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Entire Slice Regular Functions

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ISSN 2191-8198 ISSN 2191-8201 (electronic)
SpringerBriefs in Mathematics
ISBN 978-3-319-49264-3 ISBN 978-3-319-49265-0 (eBook)
DOI 10.1007/978-3-319-49265-0

Library of Congress Control Number: 2016957500

Mathematics Subject Classification (2010): 30G35, 30B10, 30D15

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

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Chapter 1
Introduction

The theory of holomorphic functions in one complex variable assumes a particular flavor when one considers functions that are holomorphic on the entire plane, namely entire functions. The reasons for the richness of results that are possible in that context are many, but certainly include the fact that for entire functions we can study growth phenomena in a cleaner way than what would be possible if one were to consider the issues introduced by the boundary of the domain of holomorphy associated with an individual function. While the beautiful issues connected with analytic continuation do not arise in the context of entire functions, the theory acquires much strength from the ability to connect the growth of the functions to the coefficients that appear in their Taylor series.

The study of entire functions, in addition, has great relevance for the study of convolution equations, where different spaces of entire functions arise naturally from the application of the Paley Wiener theorem (in its various forms), and from the topological vector space approach that was so instrumental in the work of Schwarz, Malgrange, Ehrenpreis, Palamodov, and Hörmander.

Further arguments in support of the study of entire functions are supplied by the consideration that not only the most important elementary functions are entire (polynomials, exponentials, trigonometric and hyperbolic functions), but also that many of the great special functions of analysis (the Jacobi theta function, the Weierstrass sigma function, the Weierstrass doubly periodic functions, among others) are entire. Finally, one knows that the solutions of linear differential equations with polynomial coefficients and with constant highest derivative coefficient are entire as well (this includes the elementary functions, as well as the Airy functions, for example).

While many texts exist, which are devoted exclusively to the study of entire functions (see for example [34, 126, 134]), one of the best ways to understand the importance and the beauty of entire functions is the article by de Branges [82].

One might therefore ask whether an analogous analysis can be done for functions defined on the space $\mathbb{H}$ of quaternions. As it is well known, there are several different definitions of holomorphicity (or regularity, as it is often referred to) for functions defined on quaternions. The most well-known definition is probably the one due to Fueter, who expanded on the work of Moisil, who defined regular functions as those that satisfy a first-order system of linear differential equations that generalizes the

F. Colombo et al., *Entire Slice Regular Functions*,
SpringerBriefs in Mathematics, DOI 10.1007/978-3-319-49265-0_1

one of Cauchy–Riemann. These functions are often referred to as Fueter regular, and their theory is well developed, see e.g. [69, 127]. It is therefore possible to consider those functions which are Fueter regular on the entire quaternionic space $\mathbb{H}$, and see whether there are important properties that can be deduced. However, the mathematician interested in this generalization would immediately encounter the problem that these functions do not admit a natural power series expansion. While this comment would need to be better qualified, it is the basic reason why most of what is known for holomorphic entire functions cannot be extended to the case of Fueter regular entire functions.

In the past ten years, however, many mathematicians have devoted significant attention to a different notion of regularity, known as slice regularity or slice hyperholomorphy. This theory began as a theory of functions from the space of quaternions $\mathbb{H}$ to itself whose restriction to any complex plane contained in $\mathbb{H}$ was in the kernel of the corresponding Cauchy–Riemann operator, see [110, 111]. It immediately appeared that this class of functions, at least the ones defined on balls centered at the origin, coincides with the class of polynomials and convergent power series of the quaternionic variable, previously introduced in [83]. In particular, there are very natural generalizations of the exponential and trigonometric functions that happen to be entire slice regular functions. Further studies showed that on suitable open sets called axially symmetric slice domains this class of functions coincides with the class of functions of the form $f(q) = f(x + Iy) = \alpha(x, y) + I\beta(x, y)$ when the quaternion q is written in the form $x + Iy$ (I being a suitable quaternion such that $I^2 = -1$) and the pair (α, β) satisfies the Cauchy–Riemann system and the conditions $\alpha(x, -y) = \alpha(x, y)$, $\beta(x, -y) = -\beta(x, y)$. This class of functions when α and β are real, quaternionic or, more in general, Clifford algebra valued is well known: they are the so-called holomorphic functions of a paravector variable, see [127, 150], which were later studied in the setting of real alternative algebras in [117].

