# Complex Analysis

## Several Complex Variables and Connections with PDE Theory and Geometry

BIRKHÄUSER

Peter Ebenfelt
Norbert Hungerbühler
Joseph J. Kohn
Ngaiming Mok
Emil J. Straube
Editors

Trends in Mathematics

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# Preface

Norbert Hungerbühler

The idea to organize a conference in honour of Linda Rothschild emerged in 2006. This idea began to substantiate in 2007 when the Swiss Mathematical Society assigned the traditional Spring Meeting to the University of Fribourg. An organizing committee was quickly formed:

**Organizing committee**

| | |
|---|---|
| Norbert Hungerbühler | University of Fribourg, Switzerland |
| Frank Kutzschebauch | University of Berne, Switzerland |
| Bernhard Lamel | University of Vienna, Austria |
| Francine Meylan | University of Fribourg, Switzerland |
| Nordine Mir | Université de Rouen, France |

In order to ensure a high-quality conference program, the search for a scientific committee began. Soon after, a distinguished group was found who started working right away:

**Scientific committee**

| | |
|---|---|
| Peter Ebenfelt | University of California, San Diego, USA |
| Franc Forstnerič | University of Ljubljana, Slovenia |
| Joseph J. Kohn | Princeton University, USA |
| Emil J. Straube | Texas A&M University, USA |

Spring Meeting of the Swiss Mathematical Society
**Conference on Complex Analysis 2008**
Several Complex Variables and Connections with PDEs and Geometry
In honour of Linda Rothschild, Fribourg, July 7–11

Only a little while later it became clear that the subject and the top-class speakers who agreed to participate in the conference called for a proceedings volume to make the presented results available shortly after the conference. This project was carried out under the direction of the editorial board:



## Focus on youth

The aim of the conference was to gather worldwide leading scientists, and to offer the occasion to PhD students and postdocs to come into contact with them. The committees explicitly encouraged young scientists, doctoral students and postdocs to initiate scientific contact and to aim at an academic career. The topic of the conference was apparently very attractive for young scientists, and the event an ideal platform to promote national and international doctoral students and postdocs. This aspect became manifest in a poster session where junior researchers presented their results.

The conference was intended to have a strong component in instruction of PhD students: Three mini courses with introductory character were held by Pengfei Guan, Mei-Chi Shaw and Ngaiming Mok. These three mini courses have been very well received by a large audience and were framed by the series of plenary lectures presenting newest results and techniques.

The participation of junior female researchers, PhD students and mathematicians from developing countries has been encouraged in addition by offering grants for traveling and accommodation.

## The subject

The conference *Complex Analysis 2008* has been devoted to the subject of *Several Complex Variables and Connections with PDEs and Geometry*. These three main subject areas of the conference have shown their deep relations, and how techniques from each of these fields can influence the others. The conference has stimulated further interaction between these areas.

The conference was held in honor of Prof. Linda Rothschild who is one of the most influential contributors of the subject during the last decades. A particular aim was to encourage female students to pursue an academic career. In fact, female mathematicians have been well represented among the speakers, in the organizing committee and in the poster sessions.

Several Complex Variables is a beautiful example of a field requiring a wide range of techniques coming from diverse areas in Mathematics. In the last decades, many major breakthroughs depended in particular on methods coming from Partial Differential Equations and Differential and Algebraic Geometry. In turn, Several Complex Variables provided results and insights which have been of fundamental importance to these fields. This is in particular exemplified by the subject of Cauchy-Riemann geometry, which concerns itself both with the tangential Cauchy-Riemann equations and the unique mixture of real and complex geometry that real objects in a complex space enjoy. CR geometry blends techniques from algebraic geometry, contact geometry, complex analysis and PDEs; as a unique meeting point for some of these subjects, it shows evidence of the possible synergies of a fusion of the techniques from these fields.

The interplay between PDE and Complex Analysis has its roots in Hans Lewy's famous example of a locally non solvable PDE. More recent work on PDE has been similarly inspired by examples from CR geometry. The application of analytic techniques in algebraic geometry has a long history; especially in recent years, the analysis of the $\bar{\partial}$-operator has been a crucial tool in this field. The $\bar{\partial}$-operator remains one of the most important examples of a partial differential operator for which regularity of solutions under boundary constraints have been extensively studied. In that respect, CR geometry as well as algebraic geometry have helped to understand the subtle aspects of the problem, which is still at the heart of current research.

Summarizing, our conference has brought together leading researchers at the intersection of these fields, and offered a platform to discuss the most recent developments and to encourage further interactions between these mathematicians. It was also a unique opportunity for younger people to get acquainted with the current research problems of these areas.

## Organization

The conference was at the same time the 2008 Spring Meeting of the Swiss Mathematical Society. The event has profited from the organizational structures of the SMS and the embedding in the mathematical community of Switzerland. The University of Fribourg has proven to be the appropriate place for this international event because of its tradition in Complex Analysis, the central geographic location, and its adequate infrastructure. In turn, its reputation and that of the region has benefited from this conference.

The conference has been announced internationally in the most important conference calendars and in several journals. Moreover, the event has been advertised by posters in numerous mathematics institutes worldwide, by e-mails and in the regular announcements of the Swiss Mathematical Society.

