(v, v) \geq 0$ for all timelike $v \in TM$. Let Σ be a connected smooth spacelike hypersurface contained in a "sufficiently small" nice neighborhood of $\gamma(t)$. Assume the mean curvature of Σ is nonnegative, $H_\Sigma \geq 0$. If b_γ achieves a minimum along Σ then b_γ is constant along Σ.*

Remark 7.5 By a "sufficiently small" neighborhood we will understand a neighborhood small enough to ensure that all properties along the argument below are satisfied.

Sketch of proof. Assume by contradiction that b_γ achieves a minimum along Σ, but b_γ is not constant along Σ. Let B be an open coordinate ball $B \subset \Sigma$ centered at q such that $b_\gamma |_{\partial B}$ is not constantly equal to the minimum value. Choosing B sufficiently small, we can construct a smooth function h on Σ conveniently chosen such that, in particular, $f_{\epsilon,r} = b_r + \epsilon h$ achieves a minimum on B, at p, for large r. Define $\beta_{p,r}(x) = r - (l/2 + d(x, y_r))$, where $y_r = \eta_r(l/2)$ and $\eta_r : [0, l] \to M$ is maximal geodesic segment from p to $\gamma(r)$. The restrictions on h ensure that $\varphi_{\epsilon,r} = \beta_{p,r} + \epsilon h$ is an upper support function of $f_{\epsilon,r}$ at p. Hence, $\varphi_{\epsilon,r}$ is smooth in some neighborhood of p, and achieves minimum at p. On the other hand, it can be proved that Hess $d^s_{q,r}(w, w)$ is bounded from below uniformly in q and r, where $d^s_{q,r}(\cdot) = d(\cdot, \eta_{q,r}(s))$. From this property, the nonnegative mean curvature assumption and the restrictions on h, we deduce that $\Delta_\Sigma \varphi_{\epsilon,r}(p)$ must be negative for ϵ small and r large, in contradiction to the fact that $\varphi_{\epsilon,r}$ achieves a minimum at p. □

As a direct consequence of this convexity result we deduce:

Corollary 7.6 *Let Σ be a smooth maximal spacelike hypersurface whose closure is contained in a sufficiently small nice neighborhood U of $\gamma(t)$. Assume Σ is achronal in U and $\overline{\Sigma}$ is compact. If* $\mathrm{edge}(\Sigma) \subset \{b_\gamma \geq c\}$ *then* $\Sigma \subset \{b_\gamma \geq c\}$.

Proof Otherwise, b_γ achieves a minimum value $c' < c$. From Lemma 7.4, it is $b_\gamma \equiv c'$, in contradiction with the hypothesis $\mathrm{edge}(\Sigma) \subset \{b_\gamma \geq c\}$. □

7.4 Proof of the Theorem

We are now in conditions to prove the Lorentzian Splitting Theorem. This will be overcome in six steps.

Step 1: *Existence of some spacelike hypersurface Σ with $b_\pm |_\Sigma = 0$.*

By using a similar argument to the Riemannian case (see Sect. 3), one deduces the following relations for the Busemann functions:

$$b_+ + b_- \geq 0 \text{ on } I[\gamma] \qquad \text{and} \qquad b_+ + b_- \equiv 0 \text{ on } \gamma. \tag{14}$$

Denote $S^\pm = \{b_\pm = 0\} \cap U$, with U a nice neighborhood for $\gamma_\pm$. From Lemma 7.3, S^+ is a partial Cauchy surface in U; in particular, it is an imbedded topological hypersurface (see [21, pp. 413–415]). Let W be a small coordinate ball in S^+ centered at $\gamma(0)$, with $\overline{W} \subset S^+$. By a fundamental existence result by Bartnik, concerning the existence and regularity of solutions to the Dirichlet problem for the prescribed mean curvature equation with rough boundary data [2, Theorem 4.1], there exists a smooth maximal spacelike hypersurface Σ which is achronal in U, $\overline{\Sigma}$ compact, $\text{edge}(\Sigma) = \text{edge}(W)$, and Σ meets γ. (In principle, Σ may present a *singularity set* as defined in [2], where Σ can fail to be smooth, but the acausality of S^+ ensures that it must be empty). In particular, $\text{edge}(\Sigma) \subset \{b_\pm \geq 0\}$ (recall (14)). By applying Corollary 7.6 to both b^+ and b^- we conclude $\Sigma \subset \{b_+ \geq 0\} \cap \{b_- \geq 0\}$. This joined to (14) implies $b_\pm \mid_\gamma = 0$, and so, Σ is forced to meet γ at $\gamma(0)$. Since $b_+(\gamma(0)) = b_-(\gamma(0)) = 0$, Lemma 7.4 implies $b_+ = b_- = 0$ on Σ.

Step 2: *There is a line α with $b_+(\alpha(t)) = t$, $b_-(\alpha(t)) = -t$.*

Let $B \subset \Sigma$ be a geodesic ball in Σ centered at $\gamma(0)$ of radius R. From each point of B, there exist timelike asymptotes $\alpha_\pm$ to $\gamma_\pm$, resp. Let $\alpha : \mathbb{R} \to M$ be the (possibly) broken geodesic given by:

$$\alpha(t) = \begin{cases} \alpha_-(-t) & -\infty < t \leq 0 \\ \alpha_+(t) & 0 \leq t < \infty. \end{cases}$$

From (11) and the fact that $b_\pm \mid_\Sigma = 0$, we have:

$$\begin{matrix} b_+(\alpha_+(t)) = b_+(\alpha_+(0)) + t = t \\ b_-(\alpha_-(t)) = b_-(\alpha_-(0)) + t = t \end{matrix} \quad \text{if } t \geq 0. \tag{15}$$

From (15), (10) and the fact that α is a ray, we have:

$$t = b_+(\alpha_+(t)) \geq b_+(\alpha_-(t)) + d(\alpha_-(t), \alpha_+(t)) = b_+(\alpha_-(t)) + 2t.$$

Hence,

$$b_+(\alpha_-(t)) \leq -t, \quad \text{if } t \geq 0. \tag{16}$$

From (14), (15), (16):

$$0 \leq b_+(\alpha_-(t)) + b_-(\alpha_-(t)) \leq -t + t = 0.$$

Therefore,

$$b_+(\alpha_-(t)) = -b_-(\alpha_-(t)) = -t, \quad \text{if } t \geq 0.$$

Summarizing:

$$\begin{matrix} b_+(\alpha(t)) = b_+(\alpha_+(t)) = t & \text{if } t \geq 0 \\ b_+(\alpha(t)) = b_+(\alpha_-(-t)) = -(-t) = t & \text{if } t \leq 0. \end{matrix} \tag{17}$$

The expression $b_-(\alpha(t)) = -t$ is deduced similarly.

Finally, in order to show that α is a line, we deduce from (17) and (10):

$$\text{length}(\alpha\mid_{[t_1,t_2]}) = t_2 - t_1 = b_+(\alpha(t_2)) - b_+(\alpha(t_1)) \geq d(\alpha(t_1), \alpha(t_2)).$$

Therefore, α realizes the distance, and so, it is an (unbroken) line.
Step 3: *The line α is normal to B.*

From inequality (10) we deduce that functions

$$b^+_{q,r}(x) = r - d(x, \alpha(r)), \quad b^-_{q,r}(x) = -r + d(\alpha(-r), x)$$

are upper and lower support functions, respectively, of b_+ at $q = \alpha(0)$ for $r > 0$ sufficiently small, i.e.

$$b^+_{q,r} \geq b_+ \geq b^-_{q,r} \quad \text{and} \quad b^+_{q,r}(q) = b_+(q) = b^-_{q,r}(q).$$

Since $b^\pm_{q,r}$ are smooth at q, and $\nabla b^\pm_{q,r} = -\dot{\alpha}(0)$, necessarily b_+ is once differentiable at q, and $\nabla b_+(q) = -\dot{\alpha}(0)$. Therefore, the claim follows by noting that $\nabla b_+(q)$ is perpendicular to $B \subset \Sigma$.
Step 4: *The map $E : U \to E(U)$, $U = \mathbb{R} \times B$, $E(t, q) = \exp(tN_q)$, N_q unitary normal to B a*t *q, is a diffeomorphism.*

It suffices to show that E is injective and nonsingular. Notice that E injective if and only if the normal geodesics to B do not intersect. The future normal geodesics from B are asymptotes to γ_+. Then, by applying a standard "rounding the corner" argument we deduce that they do not intersect. The same happens for the past normal geodesics from B. Finally, future and past normal geodesics cannot intersect either, since, otherwise, a convolution of them at the intersection point violates the achronality of the level surface $b_+ = 0$.

In order to prove that E is nonsingular, assume by contradiction that $\alpha(a)$, $a > 0$, is the first focal point to $p \in B$ along some asymptote α. Then, there exists some neighborhood $V \subset \mathbb{R} \times B$ of $[0, a) \times \{p\}$ such that $E : V \to V'$ is diffeomorphism. Moreover, $b_+(\exp tN_q) = t$ on V. Hence, b_+ is smooth on V, and $\Delta b_+ = H_{\Sigma_t}$, $\Sigma_t = \{b_+ = t\} \cap V$. On the one hand, we deduce from Lemma 7.1 that $H_{\Sigma_t} = \Delta b_+ \leq 0$ along $\alpha\mid_{[0,a)}$. On the other hand, since $\alpha(a)$ is a focal point, necessarily $\limsup_{t\to a} H_{\Sigma_t} = \infty$, a contradiction.
Step 5: *The map $E : U \to E(U)$ given above is an isometry (Local Splitting).*

We have proved that $b_\pm(\exp tN_q) = \pm t$. Therefore, functions $b_\pm$ are smooth. From Lemma 7.1, we also have $\Delta b_\pm \leq 0$ on U. Hence, taking into account that $b_+ = -b_-$, we deduce $\Delta b_+ = 0$ on U. Notice also that ∇b_+ is the (past directed) unit vector field tangent to the normal geodesics from B. Therefore, b_+ obeys the well-known formula

$$-\nabla b_+(\Delta b_+) = Ric(\nabla b_+, \nabla b_+) + |\text{Hess}\, b_+|^2.$$

This equation, together with condition $Ric(v, v) \geq 0$ for all timelike v and the vanishing of Δb_+, implies Hess $b_+ = 0$ on U. Hence, ∇b_+ is parallel on U, and thus, E is an isometry.
Step 6: *The Local Splitting can be extended to a Global Splitting.*

Finally, one needs to achieve the global splitting from the local one. To this aim, some previous definitions are needed:

A *flat strip* is a totally geodesic isometric immersion f of $(\mathbb{R} \times I, -dt^2 + ds^2)$ into (M, g) such that $f\,|_{\mathbb{R}\times\{s\}}$ is line for all $s \in I$. Two lines γ_1, γ_2 are said *strongly parallel* if they bound a flat strip. They are said *parallel* if there exists a finite sequence of consecutively strongly parallel lines such that $\gamma_1 = \beta_0, \beta_1, \ldots, \beta_k = \gamma_2$.

Let $c : [0, 1] \to M$ be any geodesic starting from a line γ. By using the local splitting provided by step 5, there exists a flat strip containing both, γ and c. On the other hand, if γ_1, γ_2 are parallel lines then $I[\gamma_1] = I[\gamma_2]$ and the corresponding Busemann functions agree. Denote by $P_\gamma \subset M$ the set of points which lie on a line which is parallel to γ. From previous property and (10), one deduces that b_+ is differentiable at P_γ and there exists one parallel line γ_q passing through any $q \in P_\gamma$. From the local splitting, P_γ is open, and indeed, it becomes a connected component of M. Hence, there exists one line γ_q parallel to γ, through every $q \in M$. Again by the local splitting, this defines a parallel timelike vector field V on M. Therefore, $V^\perp$ is a parallel distribution, and so, it is integrable. In conclusion, let H be the maximal integral leave through $p = \gamma(0)$. The map

$$I : \mathbb{R} \times H \to M, \qquad I(t, q) = \gamma_q(t)$$

is the desired isometry. □

8 Open Problem: Bartnik's Conjecture

In this last section we are going to recall an open problem, which is closely related to the Splitting Lorentzian Theorem and has implications in Relativity. First, let us consider the following prototype singularity theorem:

Prototype Singularity Theorem. *Suppose that the spacetime (M, g), of dimension $n > 2$, satisfies the following conditions:*

(1) *(M,g) contains a compact Cauchy surface,*
(2) $Ric(v, v) \geq 0$ *for all timelike* $v \in TM$,
(3) *every inextendible causal geodesic satisfies the generic condition.*[1]

Then (M, g) contains an incomplete causal geodesic.

This is a typical singularity theorem because it ensures the causal geodesic incompleteness of the spacetime (which uses to be associated to the existence of a singularity in the universe) under certain physically reasonable conditions. Here, condition (3) plays the role of "strict curvature condition". So, reasoning as in the Introduction, we can ask what happen if we suppress this condition and additionally assume that

[1]Certain curvature quantity is nonzero at some point of each inextendible causal geodesic [3, Definition 12.7, Theorem 12.18].

the thesis of the singularity theorem does not hold. In 1988, Bartnik conjectured that under these new conditions a rigidity behavior arises. Concretely, he established [1]:
Bartnik's Conjecture. *Suppose that the spacetime* (M, g), *of dimension* $n > 2$, *satisfies the following conditions:*

(1) *(M,g) contains a compact Cauchy surface,*
(2) $Ric(v, v) \geq 0$ *for all timelike* $v \in TM$.

Then either (M, g) *is timelike geodesically incomplete, or else* (M, g) *splits isometrically as a product* $(\mathbb{R} \times M_1, -dt^2 \oplus g_1)$, *where* (M_1, g_1) *is a compact Riemannian manifold.*

This conjecture has been proved under some additional assumptions in [1, 11, 15] and, more recently, in [22], where the authors apply some new results about the level sets of Busemann functions for spacetimes. However, as far as we know, the conjecture remains unsolved in its full generality.

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Periodic Trajectories of Dynamical Systems Having a One-Parameter Group of Symmetries

Roberto Giambò and Paolo Piccione

Abstract We study a class of dynamical systems on a compact (semi-)Riemannian manifold endowed with a non trivial 1-parameter (pre-compact) group of symmetries, and we determine the existence of a class of periodic trajectories of these systems.

1 Introduction

The present was originally meant to be the note of an invited lecture given by the second author at the *International Research School "Differential Geometry and Symmetry"*, held at the *Universidad de Murcia*, Spain, in March 2009. During that lecture, emphasis was given mostly to the study of topological and geometrical properties of compact Lorentzian manifolds endowed with a Killing vector field which is time-like somewhere. The main results presented concern some questions of compactness for 1-parameter subgroups of the isometry group of such manifolds, and a proof of existence of non trivial periodic geodesics. The material of the talk is almost entirely contained in references [8, 20].

Actually, some of the techniques employed in [8] to prove the existence of non trivial periodic geodesics in compact Lorentzian manifolds, apply as well in the more general case of periodic solutions of dynamical systems. In this note we will show how to extend the results of [8] to this more general situation using suitable notions of symmetry, thus fitting in the general theme of the *Research School*. We will consider here two types of dynamical systems whose configuration space is a

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R. Giambò (✉)
Scuola di Scienze e Tecnologie, Università di Camerino,
Via Madonna delle Carceri, 9, 62032 Camerino, MC, Italy
e-mail: roberto.giambo@unicam.it

P. Piccione
Departamento de Matemática, Universidade de São Paulo,
Rua do Matão 1010, São Paulo, SP 05508-900, Brazil
e-mail: piccione.p@gmail.com

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(compact) Riemannian or semi-Riemannian manifold (M, g), namely, *conservative* systems, i.e., of the type kinetic energy plus potential, and *exact magnetic* dynamical systems. Trajectories of these systems are curves $x : [0, L] \to M$ that are solutions of a certain second order differential equation of the type $\frac{\mathrm{D}}{\mathrm{d}t}\dot{x} = F(x, \dot{x})$, where $\frac{\mathrm{D}}{\mathrm{d}t}$ is the operator of covariant differentiation for vector fields along x induced by the Levi–Civita connection of g, and $F : TM \to \mathbb{R}$ is a smooth map defined by the potential energy or the magnetic field. When $F \equiv 0$, then solutions of the dynamical system are geodesics of (M, g). A trajectory $x : [0, L] \to M$ of the system is *periodic* if $x(0) = x(L)$ and $\dot{x}(0) = \dot{x}(L)$.

We will define a notion of *symmetry* for such systems (Definitions 2.1, 3.1), which is an isometry of the base manifold that preserves the potential energy or the magnetic field. The first key observation here is that when the dynamical systems admits a non trivial 1-parameter group of isometries, or, equivalently, a Killing vector field whose flow preserves the potential energy or the magnetic field, then some of the flow lines of the group are trajectories of the system. Such special flow lines have a variational characterization, i.e., they are those flow lines passing through the critical point of some natural smooth function on the base manifold (Propositions 2.6, 2.7, 3.6). In particular, being solutions of a *first order* differential equation, such special trajectories do not have self-intersections. It is interesting to observe that infinitesimal symmetries of dynamical systems produce conservation laws for the solutions of such systems (Lemmas 2.4 and 3.4); these are special cases of *Noether's theorem* first theorem on conserved quantities from symmetries, see [18].

When the base manifold (M, g) is compact and Riemannian, then its isometry group is compact. The second important observation is that the compactness of the isometry group implies that when the dynamical system admits a non trivial one-parameter group of symmetries, then it also admits a non trivial one-parameter group of symmetries all of whose flow lines are *closed* (Proposition 4.5). The proof of this fact is based on elementary Lie group techniques; it implies in particular that if there exists a non trivial one-parameter group of symmetries, then the manifold M has the topology of a *generalized Seifert space*, i.e., it admits a smooth circle action without fixed points (Proposition 5.1). Using these two observations, periodic trajectories of dynamical systems on compact Riemannian manifolds are obtained from flow lines of the group of symmetries. Multiplicity of periodic trajectories can be studied using *equivariant Ljusternik–Schnirelmann category* theory, which provides a lower bound for critical orbits of a smooth function on a compact manifold invariant by the action of a compact group of transformations (Sect. 5).

The very same conclusions can be drawn for dynamical systems on arbitrary compact semi-Riemannian manifolds (M, g) having a non trivial one-parameter group of symmetries which is *pre-compact* in the isometry group of (M, g). Also in this situation one has the existence of a non trivial 1-parameter group of symmetries all of whose flows lines are closed. Recall that, unlike the Riemannian case, the (connected component of the identity of the) isometry group of an arbitrary compact semi-Riemannian manifold is in general not compact, and thus the pre-compactness of 1-parameter subgroups has to be explicitly assumed. However, there are important situations where this property is satisfied. For instance, when the compact manifold