But there are deeper reasons why these functions are a relevant subject of study. Maybe the most important point is to notice that slice regular functions and their Clifford valued companions, the slice monogenic functions (see [72, 73]), have surprising applications in operator theory. In particular, one can use these functions (collectively referred to as slice hyperholomorphic functions) to define a completely new and very powerful slice hyperholomorphic functional calculus (which extends the Riesz–Dunford functional calculus to quaternionic operators and to n-tuples of noncommuting operators). These possible applications in operator theory have given great impulse to the theory of functions.

In particular, the Cauchy formulas with slice hyperholomorphic kernels are the basic tools needed to extend the Riesz–Dunford functional calculus to quaternionic operators and to n-tuples of not necessarily commuting operators. They also lead to the notion of S-spectrum which turned out to be the correct notion of spectrum for all applications in different areas of quaternionic functional analysis and in particular to the quaternionic spectral theorem.

The function theory of slice regular and slice monogenic functions was developed in a number of papers, see the list of references (note that some of the references are not explicitly quoted in the text) and the comments below. It has also been extended to vector-valued and more in general operator-valued functions, see [23] and the references therein. The monographs [23, 76, 109] are the main sources for the study of slice hyperholomorphic functions and their applications.

It should be pointed out that the theory of slice hyperholomorphic functions is different from the more classical theory of functions in the kernel of the Dirac operator [35, 69, 84], the so-called monogenic functions. While the latter is a refinement of harmonic analysis in several variables, the former has many applications among which one of the most important is in quaternionic quantum mechanics, see [3, 32, 89, 91, 128]. Slice hyperholomorphic functions and monogenic functions can be related using the Fueter theorem, see [51, 56, 64, 86] and the Radon transform, see [149] and the more recent [36].

To give a flavor of the several results available in the literature in the context of slice regular functions, with no claim of completeness, we gather the references in the various areas of research.

Function theory. The theory of slice regular functions was developed in [47, 105, 107, 108, 110, 111, 113, 114], in particular, their zeros were treated in [104, 106, 112] while further properties can be found in [33, 44, 85, 93, 94, 103, 141, 142, 151, 152, 153, 155]. Slice monogenic functions with values in a Clifford algebra and their main properties were studied in [58, 72, 73, 74, 75, 78, 81, 123, 156]. Approximation of slice hyperholomorphic functions is collected in [77, 95, 96, 97, 98, 99, 100, 101, 146]. The case of several variables was treated in [2, 79], but a lot of work has still to be done in this direction since the theory is at the beginning. The generalization of slice regularity to real alternative algebras was developed in [26, 117, 118, 119, 120] and, finally, some results related to the global operator associated with slice hyperholomorphicity are in [53, 70, 80, 121].

Function spaces. Several function spaces have been studied in this framework. In particular, the quaternionic Hardy spaces $H^2(\Omega)$, where Ω is the quaternionic unit ball $\mathbb{B}$ or the half space $\mathbb{H}^+$ of quaternions with positive real part, together with the Blaschke products are in [14, 15, 16] and further properties are in [28, 29, 30]. The Hardy spaces $H^p(\mathbb{B})$, $p > 2$, are considered in [145]. The Bergman spaces can be found in [50, 52, 54] and the Fock space in [24]. Weighted Bergman spaces, Bloch, Besov, and Dirichlet spaces on the unit ball $\mathbb{B}$ are considered in [38]. Inner product spaces and Krein spaces in the quaternionic setting are studied in [21].

Groups and semigroups of operators. The theory of groups and semigroups of quaternionic operators has been developed and studied in [22, 62, 125].

Functional calculi. There exists at least five functional calculi associated with slice hyperholomorphicity. For each one we have a quaternionic version and a version for n-tuples of operators. The S-functional calculus, see [9, 45, 48, 49, 59, 60, 61, 71], is the analog of the Riesz–Dunford functional calculus in the quaternionic setting. Further developments are in [40, 42]. The SC-functional calculus, see [65], is the commutative version of the S-functional calculus. For the functional calculus for groups of quaternionic operators based on the Laplace–Stieltjes transform, see [8]. The H^∞ functional calculus based on the S-spectrum, see [25], is the analog, in this setting, of the calculus introduced by A. McIntosh, see [138]. The F-functional calculus, see [18, 41, 63, 67, 68], which is based on the Fueter mapping theorem in integral form, is a monogenic functional calculus in the spirit of the one developed in [129, 130, 131, 135, 139], but it is associated with slice hyperholomorphicity. Finally, the W-functional calculus, see [57], is a monogenic plane wave calculus based on slice hyperholomorphic functions.