## Acknowledgment

It becomes increasingly difficult to find sponsors for conferences of the given size, in particular in mathematics. We are all the more grateful to our sponsors who have generously supported the conference, and the proceedings volume in hand:

**List of Sponsors**

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- Centre Interfacultaire Bernoulli CIB, EPFL
- Department of Mathematics, University of Fribourg
- Faculty of Sciences, University of Fribourg
- Rectorate, University of Fribourg
- Swiss National Science Foundation
- Walter Haefner Stiftung
- Swiss Doctoral Program in Mathematics

In the name of the conference committees and of all participants, we would like to thank all sponsors – foundations, institutions and companies – very cordially for their contributions and the shown appreciation for our work as mathematicians: Thank you!

We also thank the team of Dr. Thomas Hempfling of the Birkhäuser publishing company for their help and professional expertise during the production process of these proceedings.

Finally, we would like to thank Elisabeth François and Claudia Kolly who assumed the secretariat of the conference.

Fribourg, August 2009 Norbert Hungerbühler

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# Extended Curriculum Vitae of Linda Preiss Rothschild

Linda Rothschild was born February 28, 1945, in Philadelphia, PA. She received her undergraduate degree, magna cum laude, from the University of Pennsylvania in 1966 and her PhD in mathematics from MIT in 1970. Her PhD thesis was "On the Adjoint Action of a Real Semisimple Lie Group". She held positions at Tufts University, Columbia University, the Institute for Advanced Study, and Princeton University before being appointed an associate professor of mathematics at the University of Wisconsin-Madison in 1976. She was promoted to full professor in 1979. Since 1983 she has been professor of mathematics at the University of California at San Diego, where she is now a Distinguished Professor.

Rothschild has worked in the areas of Lie groups, partial differential equations and harmonic analysis, and the analytic and geometric aspects of several complex variables. She has published over 80 papers in these areas. Rothschild was awarded an Alfred P. Sloan Fellowship in 1976. In 2003 she won the Stefan Bergman Prize from the American Mathematical Society (jointly with Salah Baouendi). The citation read in part:

> *"The Bergman Prize was awarded to Professors Salah Baouendi and Linda Rothschild for their joint and individual work in complex analysis. In addition to many important contributions to complex analysis they have also done first rate work in the theory of partial differential equations. Their recent work is centered on the study of CR manifolds to which they and their collaborators have made fundamental contributions.*
>
> *Rothschild, in a joint paper with E. Stein, introduced Lie group methods to prove $L^p$ and Hölder estimates for the sum of squares operators as well as the boundary Kohn Laplacian for real hypersurfaces. In later joint work with L. Corwin and B. Helfer, she proved analytic hypoellipticity for a class of first-order systems. She also proved the existence of a family of weakly pseudoconvex hypersurfaces for which the boundary Kohn Laplacian is hypoelliptic but does not satisfy maximal $L^2$ estimates."*

In 2005, Rothschild was elected a Fellow of the American Academy of Arts and Sciences, and in 2006 she was an invited speaker at the International Congress of Mathematics in Madrid.

Rothschild served as President of the Association for Women in Mathematics from 1983 to 1985 and as Vice-President of the American Mathematical Society from 1985 to 1987. She served on the editorial committees of the Transactions of the AMS and Contemporary Mathematics. She is also an editorial board member of Communications in Partial Differential Equations and co-founder and co-editor-in-chief of Mathematical Research Letters. She has served on many professional committees, including several AMS committees, NSF panels, and an organization committee for the Special Year in Several Complex Variables at the Mathematical Sciences Research Institute. She presented the 1997 Emmy Noether Lecture for the AWM. Rothschild has a keen interest in encouraging young women who want to study mathematics. A few years ago she helped establish a scholarship for unusually talented junior high school girls to accelerate their mathematical training by participating in a summer program.

## Educational Background

| | |
|---|---|
| B.A. | University of Pennsylvania, 1966 |
| Ph.D. | in mathematics, Massachusetts Institute of Technology, 1970<br>Dissertation: *On the Adjoint Action of a Real Semisimple Lie Group*<br>Advisor: Isadore Manual Singer |

## Professional Employment

| | |
|---|---|
| 1982– | Professor, University of California, San Diego |
| 2001–05 | Vice Chair for Graduate Affairs, Mathematics Dept., UCSD |
| 1979–82 | Professor, University of Wisconsin |
| 1981–82 | Member, Institute for Advanced Study |
| 1978 | Member, Institute for Advanced Study |
| 1976–77 | Associate Professor, University of Wisconsin |
| 1975–76 | Visiting Assistant Professor, Princeton University |
| 1974–75 | Member, Institute for Advanced Study |
| 1972–74 | Ritt Assistant Professor, Columbia University |
| 1970–72 | Assistant Professor, Tufts University |
| 1970–72 | Research Staff, Artificial Intelligence Laboratory, M.I.T. |

## Honors and Fellowships

| | |
|---|---|
| 2005 | Fellow, American Academy of Arts and Sciences |
| 2003 | Stefan Bergman Prize |
| 1976–80 | Alfred P. Sloan Foundation Fellow |
| 1966–70 | National Science Foundation Graduate Fellow |

## Selected Invited Lectures

- Invited address, International Congress of Mathematicians, Madrid, August 2006
- "Frontiers in Mathematics" Lecturer, Texas A&M University, September 1999
- Invited hour speaker, Sectional joint meeting of American Mathematical Society and Mathematical Association of America, Claremont, October 1997
- Emmy Noether Lecturer (Association for Women in Mathematics), Annual Joint Mathematics Meetings, San Diego January 1997
- Invited hour lecturer, Annual Joint Mathematics Meetings, Orlando, January 1996
- Invited hour speaker, Annual Summer meeting of American Mathematics Society, Pittsburgh, August1981

