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Thomas Weiss
Patrik Ferrari
Herbert Spohn

Reflected Brownian Motions in the KPZ Universality Class

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Thomas Weiss · Patrik Ferrari
Herbert Spohn

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Thomas Weiss
Zentrum Mathematik
Technische Universität München
Garching
Germany

Herbert Spohn
Zentrum Mathematik, M5
Technische Universität München
Munich
Germany

Patrik Ferrari
Institut für Angewandte Mathematik
Universität Bonn
Bonn
Germany

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Preface

In our notes, we study a model of interacting one-dimensional Brownian motions in considerable detail. At first sight, the dynamics look simple: the Brownian motions are ordered, and Brownian motion with label j is reflected from the one with label $j-1$. This model belongs to the Kardar–Parisi–Zhang universality class. Adding to the results as presented in our notes a recent contribution by Kurt Johansson, this model allows for the so far most complete asymptotic analysis. The limiting expressions are likely to be valid for all models in the KPZ class.

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Bonn, Germany	Thomas Weiss
Munich, Germany	Patrik Ferrari
September 2016	Herbert Spohn

Contents

Chapter 1
Introduction

Back in 1931 Hans Bethe diagonalized the hamiltonian of the one-dimensional Heisenberg spin chain through what is now called the "Bethe ansatz" (Bethe 1931). At that time physicists were busy with other important developments and hardly realized the monumental step: for the first time a strongly interacting many-body system had been "solved exactly". In the 1960s Lieb and Liniger (1963), Yang (1967), and many more (Sutherland 2004) discovered that other quantum systems can be handled via Bethe ansatz, which triggered a research area known as quantum integrability. More details on the history of the Bethe ansatz can be found in Batchelor (2007). Even with the Bethe ansatz at one's disposal, it is a highly non-trivial task to arrive at predictions of physical interest. This is why efforts in quantum integrability continue even today, reenforced by the experimental realization of such chains through an array of cold atoms (Simon et al. 2011).

On a mathematical level, quantum hamiltonians and generators of Markov processes have a comparable structure. Thus one could imagine that the Bethe ansatz is equally useful for interacting stochastic systems with many particles. The first indication came somewhat indirectly from the Kardar-Parisi-Zhang equation (Kardar et al. 1986), for short KPZ, in one dimension. We refer to books (Barabasi and Stanley 1995; Meakin 1998), lecture notes (Johansson 2006; Spohn 2006; Quastel 2011; Borodin and Gorin 2012; Borodin and Petrov 2014; Spohn 2015), and survey articles (Halpin-Healy and Zhang 1995; Krug 1997; Sasamoto and Spohn 2011; Ferrari and Spohn 2011; Corwin 2012; Takeuchi 2014; Quastel and Spohn 2015; Halpin-Healy and Takeuchi 2015). The KPZ equation is a stochastic PDE governing the time-evolution of a height function $h(x, t)$ at spatial point x and time t, $h \in \mathbb{R}$, $x \in \mathbb{R}$, $t \geq 0$. The equation reads

$$\partial_t h = \tfrac{1}{2}(\partial_x h)^2 + \tfrac{1}{2}\partial_x^2 h + W \tag{1.1}$$

with $W(x, t)$ normalized space-time white noise. We use here units of height, space, and time such that all coupling parameters take a definite value. For a solution of (1.1), the function $x \mapsto h(x, t)$ is locally like a Brownian motion, which is too singular for the nonlinearity to be well-defined as written. This difficulty was

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resolved through the regularity structures of Hairer (2013), see also Gubinelli and Perkowski (2015) for a somewhat different approach. Kardar (1987) noted one link to quantum integrability. He considered the moments of e^h and established that they are related to the δ-Bose gas with attractive interactions, which is an integrable quantum many-body system (Lieb and Liniger 1963). More precisely, one defines

$$\mathbb{E}\Big(\prod_{\alpha=1}^{n} e^{h(y_\alpha,t)}\Big) = f_t(\vec{y}) \tag{1.2}$$

with $\vec{y} = (y_1, ..., y_n)$. Then

$$\partial_t f_t = -H_n f_t, \tag{1.3}$$

where H_n is the n-particle Lieb-Liniger hamiltonian

$$H_n = -\tfrac{1}{2}\sum_{\alpha=1}^{n} \partial_{y_\alpha}^2 - \tfrac{1}{2}\sum_{\alpha\neq\alpha'=1}^{n} \delta(y_\alpha - y_{\alpha'}). \tag{1.4}$$

Almost thirty years later the generator of the asymmetric simple exclusion process (ASEP) was diagonalized through the Bethe ansatz. In case of N sites, the ASEP configuration space is $\{0, 1\}^N$ signalling a similarity with quantum spin chains. In fact, the ASEP generator can be viewed as the Heisenberg chain with an imaginary XY-coupling. For the totally asymmetric limit (TASEP) and half filled lattice, Gwa and Spohn (1992) established that the spectral gap of the generator is of order $N^{-3/2}$. The same order is argued for the KPZ equation. This led to the strong belief that, despite their very different set-up, both models have the same statistical properties on large space-time scales. In the usual jargon of statistical mechanics, both models are expected to belong to the same universality class, baptized KPZ universality class according to its most prominent representative.

The KPZ equation is solved with particular initial conditions. Of interest are (i) *sharp wedge*, $h(x, 0) = -c_0|x|$ in the limit $c_0 \to \infty$, (ii) *flat*, $h(x, 0) = 0$, and (iii) *stationary*, $h(x, 0) = B(x)$ with $B(x)$ two sided Brownian motion. The quantity of prime interest is the distribution of $h(0, t)$ for large t. More ambitiously, but still feasible in some models, is the large time limit of the joint distribution of $\{h(x_\alpha, t), \alpha = 1, ..., n\}$.

In our notes we consider an integrable system of interacting diffusions, which is governed by the coupled stochastic differential equations

$$dx_j(t) = \beta e^{-\beta(x_j(t)-x_{j-1}(t))}dt + dB_j(t), \tag{1.5}$$

where $\{B_j(t)\}$ a collection of independent standard Brownian motions. For the parameter $\beta > 0$ we will eventually consider only the limit $\beta \to \infty$. But for the purpose of our discussion we keep β finite for a while. In fact there is no choice, no other system of this structure is known to be integrable. The index set depends on the problem, mostly we choose $j \in \mathbb{Z}$. Note that x_j interacts only with its left index neighbor

x_{j-1}. The drift depends on the slope, as it should be for a proper height function. But the exponential dependence on $x_j - x_{j-1}$ is very special, however familiar from other integrable systems. The famous Toda chain (Toda 1967) is a classical integrable system with exponential nearest neighbour interaction. Its quantized version is also integrable (Sutherland 1978).

Interaction with only the left neighbor corresponds to the total asymmetric version. Partial asymmetry would read

$$\mathrm{d}x_j(t) = \big(p\beta \mathrm{e}^{-\beta(x_j - x_{j-1})} - (1-p)\beta \mathrm{e}^{-\beta(x_{j+1}-x_j)}\big)\mathrm{d}t + \mathrm{d}B_j(t) \tag{1.6}$$

with $0 \le p \le 1$. These are non-reversible diffusion processes. Only in the symmetric case, $p = \frac{1}{2}$, the drift is the gradient of a potential and the diffusion process is reversible. Then the model is no longer in the KPZ universality class and has very distinct large scale properties, see Guo et al. (1988); Chang and Yau (1992), for example.

Equations (1.5) and (1.6) should be viewed as a discretization of (1.1). The independent Brownian motions are the natural spatial discretization of the white noise $W(x,t)$. For the drift one might have expected the form $(x_j - x_{j-1})^2 + x_{j+1} - 2x_j + x_{j-1}$, but integrability forces another dependence on the slope. For $p = 1$ the similarity with the KPZ equation is even stronger when considering the exponential moments

$$\mathbb{E}\Big(\prod_{\alpha=1}^{n} \mathrm{e}^{\beta x_{m_\alpha}(t)}\Big) = f_t(\vec{m}), \tag{1.7}$$

$\vec{m} \in \mathbb{Z}^n$. Differentiating in t one obtains

$$\beta^{-2}\frac{\mathrm{d}}{\mathrm{d}t} f_t(\vec{m}) = \sum_{\alpha=1}^{n} \partial_\alpha f_t(\vec{m}) + \tfrac{1}{2}\sum_{\alpha,\alpha'=1}^{n} \delta(m_\alpha - m_{\alpha'}) f_t(\vec{m}) - n f_t(\vec{m}), \tag{1.8}$$

where δ is the Kronecker delta and $\partial_\alpha f(\vec{m}) = f(\ldots, m_\alpha, \ldots) - f(\ldots, m_\alpha - 1, \ldots)$. When comparing with (1.4), instead of ∂_α one could have guessed the discrete Laplacian $\Delta_\alpha = -\partial_\alpha^{\mathrm{T}}\partial_\alpha$ with $^{\mathrm{T}}$ denoting the transpose. To obtain such a result, the drift in (1.5) would have to be replaced by $\beta \mathrm{e}^{-\beta(x_j - x_{j-1})} + \beta \mathrm{e}^{-\beta(x_j - x_{j+1})}$. But the linear equations for the exponential moments are no longer Bethe integrable. Note however that the semigroups $\exp[\partial_\alpha t]$ and $\exp[\Delta_\alpha t]$ differ on a large scale only by a uniform translation proportional to t. With such a close correspondence one would expect system (1.5) to be in the KPZ universality class for any $\beta > 0$, as has been verified to some extent (Amir et al. 2011; Sasamoto and Spohn 2010; Borodin et al. 2015). Also partial asymmetry, $\frac{1}{2} < p < 1$, should be in the KPZ universality class. In fact, the true claim is by many orders more sweeping: the exponential in (1.6), $p \neq \frac{1}{2}$, can be replaced by "any" function of $x_j - x_{j-1}$, except for the linear one, and the system is still in the KPZ universality class. There does not seem to be a promising idea around of how to prove such a property. Current techniques heavily rely on integrability.

This is a good opportunity to reflect another difficulty. Bethe ansatz is like a first indication. But an interesting asymptotic analysis is yet another huge step. This is well illustrated by the KPZ equation. Solving the n-particle Eq. (1.2) yields the exponential moment $\mathbb{E}\big(\mathrm{e}^{nh(0,t)}\big)$. However these moments grow rapidly as $\exp(n^3)$, much too fast to determine the distribution of $h(0,t)$. Because of the underlying lattice, for system (1.5) the exponential moments grow only as $\exp(n^2)$, still too fast. In replica computations one nevertheless continues formally, often with correct results (Calabrese et al. 2010; Imamura and Sasamoto 2013; Dotsenko 2010). A proof must exploit integrability, but cannot use exponential moments directly (Borodin and Corwin 2013).

The system (1.5) simplifies substantially in the limit $\beta \to \infty$. Then one arrives at interacting Brownian motions, where the Brownian motions maintain their ordering and Brownian motion with label j is reflected from its left neighboring Brownian motion with label $j-1$, see Sasamoto and Spohn (2015), Appendix B. These are the reflected Brownian motions of the title. The proper definition of their dynamics requires martingales involving local time, as will be discussed in Chap. 2. Our note discusses exclusively this limit case. Thereby we arrive at a wealth of results on universal statistical properties. Only for the TASEP a comparably detailed analysis has been carried out (Ferrari and Spohn 2011), which does not come as a surprise, since in the limit of low density, under diffusive rescaling of space-time and switching to a moving frame of reference, the TASEP converges to system (1.5) (Karatzas et al. 2016). We will not exploit this limit. Our philosophy is to work in a framework which uses only interacting diffusions.

The limit $\beta \to \infty$ is meaningful also for $p \neq 0, 1$. Then the order of Brownian particles is still preserved, but the reflection between neighbors is oblique. The symmetric version, $p = \frac{1}{2}$, corresponds to independent Brownian motions, maintaining their order, a case which has been studied quite some time ago (Harris 1965). The partially asymmetric version of the model is still Bethe integrable, but less tractable. Only for the half-Poisson initial condition, an expression sufficiently compact for asymptotic analysis has been obtained (Sasamoto and Spohn 2015).

The notion of integrability was left on purpose somewhat vague. In the $\beta = \infty$ limit for (1.5), integrability can be more concretely illustrated. For this case, let us set $j = 2, \ldots, n$ with $x_1(t)$ a standard Brownian motion. Then the transition probability from $\vec{x}$ to $\vec{y}$ at time t is given by

$$\mathbb{P}\big(\vec{x}(t) \in \mathrm{d}\vec{y} \,\big|\, \vec{x}(0) = \vec{x}\,\big) = \det\{\Phi_t^{(i-j)}(y_j - x_i)\}_{1 \le i,j \le n}\, \mathrm{d}\vec{y}, \tag{1.9}$$

where

$$\Phi_t^{(j)}(\xi, t) = \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\delta} \mathrm{d}w\, e^{tw^2/2+\xi w}\, w^{-j} \tag{1.10}$$

with $\delta > 0$ as first established by Sasamoto and Wadati (1998). There is a similar formula for the TASEP (Schütz 1997). Such formuli nourish the hope to uncover interesting features of the model.

The three initial conditions of particular interest, wedge, flat, and stationary, are easily transcribed to system (1.5) and become (i) *packed*, half-infinite system with $x_j(0) = 0$ for $j = 1, 2, \ldots$, (ii) *periodic*, $x_j(0) = j$ for $j \in \mathbb{Z}$, (iii) *Poisson*, $\{x_j(0), j \in \mathbb{Z}\}$ is a Poisson process with constant density. According to our discussion, in the latter case one might think that the quantity of prime interest is $x_1(t)$. But the reflection induces a propagation of statistical fluctuations, as can also be seen from (1.7) together with (1.8). Their propagation speed is 1 and the correct quantity is $x_{\lfloor t \rfloor}(t)$ with $\lfloor t \rfloor$ denoting integer part. Along other space-time observation rays a central limit type behavior would be observed.

As for other models in the KPZ universality class, our asymptotic analysis is limited to a single time and arbitrary number of spatial, resp. index points. Only recently Johansson (2016) posted a result on the joint distribution of $(x_{\lfloor t \rfloor}(t), x_{\lfloor \alpha t \rfloor}(\alpha t))$, $\alpha > 0$, and identified its universal limit. Possibly such progress will lead eventually to a complete understanding of the Airy sheet and the KPZ fixed point (Corwin et al. 2015).

Following the format of Springerbriefs, there is a short summary for each chapter. But let us still explain of how our material is organized as a whole. The following three chapters provide background material. In Chap. 2 we properly define the infinite system of reflected Brownian motions as the solution of a martingale problem and provide a variational formulation of this solution. Also the uniform Poisson process is identified as stationary measure. In Chap. 3 we introduce the theory of determinantal point processes and some related material on Fredholm determinants. At first sight this looks unconnected. But to study the quantities of prime interest one first identifies a "hidden" signed determinantal process which leads to an analytically more tractable representation. In the long time limit we will arrive at a stochastic process which describes the limiting spatial statistics. Such a process has been baptised Airy process, in analogy to the Airy kernel and Airy operator which are one of the defining elements. In fact there are several Airy processes depending on the initial conditions and on the window of observation. The literature on Airy processes is somewhat dispersed. Chapter 4 provides a streamlined account.

In Chap. 5 we investigate the two deterministic initial data, packed and periodic, while in Chap. 6 we study random initial data as defined through a Poisson process. These results will be used to discuss more general initial data, which should be viewed as an open ended enterprise. One natural choice is to have in the left half lattice either packed, periodic, or Poisson join up with either one of them in the right half lattice. An example would be to have periodic to the left and Poisson to the right. This then leads to distinct cross over processes. The mixed cases are studied in Chap. 7. Slow decorrelation, referring to space-like paths more general than fixed time, is a further topic. Each core chapter builds on a suitable asymptotic analysis. In the early days the required techniques were developed ad hoc for the particular model. Over the years a common strategy based on contour integrations has been established, which will be also used here. Thus on the basis of a specific example one can learn a technique applicable also to other models.

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