Spectral theory. The spectral theorem based on the S-spectrum for bounded and for unbounded quaternionic normal operators on a Hilbert space was developed in [10, 11, 116]. The case of quaternionic normal matrices was proved in [90] and it is based on the right spectrum. Note that the right spectrum is equal to the S-spectrum in the finite dimensional case. The continuous slice functional calculus in quaternionic Hilbert spaces is studied in [115].

Schur Analysis. This is a very wide field that has been developed in the last five years in the slice hyperholomorphic setting. Schur analysis originates with the works of Schur, Herglotz, and others and can be seen as a collection of topics pertaining to Schur functions and their generalizations; for a quick introduction in the classical case see for example [5]. For the slice hyperholomorphic case see [1, 2, 6, 7, 12, 13, 14, 15, 16, 17, 19, 20] and also the forthcoming monograph [23].

Since the literature in the field is so vast, it is natural to ask whether the deep theory of holomorphic entire functions can be reconstructed for slice regular functions. As this book will demonstrate, the answer is positive, and our contribution here consists in showing the way to a complete theory of entire slice regular functions. It is probably safe to assert that this monograph is only the first step, and in fact we only have chosen a fairly limited subset of the general theory of holomorphic entire functions, to demonstrate the feasibility of our project. We expect to return to other important topics in a subsequent volume.

This monograph contains four chapters, besides this introduction. In Chapters 2 and 3 we introduce and discuss the algebra and the analysis of slice regular functions. While most of the results in those chapters are well known, and can be found in the literature, see e.g. [23, 76, 109], we repeated them to make the monograph self-contained. However, there are a few new observations (e.g., in Section 2.5 and 3.2 where we tackle the composition of slice regular functions and also the Riemann mapping theorem) that do not appear in the aforementioned monographs. There are also a few new results (for example we complete the discussion on lower bounds for slice regular functions initiated with the Ehrenpreis–Malgrange lemma in Section 3.4, by adding a brand-new Cartan-type theorem in Section 3.5).

Chapter 4 deals with infinite products of slice regular functions. The results in this chapter are known, but at least the Weierstrass theorem receives here a treatment that is different from the one originally given in [114], see also [109]. This treatment leads also to the definition of genus of a canonical product. The core of the work, however, is Chapter 5, where we study the growth of entire slice regular functions, and we show how such growth is related to the coefficients of the power series expansions that these functions have. This chapter contains new results, the only exception is Section 5.5 which is taken from [100]. It should be noted that the proofs we offer are not simple reconstructions (or translations) of the holomorphic case. Indeed, the noncommutative setting creates a series of nontrivial problems that force us to define composition and multiplication in ways that are not conducive to a simple repetition of the complex case. Also, the counting of the zeros is not trivial because of the presence of spherical zeros which have infinite cardinality.

We believe that much work still needs to be done in this direction, and we hope that our monograph will inspire others to turn their attention to this nascent, and already so rich, new field of noncommutative analysis.

Acknowledgements The first two authors are grateful to Chapman University for the kind hospitality and the support during the periods in which this work has been written. The second author thanks the project FIRB 2012 "Geometria Differenziale e Teoria Geometrica delle Funzioni" for partially supporting this research.

Chapter 2
Slice Regular Functions: Algebra

2.1 Definition and Main Results

In this chapter, we will present some basic material on slice regular functions, a generalization of holomorphic functions to the quaternions.

The skew field of quaternions $\mathbb{H}$ is defined as

$$\mathbb{H} = \{q = x_0 + ix_1 + jx_2 + kx_3 \ : \ x_0, \dots, x_3 \in \mathbb{R}\},$$

where the imaginary units i, j, k satisfy

$$i^2 = j^2 = k^2 = -1, \ ij = -ji = k, \ jk = -kj = i, \ ki = -ik = j.$$

It is a noncommutative field and since $\mathbb{C}$ can be identified (in a nonunique way) with a subfield of $\mathbb{H}$, it extends the class of complex numbers. On $\mathbb{H}$, we define the Euclidean norm

$$|q| = \sqrt{x_0^2 + x_1^2 + x_2^3 + x_3^2}.$$

The symbol $\mathbb{S}$ denotes the unit sphere of purely imaginary quaternion, i.e.,

$$\mathbb{S} = \{q = ix_1 + jx_2 + kx_3, \text{ such that } x_1^2 + x_2^2 + x_3^3 = 1\}.$$

Note that if $I \in \mathbb{S}$, then $I^2 = -1$. For this reason, the elements of $\mathbb{S}$ are also called imaginary units. For any fixed $I \in \mathbb{S}$ we define

$$\mathbb{C}_I := \{x + Iy \ : \ x, y \in \mathbb{R}\}.$$

It is easy to verify that $\mathbb{C}_I$ can be identified with a complex plane, moreover $\mathbb{H} = \bigcup_{I \in \mathbb{S}} \mathbb{C}_I$. The real axis belongs to $\mathbb{C}_I$ for every $I \in \mathbb{S}$ and thus a real quaternion can be associated with any imaginary unit I. Any nonreal quaternion $q = x_0 + ix_1 + jx_2 + kx_3$ is uniquely associated to the element $I_q \in \mathbb{S}$ defined by

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$$I_q := \frac{ix_1 + jx_2 + kx_3}{|ix_1 + jx_2 + kx_3|}.$$

It is obvious that q belongs to the complex plane $\mathbb{C}_{I_q}$.