## Students

| | | |
|---|---|---|
| Mark Marson | University of California, San Diego, | 1990 |
| Joseph Nowak | University of California, San Diego, | 1994 |
| John Eggers | University of California, San Diego, | 1995 |
| Bernhard Lamel | University of California, San Diego, | 2000 |
| Slobodan Kojcinovic | University of California, San Diego, | 2001 |
| Robert Kowalski | University of California, San Diego, | 2002 |

## Selected National Committees and Offices

### National Science Foundation, Mathematics Division

- Advisory Panel, 1984–87 and other panels 1997–99, 2004

### American Mathematical Society (AMS)

- Bocher Prize Committee 2001–04
- National Program Committee 1997–2000
  Chair 1998–1999
- Nominating Committee, 1982–84, 1994–96
- Committee on Science Policy, 1979–82, 92–9
- AMS Vice President, 1985–87
- Committee on Committees, 1977–79, 1979–81
- Executive Committee, 1978–80
- Council of the AMS, 1977–80

**Association for Women in Mathematics (AWM)**

- Noether Lecture Committee 1988–90, 1994–1997
  Chair 1989–90
- Schafer Prize Committee 1993–94
- AWM President, 1983–85.

**Mathematical Association of America**

- Chauvenet Prize Committee, 1998–2000

**Mathematical Sciences Research Institute**

- Board of Trustees, 1996–1999
- Budget Committee 1996–1998

**California Science Museum**

- Jury to select California Scientist of the Year Award, 1995–1999

**Institute for Pure and Applied Mathematics (IPAM)**

- Board of Trustees, 2002–2005

## Editorial Positions

- Co-Editor-in-Chief, Mathematical Research Letters, 1994–
- Editorial Board, Journal of Mathematical Analysis and Applications, 2001–
- Editorial Board, Communications in Partial Differential Equations, 1984–
- Editorial Board, Contemporary Mathematics, 1990–1994
- Editor for complex and harmonic analysis, Transactions of the American Mathematical Society, 1983–1986

## Publication List of Linda Preiss Rothschild


[1] Peter Ebenfelt and Linda P. Rothschild. New invariants of CR manifolds and a criterion for finite mappings to be diffeomorphic. *Complex Var. Elliptic Equ.*, 54(3-4):409–423, 2009. ISSN 1747-6933.

[2] M.S. Baouendi, Peter Ebenfelt, and Linda P. Rothschild. Transversality of holomorphic mappings between real hypersurfaces in different dimensions. *Comm. Anal. Geom.*, 15(3):589–611, 2007. ISSN 1019-8385.

[3] Peter Ebenfelt and Linda P. Rothschild. Analyticity of smooth CR mappings of generic submanifolds. *Asian J. Math.*, 11(2):305–318, 2007. ISSN 1093-6106.

[4] Peter Ebenfelt and Linda P. Rothschild. Images of real submanifolds under finite holomorphic mappings. *Comm. Anal. Geom.*, 15(3):491–507, 2007. ISSN 1019-8385.

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Complex Analysis
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# Oblique Polar Lines of $\int_X |f|^{2\lambda}|g|^{2\mu}\square$

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**Abstract.** Existence of oblique polar lines for the meromorphic extension of the current valued function $\int |f|^{2\lambda}|g|^{2\mu}\square$ is given under the following hypotheses: $f$ and $g$ are holomorphic function germs in $\mathbb{C}^{n+1}$ such that $g$ is non-singular, the germ $\Sigma := \{\,\mathrm{d}f \wedge \mathrm{d}g = 0\}$ is one dimensional, and $g$ is proper and finite on $S := \{df = 0\}$. The main tools we use are interaction of strata for $f$ (see [4]), monodromy of the local system $H^{n-1}(u)$ on $S$ for a given eigenvalue $\exp(-2i\pi u)$ of the monodromy of $f$, and the monodromy of the cover $g|_S$. Two non-trivial examples are completely worked out.

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## Introduction

Given an open subset $Y$ in $\mathbb{C}^m$, two holomorphic functions $f, g$ on $Y$ and a $\mathcal{C}^\infty$ compactly support $(m,m)$-form $\phi$ in $Y$, the integral $\int_Y |f|^{2\lambda}|g|^{2\mu}\phi$, for $(\lambda,\mu)$ in $\mathbb{C}^2$ with $\Re\lambda$ and $\Re\mu > 0$, defines a holomorphic function in that region. As a direct consequence of the resolution of singularities, this holomorphic function extends meromorphically to $\mathbb{C}^2$, see Theorem 1.1. The polar locus of this extension is contained in a union of straight lines with rational slopes (see [8] for other results on this integral). In this paper we look for geometric conditions that guarantee a true polar line of this extension for at least one $\phi \in \Lambda^{m,m}C_c^\infty(Y)$, in other words a true polar line of the meromorphic extension of the holomorphic current valued function

$$(\lambda,\mu) \mapsto \int_Y |f|^{2\lambda}|g|^{2\mu}\square.$$

Since existence of horizontal or vertical polar lines follows directly from existence of poles of $\int_Y |g|^{2\mu}\square$ or $\int_Y |f|^{2\lambda}\square$ that have been extensively studied in [1], [2] and [3], we will concentrate on oblique polar lines. Because desingularization is quite hard to compute, it is not clear how to determine these polar lines. Moreover, only a few of the so obtained candidates are effectively polar and no geometric conditions are known to decide it in general.

In Section 2, we expose elementary properties of meromorphic functions of two variables that are used later for detecting oblique polar lines. Four examples of couples $(f, g)$ for which these results apply are given.

In Sections 3 and 4 we give sufficient criteria to obtain oblique polar lines in rather special cases, but with a method promised to a large generalization. They rely on results which give realization in term of holomorphic differential forms of suitable multivalued sections of the sheaf of vanishing cycles along the smooth part of the singular set $S$ (assumed to be a curve) of the function $f$. The second function $g$ being smooth and transversal to $S$ at the origin. The sufficient condition is then given in term of the monodromy on $S^* := S \setminus \{0\}$ on the sheaf of vanishing cycles of $f$ for the eigenvalue $\exp(-2i\pi u)$ assuming that the meromorphic extension of $\int_X |f|^{2\lambda}\square$ has only simple poles at $-u-q$ for all $q \in \mathbb{N}$ (see Corollary 4.3).

To be more explicit, recall the study of $\int |f|^{2\lambda}\square$ started in [4] and completed in [5], for a holomorphic function $f$ defined in an open neighbourhood of $0 \in \mathbb{C}^{n+1}$ with one-dimensional critical locus $S$. The main tool was to restrict $f$ to hyperplane sections transverse to $S^*$ and examine, for a given eigenvalue $\exp(-2i\pi u)$ of the monodromy of $f$, the local system $H^{n-1}(u)$ on $S^*$ formed by the corresponding spectral subspaces. Higher-order poles of the current valued meromorphic function $\int |f|^{2\lambda}\square$ at $-u-m$, some $m \in \mathbb{N}$, are detected using the existence of a uniform section of the sheaf $H^{n-1}(u)$ on $S^*$ which is not extendable at the origin. So an important part of this local system remained unexplored in [4] and [5] because only the eigenvalue 1 of the monodromy $\Theta$ of the local system $H^{n-1}(u)$ on $S^*$ is involved in the exact sequence

$$0 \to H^0(S, H^{n-1}(u)) \to H^0(S^*, H^{n-1}(u)) \to H^1_{\{0\}}(S, H^{n-1}(u)) \to 0.$$

In this paper, we will focus on the other eigenvalues of $\Theta$.

Let us assume the following properties:

(1) the function $g$ is non-singular near $0$;
(2) the set $\Sigma := \{\, df \wedge dg = 0\}$ is a curve;
(3) the restriction $g|_S : S \to \mathbb{D}$ is proper and finite;
(4) $g|_S^{-1}(0) = \{0\}$ and $g|_{S^*}$ is a finite cover of $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$.

Condition (2) implies that the singular set $S := \{\, df = 0\}$ of $f$ has dimension $\leqslant 1$. We are interested in the case where $S$ is a curve.

Remark that condition (4) may always be achieved by localization near $0$ when conditions (1), (2) and (3) are satisfied. These conditions hold in a neighbourhood of the origin if $(f, g)$ forms an isolated complete intersection singularity (icis) with one-dimensional critical locus, assuming $g$ smooth. But we allow also the case where $\Sigma$ has branches in $\{f = 0\}$ not contained in $S$.

The direct image by $g$ of the constructible sheaf $H^{n-1}(u)$ supported in $S$ will be denoted by $\mathcal{H}$; it is a local system on $\mathbb{D}^*$. Let $\mathcal{H}_0$ be the fibre of $\mathcal{H}$ at $t_0 \in \mathbb{D}^*$ and $\Theta_0$ its monodromy which is an automorphism of $\mathcal{H}_0$. In case where $S$ is smooth, it is possible to choose the function $g$ in order that $g|_S$ is an isomorphism and $\Theta_0$ may be identified with the monodromy $\Theta$ of $H^{n-1}(u)$ on $S^*$. In general, $\Theta_0$ combines $\Theta$ and the monodromy of the cover $g|_{S^*}$.

Take an eigenvalue[1] $\exp(-2i\pi l/k) \neq 1$ of $\Theta$, with $l \in [1, k-1]$ and $(l,k) = 1$. We define an analogue of the interaction of strata in this new context. The auxiliary non singular function $g$ is used to realize analytically the rank one local system on $S^*$ with monodromy $\exp(-2i\pi l/k)$. To perform this we shall assume that the degree of $g$ on the irreducible branch of $S$ we are interested in, is relatively prime to $k$. Of course this is the case when $S$ is smooth and $g$ transversal to $S$ at the origin. Using then a $k$th root of $g$ we can lift our situation to the case where we consider an invariant section of the complex of vanishing cycles of the lifted function $\tilde{f}$ (see Theorem 4.2) and then use already known results from [4]. The existence of true oblique polar lines follows now from results of Section 2.

The paper ends with a complete computation of two non-trivial examples that illustrate the above constructions.

## 1. Polar structure of $\int_X |f|^{2\lambda}\,\square$

**Theorem 1.1.** Bernstein & Gelfand. *For $m$ and $p \in \mathbb{N}^*$, let $Y$ be an open subset in $\mathbb{C}^m$, $f : Y \to \mathbb{C}^p$ a holomorphic map and $X$ a relatively compact open set in $Y$. Then there exists a finite set $P(f) \subset \mathbb{N}^p \setminus \{0\}$ such that, for any form $\phi \in \Lambda^{m,m}C_c^\infty(X)$ with compact support, the holomorphic map in the open set $\{\Re\lambda_1 > 0\} \times \cdots \times \{\Re\lambda_p > 0\}$ given by*

$$(\lambda_1, \ldots, \lambda_p) \mapsto \int_X |f_1|^{2\lambda_1} \ldots |f_p|^{2\lambda_p}\phi \tag{1.1}$$

*has a meromorphic extension to $\mathbb{C}^p$ with poles contained in the set*

$$\bigcup_{a\in P(f), l\in\mathbb{N}^*} \{\langle a \mid \lambda\rangle + l = 0\}.$$

*Proof.* For sake of completeness we recall the arguments of [10].

Using desingularization of the product $f_1 \ldots f_p$, we know [12] that there exists a holomorphic manifold $\tilde{Y}$ of dimension $m$ and a holomorphic proper map $\pi : \tilde{Y} \to Y$ such that the composite functions $\tilde{f}_j := f_j \circ \pi$ are locally expressible as

$$\tilde{f}_k(y) = y_1^{a_1^k} \ldots y_m^{a_m^k} u_k(y), 1 \leqslant k \leqslant p, \tag{1.2}$$

where $a_j^k \in \mathbb{N}$ and $u_k$ is a holomorphic nowhere vanishing function. Because $\pi^{-1}(X)$ is relatively compact, it may be covered by a finite number of open set where (1.2) is valid.

For $\varphi \in \Lambda^{m,m}C_c^\infty(X)$ and $\Re\lambda_1, \ldots, \Re\lambda_p$ positive, we have

$$\int_X |f_1|^{2\lambda_1} \ldots |f_p|^{2\lambda_p}\phi = \int_{\pi^{-1}(X)} |\tilde{f}_1|^{2\lambda_1} \ldots |\tilde{f}_p|^{2\lambda_p}\pi^*\phi.$$

[1] Note that the eigenvalues of $\Theta$ are roots of unity.

Using partition of unity and setting $\mu_k := a_k^1\lambda_1 + \cdots + a_k^p\lambda_p$, $1 \leqslant k \leqslant m$, we are reduced to give a meromorphic extension to

$$(\mu_1, \ldots, \mu_m) \to \int_{\mathbb{C}^m} |y_1|^{2\mu_1} \ldots |y_m|^{2\mu_m} \omega(\mu, y), \tag{1.3}$$

where $\omega$ is a $C^\infty$ form of type $(m, m)$ with compact support in $\mathbb{C}^m$ valued in the space of entire functions on $\mathbb{C}^m$. Of course, (1.3) is holomorphic in the set $\{\Re\mu_1 > -1, \ldots, \Re\mu_m > -1\}$.

The relation

$$(\mu_1 + 1).|y_1|^{2\mu_1} = \partial_1(|y_1|^{2\mu_1}.y_1)$$

implies by partial integration in $y_1$

$$\int_{\mathbb{C}^m} |y_1|^{2\mu_1} \ldots |y_m|^{2\mu_m} \omega(\mu, y) = \frac{-1}{\mu_1 + 1} \int_{\mathbb{C}^m} |y_1|^{2\mu_1}.y_1.|y_2|^{2\mu_2} \ldots |y_m|^{2\mu_m} \partial_1\omega(\mu, y).$$

Because $\partial_1\omega$ is again a $C^\infty$ form of type $(m, m)$ with compact support in $\mathbb{C}^m$ valued in the space of entire functions on $\mathbb{C}^m$, we may repeat this argument for each coordinate $y_2, \ldots, y_m$ and obtain

$$\int_{\mathbb{C}^m} |y_1|^{2\mu_1} \ldots |y_m|^{2\mu_m} \omega(\mu, y) =$$
$$= \frac{(-1)^m}{(\mu_1 + 1) \ldots (\mu_m + 1)} \int_{\mathbb{C}^m} |y_1|^{2\mu_1}.y_1.|y_2|^{2\mu_2}.y_2 \ldots |y_m|^{2\mu_m}.y_m.\partial_1 \ldots \partial_m\omega(\mu, y).$$

The integral on the RHS is holomorphic for $\Re\mu_1 > -3/2, \ldots, \Re\mu_m > -3/2$. Therefore the function (1.3) is meromorphic in this domain with only possible poles in the union of the hyperplanes $\{\mu_1 + 1 = 0\}, \ldots, \{\mu_m + 1 = 0\}$.

Iteration of these arguments concludes the proof. □

**Remark 1.2.** An alternate proof of Theorem 1.1 has been given for $p = 1$ by Bernstein [9], Björk [11], Barlet-Maire [6], [7]. See also Loeser [13] and Sabbah [14] for the general case.

In case where $f_1, \ldots, f_p$ define an isolated complete intersection singularity (icis), Loeser and Sabbah gave moreover the following information on the set $P(f)$ of the "slopes" of the polar hyperplanes in the meromorphic extension of the function (1.1): it is contained in the set of slopes of the discriminant locus $\Delta$ of $f$, which in this case is an hypersurface in $\mathbb{C}^p$. More precisely, take the $(p-1)$-skeleton of the fan associated to the Newton polyhedron of $\Delta$ at 0 and denote by $P(\Delta)$ the set of directions associated to this $(p-1)$-skeleton union with the set $\{(a_1, \ldots, a_p) \in \mathbb{N}^p \mid a_1 \ldots a_p = 0\}$. Then

$$P(f) \subseteq P(\Delta).$$

In particular, if the discriminant locus is contained in the hyperplanes of coordinates, then there are no polar hyperplanes with direction in $(\mathbb{N}^*)^p$.

The results of Loeser and Sabbah above have the following consequence for an icis which is proved below directly by elementary arguments, after we have introduced some terminology.

**Definition 1.3.** Let $f_1, \dots, f_p$ be holomorphic functions on an open neighbourhood $X$ of the origin in $\mathbb{C}^m$. We shall say that a polar hyperplane $H \subset \mathbb{C}^p$ for the meromorphic extension of $\int_X |f_1|^{2\lambda_1} \dots |f_p|^{2\lambda_p}\square$ *is supported by the closed set* $F \subset X$, when $H$ is not a polar hyperplane for the meromorphic extension of $\int_{X\setminus F} |f_1|^{2\lambda_1} \dots |f_p|^{2\lambda_p}\square$. We shall say that a polar direction is *supported in* $F$ if any polar hyperplane with this direction is supported by $F$.

**Proposition 1.4.** *Assume $f_1, \dots, f_p$ are quasi-homogeneous functions for the weights $w_1, \dots, w_p$, of degree $a_1, \dots, a_p$. Then if there exists a polar hyperplane direction supported by the origin for (1.1) in $(\mathbb{N}^*)^p$ it is $(a_1, \dots, a_p)$ and the corresponding poles are at most simple.*

*In particular, for $p = 2$, and if $(f_1, f_2)$ is an icis, all oblique polar lines have direction $(a_1, a_2)$.*

*Proof.* Quasi-homogeneity gives $f_k(t^{w_1}x_1, \dots, t^{w_m}x_m) = t^{a_k} f_k(x)$, $k = 1, \dots, p$.

Let $\Omega := \sum_1^m (-1)^{j-1} w_j x_j \, \mathrm{d}x_0 \wedge \cdots \wedge \widehat{\mathrm{d}x_j} \wedge \cdots \wedge \mathrm{d}x_m$ so that $\mathrm{d}\Omega = (\sum w_j)\, \mathrm{d}x$.

From Euler's relation, because $f_k^{\lambda_k}$ is quasi-homogeneous of degree $a_k\lambda_k$:

$$\mathrm{d}f_k^{\lambda_k} \wedge \Omega = a_k\lambda_k f_k^{\lambda_k}\, \mathrm{d}x,$$
$$\mathrm{d}x^\delta \wedge \Omega = \langle w \mid \delta\rangle x^\delta \, \mathrm{d}x, \ \forall \delta \in \mathbb{N}^m.$$

Take $\rho \in \mathcal{C}_c^\infty(\mathbb{C}^m)$; then, with $\mathbf{1} = (1, \dots, 1) \in \mathbb{N}^p$ and $\varepsilon \in \mathbb{N}^m$:

$$\mathrm{d}(|f|^{2\lambda} x^\delta \bar{x}^\varepsilon \rho\Omega \wedge \mathrm{d}\bar{x}) =$$
$$= (\langle a \mid \lambda\rangle + \langle w \mid \delta + \mathbf{1}\rangle)|f|^{2\lambda} x^\delta \bar{x}^\varepsilon \rho \, \mathrm{d}x \wedge \mathrm{d}\bar{x} + |f|^{2\lambda} x^\delta \bar{x}^\varepsilon \, \mathrm{d}\rho \wedge \Omega \wedge \mathrm{d}\bar{x}.$$

Using Stokes' formula we get

$$(\langle a \mid \lambda\rangle + \langle w \mid \delta + \mathbf{1}\rangle) \int |f|^{2\lambda} x^\delta \bar{x}^\varepsilon \rho \, \mathrm{d}x \wedge \mathrm{d}\bar{x} = -\int |f|^{2\lambda} x^\delta \bar{x}^\varepsilon \, \mathrm{d}\rho \wedge \Omega \wedge \mathrm{d}\bar{x}.$$

For $\rho = 1$ near $0$, $\mathrm{d}\rho = 0$, near $0$. Therefore the right-hand side has no poles supported by the origin. Now the conclusion comes from the Taylor expansion at $0$ of the test function. □

## 2. Existence of polar oblique lines

In this section, we consider two holomorphic functions $f, g : Y \to \mathbb{C}$, where $Y$ is an open subset in $\mathbb{C}^m$ and fix a relatively compact open subset $X$ of $Y$. Without loss of generality, we assume $0 \in X$. We study the possible oblique polar lines of the meromorphic extension of the current valued function

$$(\lambda, \mu) \mapsto \int_X |f|^{2\lambda}|g|^{2\mu}\square. \tag{2.1}$$

The following elementary lemma is basic.

**Lemma 2.1.** *Let $M$ be a meromorphic function in $\mathbb{C}^2$ with poles along a family of lines with directions in $\mathbb{N}^2$. For $(\lambda_0, \mu_0) \in \mathbb{C}^2$, assume*

(i) $\{\lambda = \lambda_0\}$ *is a polar line of order* $\leqslant k_0$ *of* $M$,
(ii) $\{\mu = \mu_0\}$ *is not a polar line of* $M$,
(iii) $\lambda \mapsto M(\lambda, \mu_0)$ *has a pole of order at least* $k_0 + 1$ *at* $\lambda_0$.

*Then there exists $(a, b) \in (\mathbb{N}^*)^2$ such that the function $M$ has a pole along the (oblique) line $\{a\lambda + b\mu = a\lambda_0 + b\mu_0\}$.*

*Proof.* If $M$ does not have an oblique polar line through $(\lambda_0, \mu_0)$, then the function $(\lambda, \mu) \mapsto (\lambda - \lambda_0)^{k_0} M(\lambda, \mu)$ is holomorphic near $(\lambda_0, \mu_0)$. Therefore, $\lambda \mapsto M(\lambda, \mu_0)$ has at most a pole of order $k_0$ at $\lambda_0$. Contradiction. □

It turns out that to check the first condition in the above lemma for the function (2.1), a sufficient condition is that the poles of the meromorphic extension of the current valued function

$$\lambda \mapsto \int_X |f|^{2\lambda} \square \tag{2.2}$$

are of order $\leqslant k_0$. Such a simplification does not hold for general meromorphic functions. For example,

$$(\lambda, \mu) \mapsto \frac{\lambda + \mu}{\lambda^2}$$

has a double pole along $\{\lambda = 0\}$ but its restriction to $\{\mu = 0\}$ has only a simple pole at $0$.

**Proposition 2.2.** *If the meromorphic extension of the current valued function* (2.2) *has a pole of order $k$ at $\lambda_0 \in \mathbb{R}_-$,* i.e., *it has a principal part of the form*

$$\frac{T_k}{(\lambda - \lambda_0)^k} + \cdots + \frac{T_1}{\lambda - \lambda_0},$$

*at $\lambda_0$, then the meromorphic extension of the function* (2.1) *has a pole of order*

$$k_0 := \max\{0 \leqslant l \leqslant k \mid \operatorname{supp} T_l \not\subseteq \{g = 0\}\} \tag{2.3}$$

*along the line $\{\lambda = \lambda_0\}$.*

*Proof.* As a consequence of the Bernstein identity (see [11]), there exists $N \in \mathbb{N}$ such that the extension of $\int_X |f|^{2\lambda}\phi$ in $\{\Re\lambda > \lambda_0 - 1\}$ can be achieved for $\phi \in \Lambda^{m,m}\mathcal{C}_c^N(X)$. Our hypothesis implies that this function has a pole of order $\leqslant k$ at $\lambda_0$. Because $|g|^{2\mu}\phi$ is of class $\mathcal{C}^N$ for $\Re\mu$ large enough, the function

$$\lambda \mapsto \int_X |f|^{2\lambda}|g|^{2\mu}\phi$$

has a meromorphic extension in $\{\Re\lambda > \lambda_0 - 1\}$ with a pole of order $\leqslant k$ at $\lambda_0$. We have proved that (2.1) has a pole of order $\leqslant k$ along the line $\{\lambda = \lambda_0\}$.

Near $\lambda_0$, the extension of $\int_X |f|^{2\lambda}\phi$ writes

$$\frac{\langle T_k,\phi\rangle}{(\lambda-\lambda_0)^k}+\cdots+\frac{\langle T_1,\phi\rangle}{\lambda-\lambda_0}+\cdots.$$

Hence that of $\int_X |f|^{2\lambda}|g|^{2\mu}\phi$ looks

$$\frac{\langle T_k|g|^{2\mu},\phi\rangle}{(\lambda-\lambda_0)^k}+\cdots+\frac{\langle T_1|g|^{2\mu},\phi\rangle}{\lambda-\lambda_0}+\cdots.$$

If $\operatorname{supp} T_k \subseteq \{g=0\}$, then the first term vanishes for $\Re\mu$ large enough, because $T_k$ is of finite order (see the beginning of the proof). So the order of the pole along the line $\{\lambda=\lambda_0\}$ is $\leqslant k_0$.

Take $x_0 \in \operatorname{supp} T_{k_0}$ such that $g(x_0)\neq 0$ and $V$ a neighborhood of $x_0$ in which $g$ does not vanish. From the definition of the support, there exists $\psi \in \Lambda^{m,m}C_c^\infty(V)$ such that $\langle T_{k_0},\psi\rangle \neq 0$. With $\phi := \psi|g|^{-2\mu} \in \Lambda^{m,m}C_c^\infty(V)$, we get

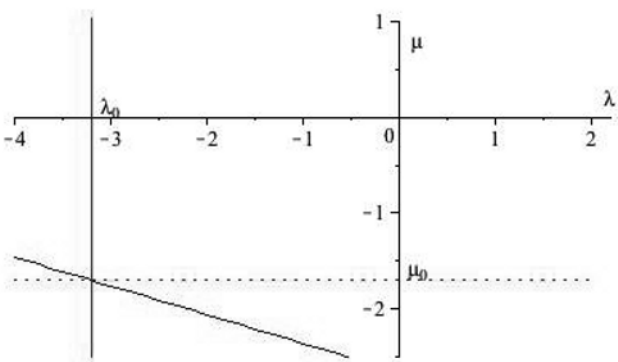
$$\langle T_{k_0}|g|^{2\mu},\phi\rangle = \langle T_{k_0},\psi\rangle \neq 0.$$

Therefore, the extension of (2.1) has a pole of order $k_0$ along the line $\{\lambda=\lambda_0\}$. □

**Corollary 2.3.** *For* $(\lambda_0,\mu_0)\in(\mathbb{R}_-)^2$, *assume*

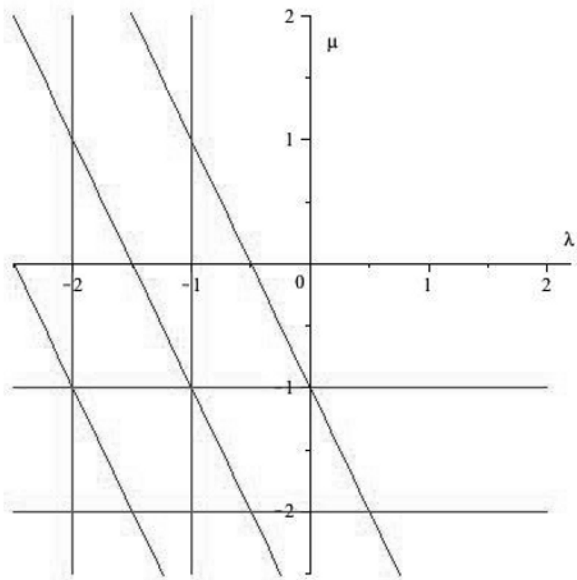
(i) *the extension of the current valued function* (2.2) *has a pole of order* $k$ *at* $\lambda_0$,
(ii) $\mu_0$ *is not an integer translate of a root of the Bernstein polynomial of* $g$,
(iii) $\lambda \mapsto \mathrm{Pf}(\mu=\mu_0, \int_X |f|^{2\lambda}|g|^{2\mu}\square)$ *has a pole of order* $l_0 > k_0$ *where* $k_0$ *is defined in* (2.3) *at* $\lambda_0$.

*Then the meromorphic extension of the current valued function* (2.1) *has at least* $l_0-k_0$ *oblique lines, counted with multiplicities, through* $(\lambda_0,\mu_0)$.

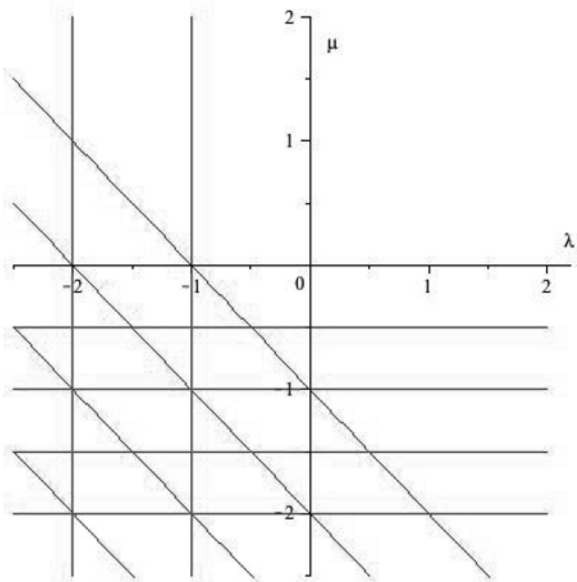



*Proof.* Use Proposition 2.2 and a version of Lemma 2.1 with multiplicities. □

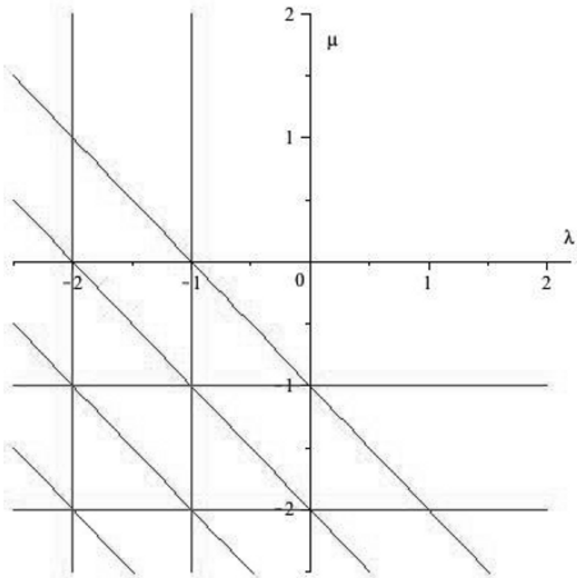
**Example 2.4.** $m = 3\,,\ f(x,y,z) = x^2 + y^2 + z^2\,,\ g(x,y,z) = z\,.$




**Example 2.5.** $m = 4\,,\ f(x,y,z) = x^2 + y^2 + z^2 + t^2\,,\ g(x,y,z,t) = t^2\,.$



**Example 2.6.** $m = 3\,,\ f(x,y,z) = x^2 + y^2\,,\ g(x,y,z) = y^2 + z^2\,.$




In this last example, Corollary 2.3 does not apply because for $\lambda_0 = -1$ we have $k_0 = l_0$. Existence of an oblique polar line through $(-1, 0)$ is obtained by computation of the extension of $\lambda \mapsto \mathrm{Pf}\left(\mu = 1/2, \int_X |f|^{2\lambda}|g|^{2\mu}\square\right)$.

## 3. Pullback and interaction

In this section, we give by pullback a method to verify condition (iii) of Corollary 2.3 when $g$ is a coordinate. As a matter of fact the function $\lambda \mapsto \int_X |f|^{2\lambda}|g|^{2\mu_0}\square$ is only known by meromorphic extension (via Bernstein identity) when $\mu_0$ is negative; it is in general difficult to exhibit some of its poles.

In $\mathbb{C}^{n+1}$, denote the coordinates by $x_1, \dots, x_n, t$ and take $g(x,t) = t$. We consider therefore only one holomorphic function $f : Y \to \mathbb{C}$, where $Y$ is an open subset in $\mathbb{C}^{n+1}$ and fix a relatively compact open subset $X$ of $Y$. Let us introduce also the finite map

$$p : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1} \text{ such that } p(x_1, \dots, x_n, \tau) = (x_1, \dots, x_n, \tau^k) \qquad (3.1)$$

for some fixed integer $k$. Finally, put $\tilde{f} := f \circ p : \tilde{X} \to \mathbb{C}$ where $\tilde{X} := p^{-1}(X)$.

**Proposition 3.1.** *With the above notations and $\lambda_0 \in \mathbb{R}_-$ suppose*

(a) *the extension of the current valued function* (2.2) *has a pole of order $\leqslant 1$ at $\lambda_0$,*

(b) $\lambda \mapsto \int_{\tilde{X}} |\tilde{f}|^{2\lambda}\square$ *has a double pole at $\lambda_0$.*

*Then there exists $l \in [1, k-1]$ such that the extension of the current valued function $\lambda \mapsto \int_X |f|^{2\lambda}|t|^{-2l/k}\square$ has a double pole at $\lambda_0$.*