Definition 2.1 Let U be an open set in $\mathbb{H}$ and let $f : U \to \mathbb{H}$ be real differentiable. The function f is said to be (left) slice regular or (left) slice hyperholomorphic if for every $I \in \mathbb{S}$, its restriction f_I to the complex plane $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$ passing through origin and containing I and 1 satisfies

$$\overline{\partial}_I f(x + Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + Iy) = 0,$$

on $U \cap \mathbb{C}_I$. The class of (left) slice regular functions on U will be denoted by $\mathscr{R}(U)$.

Analogously, a function is said to be right slice regular in U if

$$(f_I \overline{\partial}_I)(x + Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x + Iy) + \frac{\partial}{\partial y} f_I(x + Iy)I\right) = 0,$$

on $U \cap \mathbb{C}_I$.

It is immediate to verify that:

Proposition 2.1 *Let U be an open set in $\mathbb{H}$. Then $\mathscr{R}(U)$ is a right linear space on $\mathbb{H}$.*

Let $f \in \mathscr{R}(U)$. The so-called left I-derivative of f at a point $q = x + Iy$ is defined by

$$\partial_I f_I(x + Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x + Iy) - I\frac{\partial}{\partial y} f_I(x + Iy)\right),$$

and the right I-derivative of f at $q = x + Iy$ is defined by

$$\partial_I f_I(x + Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x + Iy) - \frac{\partial}{\partial y} f_I(x + Iy)I\right).$$

Let us now introduce another suitable notion of derivative:

Definition 2.2 Let U be an open set in $\mathbb{H}$, and let $f : U \to \mathbb{H}$ be a slice regular function. The slice derivative $\partial_s f$ of f, is defined by:

$$\partial_s(f)(q) = \begin{cases} \partial_I(f)(q) & \text{if } q = x + Iy,\ y \neq 0, \\ \dfrac{\partial f}{\partial x}(x) & \text{if } q = x \in \mathbb{R}. \end{cases}$$

The definition of slice derivative is well posed because it is applied only to slice regular functions, thus

$$\frac{\partial}{\partial x} f(x + Iy) = -I \frac{\partial}{\partial y} f(x + Iy), \qquad \forall I \in \mathbb{S}.$$

Similarly to what happens in the complex case, we have

$$\partial_s(f)(x + Iy) = \partial_I(f)(x + Iy) = \partial_x(f)(x + Iy).$$

We will often write $f'(q)$ instead of $\partial_s f(q)$.
It is important to note that if $f(q)$ is a slice regular function then also $f'(q)$ is a slice regular function.
Let $I, J \in \mathbb{S}$ be such that I and J are orthogonal, so that $I, J, IJ = K$ is a basis of $\mathbb{H}$ and write the restriction $f_I(x + Iy) = f(x + Iy)$ of f to the complex plane $\mathbb{C}_I$ as $f = f_0 + If_1 + Jf_2 + Kf_3$, where $f_0, \ldots, f_3$ are $\mathbb{R}$-valued. In alternative, it can also be written as $f = F + GJ$ where $f_0 + If_1 = F$, and $f_2 + If_3 = G$ are $\mathbb{C}_I$-valued. This observation immediately gives the following result:

Lemma 2.1 (Splitting Lemma) *If f is a slice regular function on U, then for every $I \in \mathbb{S}$, and every $J \in \mathbb{S}$, perpendicular to I, there are two holomorphic functions $F, G : U \cap \mathbb{C}_I \to \mathbb{C}_I$ such that for any $z = x + Iy$*

$$f_I(z) = F(z) + G(z)J.$$

Proof Since f is slice regular, we know that

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + Iy) = 0.$$

Therefore by decomposing the values of f_I into its complex components

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + Iy) = \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) F(x + Iy) + \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) G(x + Iy)J,$$

the statement immediately follows. □

We now consider slice regular functions on open balls $B(0; r)$ centered at the origin with radius $r > 0$. We have: