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Analysis in Banach Spaces

Volume I: Martingales and Littlewood-Paley Theory

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Ergebnisse der Mathematik und ihrer Grenzgebiete, now in its third sequence, aims to provide summary reports, on a high level, on important topics of mathematical research. Each book is designed as a reliable reference covering a significant area of advanced mathematics, spelling out related open questions, and incorporating a comprehensive, up-to-date bibliography.

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The four authors during a writing session in Oberwolfach in November 2013.
Left to right: Mark Veraar, Lutz Weis, Tuomas Hytönen, Jan van Neerven

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Volume I: Martingales and Littlewood-Paley Theory

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Preface

Over the past fifteen years, motivated by regularity problems in evolution equations, there has been tremendous progress in the analysis of Banach space-valued functions and processes. For so-called UMD spaces in particular, central areas of harmonic analysis, such as the theory of Fourier multipliers and singular integrals, were extended to operator-valued kernels acting on Bochner spaces, and basic estimates of stochastic analysis, including the Itô isometry and the Burkholder–Davis–Gundy inequalities, were generalised to Banach space-valued processes.

As it was long known that extensions of such sophisticated scalar-valued estimates are not possible for all Banach spaces, these results depended on essential progress in the geometry of Banach spaces during the 70s and 80s. The theory of Burkholder and Bourgain on UMD spaces became the foundation on which the recent theory we wish to report on was built; just as important are results of Kwapień, Maurey, and Pisier on type and cotype, since they link the structure of the Banach space to estimates for random sums which replace to some extent the fundamental orthogonality relations in Hilbert spaces.

For most classical Banach spaces, the UMD, type and cotype properties are readily available and therefore the results of vector-valued Analysis can be applied to many situations of interest in the theory of partial differential equations; they have already proved their value by providing sharp regularity estimates for parabolic problems. Our aim is to give a detailed and careful presentation of these topics that is useful not only as a reference book but can be used also selectively as a basis for advanced courses and seminars.

This project ranges over a broad spectrum of Analysis and includes Banach space theory, operator theory, harmonic analysis and stochastic analysis. For this reason we have divided it into three parts. The present volume develops the theory of Bochner integration, Banach space-valued martingales and UMD spaces, and culminates in a treatment of the Hilbert transform, Littlewood–Paley theory and the vector-valued Mihlin multiplier theorem.

Volume II will present a thorough study of the basic randomisation techniques and the operator-theoretic aspects of the theory, such as R-

boundedness, vector-valued square functions and radonifying operators, as well as a detailed treatment of the relevant probabilistic Banach space notions such as type, cotype, K-convexity and properties related to contraction principles. These techniques will allow us to present the theory of H^∞-functional calculus for sectorial operators and work out the main examples. This sets the stage for our final aim, a presentation of the theory of singular integral operators with operator-valued kernels and its applications to maximal regularity for deterministic and stochastic parabolic evolution equations, which will be the subject matter of Volume III.

The central theme in all volumes is the identification of the Banach spaces to which the key estimates of classical harmonic and stochastic analysis can be extended as those with the fundamental UMD property. The very definition behind this abbreviation is the unconditionality of martingale differences, a primarily probabilistic notion, and a number of different characterisations are formulated in purely probabilistic terms. However, this same property is also equivalent to the boundedness of the vector-valued Hilbert transform, the Littlewood–Paley inequality for vector-valued Fourier integrals, and several other estimates in the realm of classical harmonic analysis.

Each of these aspects of UMD spaces makes a substantial body of theory in its own right, and one could certainly produce respectable treatments of large parts of this material with a "clean" probabilistic or analytic flavour. However, rather than striving for such "purity", our aim is to emphasise the rich connections between the two worlds and the unity of the subject. For example, while martingales are traditionally regarded as a topic in Probability, we define and discuss them on σ-finite measure spaces from the beginning, so that they are immediately applicable to Analysis on the Euclidean space $\mathbb{R}^d$ without the need of auxiliary truncations or decompositions into probability spaces. Moreover, it is important to observe that even if we (or the reader) wanted to concentrate on the analytic side of UMD spaces only, we could hardly present a complete picture without an occasional reference to the probabilistic notions, at least at the present state of knowledge. For instance, although we know that both the Hilbert transform boundedness and the Littlewood–Paley inequality are equivalent to the UMD property, and therefore to each other, the only known way of proving the equivalence of these two analytic notions passes through the probabilistic UMD. There are numerous other such examples, and new frontiers of the theory have shown over and over again that it is the probabilistic definition of UMD spaces that lies at the centre and connects everything together.

So much said about the unity of Analysis and Probability (in Banach spaces), we should acknowledge the existence of a third side of the triangle, which is barely touched by the present treatise, namely: Geometry (of Banach spaces). Our choice of topics is not meant in any way to downplay the importance of this huge topic, both in its own right and in relation to analytic and probabilistic questions, but rather to admit our limits and to leave the proper account of the geometric connections for other treatments.

*

This book can be studied in a variety of ways and for different motivations. The principal, but not the only, audience that we have in mind consists of researchers who need and use Analysis in Banach spaces as a tool for studying other problems, in particular the regularity of evolution equations mentioned above. Until now, the contents of this extensive and powerful toolbox have been mostly scattered around in research papers, or in some cases monographs addressed to readerships with a rather different background from our focus, and we feel that collecting this diverse body of material into a unified and accessible presentation fills a gap in the existing literature. Indeed, we regard ourselves as part of this audience, and we have written the kind of book that we would have liked to have for ourselves when working through this theory for the first time.

Aside from this, parts of the book may also offer an interesting angle to the classical analysis of scalar-valued functions, which is certainly covered as a special case, and seldom required as a prerequisite or used as a building block for the Banach space-valued theory. For a classical harmonic analyst, the approach that we take, say, to the L^p-boundedness of the Hilbert transform, is possibly exotic, but not necessarily substantially more difficult than more traditional treatments in the scalar-valued case.

*

There are a couple of technical features of the book worth mentioning. Most of the time, we are quite explicit with the constants appearing in our estimates, and we especially try to keep track of the dependence on the main parameters involved. Thus, rather than saying that a particular bound "only depends on the UMD constant $\beta_{p,X}$", we prefer to write out, say, $(\beta_{p,X})^2$, or whichever function of $\beta_{p,X}$ appears from the calculation. We often go to the extent of writing, say, "2000" instead of "c, where c is a numerical constant", although we also might write "2000" instead of "1764", when there is no reason to believe that the latter constant, although given by a particular computation, would be anywhere close to optimal. Indeed, except for a few select places, we make no claim that our explicit constants cannot be improved; however, in many places, we have made an effort to present the best (order of) bounds currently available by the existing methods. We hope that making this explicit documentation might spur some interest towards research on such quantitative issues.

We also pay more attention than many texts to the impact of the underlying scalar field (real or complex) on the results under consideration. While this is largely irrelevant for many questions, it does play a role in some others, and we try to be quite explicit in pointing out the differences when they do occur, hopefully without insisting too much on this point when they do not.

*

This project was initiated in Delft and Karlsruhe already in 2008. Critical to its eventual progress was the possibility of intensive joint working periods in the serenity provided by the Banach Center in Będlewo (2012), Mathematisches Forschungsinstitut Oberwolfach (2013), Stiftsgut Keysermühle in Klingenmünster (2014 and 2015) and Hotel 't Paviljoen in Rhenen (2015). All four of us also met twice in Helsinki (2014 and 2016), and a number of additional working sessions were held by subgroups of the author team. One of us (J.v.N.) wishes to thank Marta Sanz-Solé for her hospitality during a sabbatical leave at the University of Barcelona in 2013.

Preliminary versions of parts of the material were presented in advanced courses and lecture series at various international venues and in seminars at our departments, and we would like to thank the students and colleagues who attended these events for feedback that shaped and improved the final form of the text. Special thanks go to Jamil Abreu, Alex Amenta, Markus Antoni, Sonja Cox, Chiara Gallarati, Fabian Hornung, Luca Hornung, Marcel de Jeu, Marcel Kreuter, Nick Lindemulder, Emiel Lorist, Bas Nieraeth, Jan Rozendaal, Jonas Teuwen, and Ivan Yaroslavtsev who did detailed reading of portions of this book. Needless to say, we take full responsibility for any remaining errors. A list with errata will be maintained on our personal websites. We wish to thank Klaas Pieter Hart for LaTeX support.

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Delft, Helsinki and Karlsruhe,
November 14, 2016.

Contents

Symbols and notations

Sets

$\mathbb{N} = \{0, 1, 2, \ldots\}$ - non-negative integers
$\mathbb{Z}$ - integers
$\mathbb{Q}$ - rational numbers
$\mathbb{R}$ - real numbers
$\mathbb{C}$ - complex numbers
$\mathbb{K}$ - scalar field ($\mathbb{R}$ or $\mathbb{C}$)
$\mathbb{S} = \{z \in \mathbb{C} : 0 < \Im z < 1\}$ - unit strip
$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ - unit circle
$\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, \infty\}$ - extended integers
$\mathbb{R}_+ = (0, \infty)$ - positive real line
B_X - open unit ball
S_X - unit sphere
$B(x, r)$ - open ball centred at x with radius r

Vector spaces

BMO - space of functions of bounded mean oscillation
c_0 - space of null sequences
C - space of continuous functions
C_0 - space of continuous functions vanishing at infinity
C_{b} - space of bounded continuous functions
C_{c} - space of continuous functions with compact support
C_{c}^∞ - space of test functions with compact support
$\mathscr{C}^p$ - Schatten class
H - Hilbert space
$H^{s,p}$ - Bessel potential space
H^p - Hardy space
$\mathscr{H}(X_0, X_1)$ - space of homomorphic functions on the strip
ℓ^p - space of p-summable sequences

ℓ^p_N - space of p-summable finite sequences
L^p - Lebesgue space
$L^{p,\infty}$ - weak-L^p
$\mathscr{L}(X,Y)$ - space of bounded linear operators
$\mathfrak{M}L^p(\mathbb{R}^d;X,Y)$ - space of Fourier multipliers
$\mathfrak{M}(\mathbb{R}^d;X,Y)$ - Mihlin class
$\mathscr{S}$ - space of Schwartz functions
$\mathscr{S}'$ - space of tempered distributions
$W^{k,p}$ - Sobolev space
X, Y, $\dots$ - Banach spaces
$X_{\mathbb{C}}$ - complexification
$X^{\gamma,p}_{\mathbb{C}}$ - Gaussian complexification
X^*, Y^*, $\dots$ - dual Banach spaces
$X \otimes Y$ - tensor product
$[X_0,X_1]_\theta$ - complex interpolation space
$(X_0,X_1)_{\theta,p}$, $(X_0,X_1)_{\theta,p_0,p_1}$ - real interpolation spaces

Measure theory and probability

$\mathscr{A}$ - σ-algebra
$df_n = f_n - f_{n-1}$ - nth martingale difference
ϵ_n - signs in $\mathbb{K}$, i.e., scalars in $\mathbb{K}$ of modulus one
ε_n - Rademacher variables with values in $\mathbb{K}$
$\mathbb{E}$ - expectation
$\mathscr{F}$, $\mathscr{G}$, $\dots$ - σ-algebras
$\mathscr{F}_f$ - collection of sets in $\mathscr{F}$ on which f is integrable
$\mathbb{E}(\cdot|\cdot)$ - conditional expectation
${}^\tau f_n = f_n - f_{\tau\wedge n}$ - started martingale
$f^\tau_n = f_{\tau\wedge n}$ - stopped martingale
$f^\star_n = \sup_{k\leqslant n}\|f_n\|$ - maximal function
γ - Gaussian variable
h_I - Haar function
μ - measure
$\|\mu\|$ - variation of a measure
$(\Omega,\mathscr{A},\mathbb{P})$ - probability space
$\mathbb{P}$ - probability measure
$\mathbb{P}(\cdot|\cdot)$ - conditional probability
r_n - real Rademacher variables
$(S,\mathscr{A},\mu)$ - measure space
$\sigma(f,g,\dots)$ - σ-algebra generated by the functions $f,g,\dots$
$\sigma(\mathscr{C})$ - σ-algebra generated by the collection $\mathscr{C}$
τ - stopping time
w_α - Walsh function

Norms and pairings

$|\cdot|$ - modulus, Euclidean norm
$\|\cdot\| = \|\cdot\|_X$ - norm in a Banach space X
$\|\cdot\|_p = \|\cdot\|_{L^p}$ - L^p-norm
$\langle\cdot,\cdot\rangle$ - duality
$(\cdot|\cdot)$ - inner product in a Hilbert space
$a \cdot b$ - inner product of $a, b \in \mathbb{R}^d$

Operators

D_j - pre-decomposition
$\varDelta$ - Laplace operator
$\mathscr{D}$ - dyadic system
$\partial_j = \partial/\partial x_j$ - partial derivative with respect to x_j
∂^α - partial derivative with multi-index α
$\mathbb{E}(\cdot|\cdot)$ - conditional expectation
$\mathscr{F} f$ - Fourier transform
$\mathscr{F}^{-1} f$ - inverse Fourier transform
H - Hilbert transform
$\widetilde{H}$ - periodic Hilbert transform
J_s - Bessel potential operator
$\mathscr{L}(X,Y)$ - space of bounded operators from X to Y
$\mathscr{L}_{\rm so}(X,Y)$ - idem, endowed with the strong operator topology
$\mathscr{R}_p$ - R-bound
R_j - jth Riesz transform
S, T, ... - bounded linear operators
T^* - adjoint of the operator T
T_m - Fourier multiplier operator associated with multiplier m
T_v - martingale transform associated with predictable sequence v
$T \otimes I_X$ - tensor extension of T

Constants and inequalities

$\beta_{p,X}$ - UMD constant
$\beta^{\mathbb{R}}_{p,X}$ - UMD constant with signs ± 1
$\beta^{\pm}_{p,X}$ - upper and lower randomised UMD constant
$c_{q,X}$ - cotype q constant
$c^{\rm mart}_{q,X}$ - martingale cotype q constant
$\hbar_{p,X}$ - norm of the Hilbert transform on $L^p(\mathbb{R};X)$
$K_{p,X}$ - K-convexity constant
$\kappa_{p,q}$ - Kahane–Khintchine constant
$\tau_{p,X}$ - type p constant
$\tau^{\rm mart}_{p,X}$ - martingale type p constant
$\varphi_{p,X}(\mathbb{R}^d)$ - norm of the Fourier transform $\mathscr{F} : L^p(\mathbb{R}^d;X) \to L^{p'}(\mathbb{R}^d;X)$.

Miscellaneous

$\hookrightarrow$ - continuous embedding
$\mathbf{1}_A$ - indicator function
$a \lesssim b$ - $\exists C$ such that $a \leqslant Cb$
$a \lesssim_{p,P} b$ - $\exists C$, depending on p and P, such that $a \leqslant Cb$
C - generic constant
$\complement$ - complement
$d(x,y)$ - distance
$f^\star$ - maximal function
$\widetilde{f}$ - reflected function
$\widehat{f}$ - Fourier transform
$\breve{f}$ - inverse Fourier transform
$f * g$ - convolution
$\Im$ - imaginary part
$K(t,x) = K(t,x;X_0,X_1)$ - K-functional
Mf - Hardy–Littlewood maximal function
$M_{\mathrm{Rad}}f$ - Rademacher maximal function
$p' = p/(p-1)$ - conjugate exponent
$p^* = \max\{p,p'\}$
$\Re$ - real part
$s \wedge t = \min\{s,t\}$
$s \vee t = \max\{s,t\}$
$v \star f$ - martingale transform of f by v
x - generic element of X
x^* - generic element of X^*
$x \otimes y$ - elementary tensor

Standing assumptions

Throughout this book, two of conventions will be in force.

1. Unless stated otherwise, the scalar field $\mathbb{K}$ can be real or complex. Results which do not explicitly specify the scalar field to be real or complex are true over both the real and complex scalars.
2. In the context of randomisation, a *Rademacher variable* is a uniformly distributed random variable taking values in the set $\{z \in \mathbb{K} : |z| = 1\}$. Such variables are denoted by the letter ε. Thus, whenever we work over $\mathbb{R}$ it is understood that ε is a real Rademacher variable, i.e.,

$$\mathbb{P}(\varepsilon = 1) = \mathbb{P}(\varepsilon = -1) = \frac{1}{2},$$

   and whenever we work over $\mathbb{C}$ it is understood that ε is a complex Rademacher variable (also called a *Steinhaus variable*), i.e.,

$$\mathbb{P}(a < \arg(\varepsilon) < b) = \frac{1}{2\pi}(b - a).$$

   Occasionally we need to use real Rademacher variables when working over the complex scalars. In those instances we will always denote these with the letter r. Similar conventions are in force with respect to Gaussian random variables: a *Gaussian random variable* is a standard normal real-valued variable when working over $\mathbb{R}$ and a standard normal complex-valued variable when working over $\mathbb{C}$.

1

Bochner spaces

In this first chapter we present the essentials of the integration theory of Banach space-valued functions. We begin by exploring the various possible definitions of measurability for such functions. It turns out that for separable Banach spaces X and measurable spaces $(S, \mathscr{A})$, a function $f : S \to X$ is measurable—in the sense that the pre-images

$$f^{-1}(B) := \{f \in B\} := \{s \in S : f(s) \in B\}$$

are measurable for every Borel set B in X—if and only if the scalar-valued function $\langle f, x^* \rangle$ is measurable for every functional x^* in the dual space X^*. This is essentially the content of the Pettis measurability theorem, which is proved in Section 1.1.

In Section 1.2 we proceed with the construction of the Bochner integral, which is the analogue of the Lebesgue integral for X-valued functions. It preserves all essential aspects of the Lebesgue integral, such as the availability of approximation results, convergence theorems and Fubini's theorem. The Banach spaces $L^p(S; X)$ of Bochner integrable functions provide the basic functional framework of our work. However, occasionally we shall also need the Pettis integral, which is defined in terms of the Lebesgue integrals of the functions $\langle f, x^* \rangle$.

For the most complete statement of the duality theory of the Bochner spaces $L^p(S; X)$ in Section 1.3 we have to restrict ourselves to the class of Banach spaces for which an analogue of the Radon–Nikodým theorem holds. This class is closely connected with the almost-everywhere differentiability of Lipschitz and absolutely continuous functions with values in X, a topic that will be discussed in the next chapter.

Throughout this book, X is a Banach space over the scalar field $\mathbb{K}$ which may be either $\mathbb{R}$ or $\mathbb{C}$. If we wish to emphasise a particular choice of scalar field we will speak of *real* and *complex* Banach spaces. It is often useful to note that a complex Banach space is also a real Banach space by restricting the complex scalar multiplication to the real numbers.

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The norm of an element $x \in X$ is denoted by $\|x\|_X$, or, if no confusion can arise, by $\|x\|$. The Banach space dual of X is denoted by X^*. We shall use the notation $\langle x, x^*\rangle := x^*(x)$ to denote the duality pairing of the elements $x \in X$ and $x^* \in X^*$.

1.1 Measurability

In the context of Analysis in Banach spaces, several natural notions of measurability present themselves, such as measurability, strong measurability and weak measurability. In finite dimensions all three coincide, but in infinite dimensions they do not and our first task is to understand the way they are interrelated. The main result in this direction, and indeed one of the cornerstones of the theory, is the Pettis measurability theorem. It asserts that a function with values in a Banach space X is strongly measurable if and only if it is separably valued and weakly measurable. We shall present two versions of this result: a pointwise version for functions defined on a measurable space $(S, \mathscr{A})$ and a μ-almost everywhere version for functions defined on a measure space $(S, \mathscr{A}, \mu)$. In general we shall make some effort to present the results for arbitrary measure spaces, avoiding assumptions such as σ-finiteness whenever this is possible.

1.1.a Functions on a measurable space $(S, \mathscr{A})$

Measurability

The first definition of measurability for Banach space-valued functions that comes to mind is that of inverse images: a function with values in a Banach space X is said to be *measurable* if the pre-image $f^{-1}(B)$ of every Borel set B in X is measurable. As it turns out, in many respects this natural notion is not as useful as one might think, the reason being that the Borel σ-algebra $\mathscr{B}(X)$ is in general 'too large'. In fact, the σ-algebra generated by all continuous linear functionals on X may be strictly smaller than $\mathscr{B}(X)$. This presents an obstruction to applying the standard tools of functional analysis such as the Hahn–Banach theorem in an effective way.

Our first objective is to prove that if X is a *separable* Banach space, this problem does not occur. Given a subset Y of the dual space X^* we denote by $\sigma(Y)$ the σ-algebra generated by Y, i.e., the smallest σ-algebra in X for which every $x^* \in Y$ measurable. It is easy to see that $\sigma(Y)$ is generated by the collection $\mathscr{C}(Y)$ of all sets of the form

$$\big\{x \in X : \ \big(\langle x, x_1^*\rangle, \ldots , \langle x, x_n^*\rangle\big) \in B\big\}$$

with $n \geqslant 1$, $x_1^*, \ldots, x_n^* \in Y$ and $B \in \mathscr{B}(\mathbb{K}^n)$.

Recall that a linear subspace Y of X^* is dense with respect to the weak* topology of X (which, by definition, is the smallest topology in X^* for which the mapping $x^* \mapsto \langle x, x^* \rangle$ continuous for every $x \in X$) if and only if Y separates the points of X. We refer the reader to Appendix B for some background material on the weak* topology.

Proposition 1.1.1. *If X is separable and Y is a weak* dense linear subspace of X^*, then*

$$\sigma(Y) = \sigma(X^*) = \mathscr{B}(X).$$

Proof. Let G denote the set of all $x^* \in X^*$ having the property that the function $x \mapsto \langle x, x^* \rangle$ is $\sigma(Y)$-measurable. Then G is a linear subspace of X^* containing Y. Moreover, G is weak* sequentially closed. Therefore, $G = X^*$ by a corollary to the Krein–Šmulian theorem (Corollary B.1.14), which means that $x \mapsto \langle x, x^* \rangle$ is $\sigma(Y)$-measurable for all $x^* \in X^*$.

Now let $n \geqslant 1$, $x_1^*, \dots, x_n^* \in X^*$ and $B_1, \dots, B_n \in \mathscr{B}(\mathbb{K})$ be given. Put $B := B_1 \times \dots \times B_n$. Then $B \in \mathscr{B}(\mathbb{K}^n)$ and the set

$$\big\{x \in X : \ \big(\langle x, x_1^* \rangle, \dots, \langle x, x_n^* \rangle\big) \in B\big\} = \bigcap_{k=1}^{n} \big\{x \in X : \ \langle x, x_k^* \rangle \in B_k\big\}$$

belongs to $\sigma(Y)$. Denote by Σ the collection of all $B \in \mathscr{B}(\mathbb{K}^n)$ having the property that

$$\big\{x \in X : \ \big(\langle x, x_1^* \rangle, \dots, \langle x, x_n^* \rangle\big) \in B\big\} \in \sigma(Y). \tag{1.1}$$

Then Σ is a σ-algebra in $\mathbb{K}^n$, and by the observation just made it contains all Borel rectangles $B_1 \times \dots, \times B_n$. Therefore $\mathscr{B}(\mathbb{K}^n) \subseteq \Sigma$.

We have shown that (1.1) holds for all finite sets $x_1^*, \dots, x_n^* \in X^*$ and all Borel sets $B \in \mathscr{B}(\mathbb{K}^n)$. It follows that $\sigma(X^*) \subseteq \sigma(Y)$. Since the reverse inclusion holds trivially, it follows that $\sigma(X^*) = \sigma(Y)$. It remains to be shown that $\sigma(X^*) = \mathscr{B}(X)$. Since every open set is the countable union of open balls and every open ball is a countable union of closed balls, it is enough to show that every closed ball $B(x_0, r) := \{x \in X : \ \|x - x_0\| \leqslant r\}$ belongs to $\sigma(X^*)$. Choose a norming sequence of unit vectors $(x_n^*)_{n \geqslant 1}$ in X^*. Then

$$B(x_0, r) = \big\{x \in X : \ \|x - x_0\| \leqslant r\big\} = \bigcap_{n \geqslant 1} \big\{x \in X : \ |\langle x - x_0, x_n^* \rangle| \leqslant r\big\},$$

and this set belongs to $\sigma(X^*)$. This completes the proof. □

When X is non-separable, strict inclusion $\sigma(X^*) \subsetneq \mathscr{B}(X)$ may indeed occur; this phenomenon will be discussed in the Notes at the end of the chapter.

Corollary 1.1.2. *If X is separable, then for a function $f : S \to X$ the following assertions are equivalent:*

(1) *f is measurable;*
(2) *$\langle f, x^* \rangle$ is measurable for all $x^* \in X^*$.*

Indeed, if (2) holds, then with the notations introduced in the above proof,

$$f^{-1}(B(x_0,r)) = \bigcap_{n\geqslant 1} \big\{s\in S:\ |\langle f(s)-x_0, x_n^*\rangle| \leqslant r\big\} \in \mathscr{A}.$$

Since the balls $B(x_0,r)$ generate $\mathscr{B}(X)$, this proves that f is measurable.

Strong measurability

The essential feature used in the construction of the Lebesgue integral for scalar-valued functions is that every measurable function can be approximated pointwise by a sequence of simple functions. Since, in the converse direction, pointwise limits of measurable functions are measurable, this suggests to tie up the notion of measurability with approximation by simple functions. This is precisely the idea taken up in the definition of *strong measurability.*

As before we let $(S,\mathscr{A})$ be a measurable space.

Definition 1.1.3. *A function $f: S\to X$ is called* simple *if it is of the form $f=\sum_{n=1}^N \mathbf{1}_{A_n}\otimes x_n$ with $A_n\in\mathscr{A}$ and $x_n\in X$ for all $1\leqslant n\leqslant N$.*

Here $\mathbf{1}_A$ denotes the indicator function of the set A and we use the notation

$$(f\otimes x)(s) := f(s)x$$

for functions $f: S\to\mathbb{K}$ and elements $x\in X$. We also define

$$F\otimes X := \Big\{\sum_{n=1}^N f_n\otimes x_n : f_n\in F,\ x_n\in X,\ n=1,\dots,N;\ N=1,2,\dots\Big\},$$

whenever F is a vector space of scalar-valued functions.

Definition 1.1.4. *A function $f: S\to X$ is* strongly measurable *if there exists a sequence of simple functions $f_n: S\to X$ such that $\lim_{n\to\infty} f_n = f$ pointwise on S.*

If we wish to emphasise the underlying σ-algebra we shall speak of a *strongly $\mathscr{A}$-measurable* function.

We shall see below that if X is separable, then an X-valued function f is strongly measurable if and only if it is measurable. The next example shows that the word 'separable' cannot be omitted from this statement.

Example 1.1.5. For any non-separable Banach space X, the identity map $I: X\to X$ is continuous and hence measurable, but it fails to be strongly measurable. Suppose, for a contradiction, that $I_n: X\to X$ is a sequence of simple Borel functions converging to I pointwise. Let V be the countable set of values taken by these functions. Then every $x\in X$ is the limit of a sequence in V, which implies that X is separable. By the same argument, I even fails to be strongly $\mathscr{P}(X)$-measurable (with $\mathscr{P}(X)$ the power set of X).

A function $f : S \to X$ is called *separably valued* if there exists a separable closed subspace $X_0 \subseteq X$ such that $f(s) \in X_0$ for all $s \in S$. The function f is called *weakly measurable* if

$$s \mapsto \langle f, x^* \rangle(s) := \langle f(s), x^* \rangle,$$

is measurable for all $x^* \in X^*$.

Theorem 1.1.6 (Pettis measurability theorem, first version). *Let $(S, \mathscr{A})$ be a measurable space and let Y be a weak* dense subspace of X^*. For a function $f : S \to X$ the following assertions are equivalent:*

(1) *f is strongly measurable;*
(2) *f is separably valued and weakly measurable;*
(3) *f is separably valued and $\langle f, x^* \rangle$ is measurable for all $x^* \in Y$.*

Moreover, if f takes its values in a closed linear subspace X_0 of X, then f is the pointwise limit of a sequence of X_0-valued simple functions.

Proof. (1)⇒(2): Let $(f_n)_{n\geqslant 1}$ be a sequence of simple functions converging to f pointwise and let X_1 be the closed subspace spanned by the countably many values taken by these functions. Then X_1 is separable and f takes its values in X_1. Furthermore, each $\langle f, x^* \rangle$ is measurable, being the pointwise limit of the measurable functions $\langle f_n, x^* \rangle$.

(2)⇒(3): This implication holds trivially.

(3)⇒(2): Let X_1 be a separable closed subspace of X with the property that $f(s) \in X_1$ for all $s \in S$, and let $j_1 : X_1 \subseteq X$ be the inclusion mapping. Then for all $x^* \in X^*$ we have $\langle f, x^* \rangle = \langle f, j_1^* x^* \rangle$, where on the left we regard f as an X-valued function and on the right as an X_1-valued function. Thus is suffices to prove that $\langle f, x_1^* \rangle$ is measurable for every $x_1^* \in X_1^*$.

Let Y_1 be the subspace in X_1^* consisting of all $x_1^* \in X_1^*$ for which $\langle f, x_1^* \rangle$ is measurable. Then $j_1^*(Y) \subseteq Y_1$. Since j_1^*, being an adjoint operator, is weak* continuous, $j_1^*(Y)$ is weak* dense in X_1^*. Therefore the same is true for Y_1. Also, Y_1 is weak* sequentially closed in X_1^*. Hence by a corollary to the Krein–Šmulian theorem (Corollary B.1.14), $Y_1 = X_1^*$.

(2)⇒(1): Choose a sequence $(x_n^*)_{n\geqslant 1}$ of unit vectors in X^* that is norming for a separable closed subspace X_1 of X where f takes its values. By the weak measurability of f, for each $x \in X_1$ the real-valued function

$$s \mapsto \|f(s) - x\| = \sup_{n\geqslant 1} |\langle f(s) - x, x_n^* \rangle|$$

is measurable. Let $(x_n)_{n\geqslant 1}$ be a dense sequence in X_1 with $x_1 = 0$.

Define the functions $\phi_n : X_1 \to \{x_1, \ldots, x_n\}$ as follows. For each $y \in X_1$ let $k(n, y)$ be the least integer $1 \leqslant k \leqslant n$ with the properties that $\|x_k\| \leqslant \|y\|$ and

$$\|y - x_k\| = \min_{1\leqslant j\leqslant n} \|y - x_j\|,$$

and put $\phi_n(y) := x_{k(n,y)}$. Since $(x_n)_{n\geqslant 1}$ is dense in X_1, we obtain

$$\lim_{n\to\infty} \|\phi_n(y) - y\| = 0 \quad \text{and} \quad \|\phi_n(y)\| \leqslant \|y\| \qquad \forall y \in X_1.$$

Now define $f_n : S \to X$ by

$$f_n(s) := \phi_n(f(s)), \qquad s \in S.$$

Then for all $x \in X_1$, $\|f_n(x)\| \leqslant \|f(x)\|$. Moreover, for all $1 \leqslant k \leqslant n$ we have

$$\{f_n = x_k\} = \Big\{\|f - x_k\| = \min_{1\leqslant j\leqslant n} \|f - x_j\| < \min_{1\leqslant j<k} \|f - x_j\|\Big\}.$$

The set on the right hand side is in $\mathscr{A}$. Hence each f_n is simple, takes values in X_1, and for all $s \in S$ we have

$$\lim_{n\to\infty} \|f_n(s) - f(s)\| = \lim_{n\to\infty} \|\phi_n(f(s)) - f(s)\| = 0.$$

The final assertion follows from the fact that if X_0 has the stated properties, then $X_1 \subseteq X_0$. □

We state a number of simple corollaries. The first corollary follows from the proof of the implication (2)⇒(1).

Corollary 1.1.7. *If $f : S \to X$ is strongly measurable, then there exists a sequence of simple functions $(f_n)_{n\geqslant 1}$ such that*

$$\|f_n(x)\| \leqslant \|f(x)\| \quad \textit{and} \quad f_n(x) \to f(x) \quad \textit{for all } x \in X.$$

Corollary 1.1.8. *If $f : S \to X$ takes values in a closed subspace X_0 of X, then f is strongly measurable as a function with values in X if and only if f is strongly measurable as a function with values in X_0.*

Corollary 1.1.9. *The pointwise limit $f : S \to X$ of a sequence of strongly measurable functions $f_n : S \to X$ is strongly measurable.*

Proof. Each function f_n takes its values in a separable subspace of X. Then f takes its values in the closed linear span of these subspaces, which is again separable. The measurability of the functions $\langle f, x^*\rangle$ follows by noting that each $\langle f, x^*\rangle$ is the pointwise limit of the measurable functions $\langle f_n, x^*\rangle$. □

The next corollary gives the precise connection between measurability and strong measurability.

Corollary 1.1.10. *For a function $f : S \to X$, the following assertions are equivalent:*

(1) *f is strongly measurable;*
(2) *f is separably valued and measurable.*

Proof. (1)⇒(2): If f is strongly measurable, then f is weakly measurable and separably valued, say with values in a separable closed subspace X_0 of X. By the Hahn–Banach theorem, f is weakly measurable as an X_0-valued function, and by Corollary 1.1.2, this implies that f is measurable as an X_0-valued function. If $B \in \mathscr{B}(X)$, then $B_0 := B \cap X_0 \in \mathscr{B}(X_0)$, and

$$\{f \in B\} = \{f \in B_0\} \in \mathscr{A},$$

so that f is also measurable as an X-valued function.

(2)⇒(1): The functions $\langle f, x^* \rangle$ are measurable for all $x^* \in X^*$. The result now follows from the Pettis measurability theorem. □

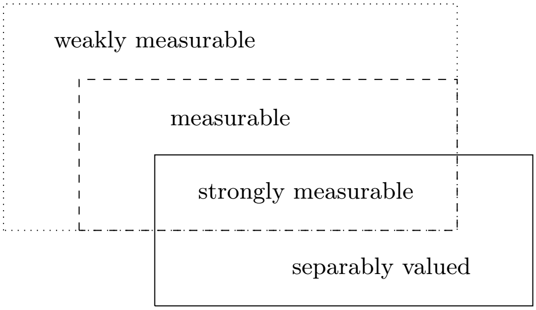

Fig. 1.1: The interrelations between different notions of measurability, as established in Theorem 1.1.6 and Corollary 1.1.10.

If $f : S \to X$ is strongly measurable and takes values in an open subset $O \subseteq X$, and $\phi : O \to Y$ is continuous, where Y is another Banach space, then $\phi \circ f$ is strongly measurable. In fact f is the pointwise limit of some simple f_n, and therefore also the pointwise limit of $\widetilde{f}_n := \mathbf{1}_{\{f_n \in O\}} f_n + \mathbf{1}_{\{f_n \in \complement O\}} x_0$, where x_0 is some fixed element of O. Then $\phi \circ \widetilde{f}_n$ is well defined, simple, and converges to $\phi \circ f$, which proves the claim.

More generally the following is true:

Corollary 1.1.11. *If $f : S \to X$ is strongly measurable and $\phi : X \to Y$ is measurable, where Y is another Banach space, then $\phi \circ f$ is strongly measurable.*

The proof of Corollary 1.1.11 uses the following topological fact.

Lemma 1.1.12. *Let E be a separable metric space and let F be a metric space. If $f : E \to F$ is measurable, then $f(E)$ is a separable subset of F.*

Proof. Suppose that $f(E)$ is non-separable. Then there exists an uncountable family of disjoint open sets $(O_i)_{i\in I}$ in F, each of which intersects $f(E)$. For every subset $I' \subseteq I$ we obtain an open set $O_{I'} := \bigcup_{i\in I'} O_i$ in F, hence a Borel set $f^{-1}(O_{I'})$ in E. If $I' \neq I''$, then also $O_{I'} \neq O_{I''}$, which shows that in E there are at least $2^{|I|}$ Borel sets. This is impossible since separable metric spaces have at most $2^{|\mathbb{N}|}$ Borel sets (see the Notes for a sketch of the proof). □

Proof of Corollary 1.1.11. It is clear that $\phi \circ f$ is measurable, and therefore by Corollary 1.1.10 it suffices to show that $\phi \circ f$ takes its values in a separable closed subspace of Y. The function f takes values in a separable closed subspace X_0 of X. Then $\phi \circ f$ takes its values in the subspace $\phi(X_0)$ of Y, which is separable by Lemma 1.1.12. □

1.1.b Functions on a measure space $(S, \mathscr{A}, \mu)$

Up to this point we have considered measurability properties of X-valued functions defined on a measurable space $(S, \mathscr{A})$. Next we consider functions defined on a measure space $(S, \mathscr{A}, \mu)$.

Definition 1.1.13. *A* μ-simple function *with values in* X *is a function of the form* $f = \sum_{n=1}^{N} \mathbf{1}_{A_n} \otimes x_n$*, where* $x_n \in X$ *and the sets* $A_n \in \mathscr{A}$ *satisfy* $\mu(A_n) < \infty$*.*

We say that a property holds *μ-almost everywhere* if there exists a μ-null set $N \in \mathscr{A}$ such that the property holds on the complement of N. Note that this definition makes no statement with regard to the validity of the property on N; the property may also hold on some subset of N, and this subset need not be in $\mathscr{A}$.

Definition 1.1.14. *A function* $f : S \to X$ *is* strongly μ-measurable *if there exists a sequence* $(f_n)_{n\geqslant 1}$ *of μ-simple functions converging to* f *μ-almost everywhere.*

When $X = \mathbb{K}$, we shall usually omit the prefix 'strongly'. Thus, a function $f : S \to \mathbb{K}$ is *μ-measurable* if it is the μ-almost everywhere limit of a sequence if μ-simple functions $f_n : S \to \mathbb{K}$.

A measure μ is said to be *σ-finite* if it admits an *exhausting sequence*, i.e., an increasing sequence $S^{(1)} \subseteq S^{(2)} \subseteq \dots$ of sets in $\mathscr{A}$ of finite μ-measure such that $\bigcup_{n\geqslant 1} S^{(n)} = S$. The next result shows that strongly μ-measurable functions are μ-essentially supported on σ-finite measure spaces:

Proposition 1.1.15. *Suppose that* $f : S \to X$ *is strongly μ-measurable. Then we have a disjoint decomposition* $S = S_0 \cup S_1$ *with* $S_0, S_1 \in \mathscr{A}$ *such that:*

(i) $f \equiv 0$ *μ-almost everywhere on* S_0*;*
(ii) μ *is σ-finite on* S_1*.*

Proof. Suppose $f_n \to f$ μ-almost everywhere, with each f_n a μ-simple function, say $f_n = \sum_{m=1}^{N_n} \mathbf{1}_{A_{mn}} \otimes x_{mn}$ with $\mu(A_{mn}) < \infty$. The sets $S_1 = \bigcup_{n\geqslant 1} \bigcup_{m=1}^{N_n} A_{mn}$ and $S_0 = \complement S_1$ have the desired properties. □

The next proposition relates the notions of 'strong measurability' and 'strong μ-measurability'.

Proposition 1.1.16. *Consider a function $f : S \to X$.*

(1) *If f is strongly μ-measurable, then f is μ-almost everywhere equal to a strongly measurable function.*
(2) *If μ is σ-finite and f is μ-almost everywhere equal to a strongly measurable function, then f is strongly μ-measurable.*

Proof. (1): Suppose that $f_n \to f$ pointwise outside the null set $N \in \mathscr{A}$, with each f_n μ-simple. Then we have $\lim_{n\to\infty} \mathbf{1}_{\complement N} f_n = \mathbf{1}_{\complement N} f$ pointwise on S, and since the functions $\mathbf{1}_{\complement N} f_n$ are simple, $\mathbf{1}_{\complement N} f$ is strongly measurable. Clearly, $f = \mathbf{1}_{\complement N} f$ μ-almost everywhere.

(2): Let $\widetilde{f}$ be a strongly measurable function and let $N \in \mathscr{A}$ be a μ-null set such that $f = \widetilde{f}$ on $\complement N$. If $(\widetilde{f}_n)_{n\geqslant 1}$ is a sequence of simple functions converging pointwise to $\widetilde{f}$, then $\lim_{n\to\infty} \widetilde{f}_n = f$ on $\complement N$, so $\lim_{n\to\infty} \widetilde{f}_n = f$ μ-almost everywhere. If $(S^{(n)})_{n\geqslant 1}$ is an exhausting sequence for μ, then the functions $f_n := \mathbf{1}_{S^{(n)}} \widetilde{f}_n$ are μ-simple and we have $\lim_{n\to\infty} f_n = f$ μ-almost everywhere. □

Part (2) is wrong without the σ-finiteness assumption:

Example 1.1.17. The constant function $\mathbf{1}$ is always strongly measurable (as an indicator function); it is strongly μ-measurable if and only if μ is σ-finite.

Remark 1.1.18. As a consequence of Proposition 1.1.16(1), a separably valued and strongly μ-measurable function $f : S \to X$ is strongly $\mathscr{A}_\mu$-measurable, where $\mathscr{A}_\mu$ is the *completion* of $\mathscr{A}$ with respect to μ, i.e., the σ-algebra generated by $\mathscr{A}$ and the collection of all subsets of the μ-null sets in $\mathscr{A}$. The converse holds if μ is σ-finite.

Remark 1.1.19 (Pre-images with respect to strongly μ-measurable functions). If $f : S \to X$ is strongly μ-measurable and $\widetilde{f} : S \to X$ is strongly measurable such that $f = \widetilde{f}$ μ-almost everywhere, then by Corollary 1.1.10 the set

$$\{\widetilde{f} \in B\} := \{s \in S : \ \widetilde{f}(s) \in B\}$$

belongs to $\mathscr{A}$ for all Borel sets $B \in \mathscr{B}(X)$. The μ-measure of the set $\{\widetilde{f} \in B\}$ does not depend on the particular choice of measurable function $\widetilde{f}$. This justifies the notation

$$\mu\{f \in B\} := \mu\{\widetilde{f} \in B\}$$

which often use without further notice.

An X-valued function f is said to be *μ-essentially separably valued* if there exists a closed separable subspace X_0 of X such that $f(s) \in X_0$ for μ-almost all $s \in S$, and *weakly μ-measurable* if $\langle f, x^* \rangle$ is μ-measurable for all $x^* \in X^*$.

Theorem 1.1.20 (Pettis measurability theorem, second version). *For a function $f : S \to X$ the following assertions are equivalent:*

(1) *f is strongly μ-measurable;*
(2) *f is μ-essentially separably valued and weakly μ-measurable;*
(3) *f is μ-essentially separably valued and there exists a weak* dense subspace Y of X^* such that $\langle f, x^* \rangle$ is μ-measurable for all $x^* \in Y$.*

Moreover, if f takes its values μ-almost everywhere in a closed linear subspace X_0 of X, then f is the μ-almost everywhere pointwise limit of a sequence of X_0-valued simple functions.

Proof. The implications (1)⇒(2)⇔(3) are proved in the same way as in Theorem 1.1.6. For the implication (2)⇒(1) we have to be a bit more careful: the corresponding proof in Theorem 1.1.6 produces a sequence of simple functions, but not necessarily a sequence of μ-simple functions.

Let X_1 be a separable closed subspace in which f takes μ-almost all of its values, and let $(x_k^*)_{k\geqslant 1}$ be a norming sequence for X_1. The functions $g_k = \langle f, x_k^* \rangle$ are μ-measurable, and therefore by Proposition 1.1.15 we find decompositions $S = S_{k,0} \cup S_{k,1}$ such that $g_k \equiv 0$ μ-almost everywhere on $S_{k,0}$ and μ is σ-finite on $S_{k,1}$. Put $S_0 = \bigcap_{k\geqslant 1} S_{k,0}$ and $S_1 := \complement S_0$. Then $f \equiv 0$ μ-almost everywhere on S_0 and μ is σ-finite on S_1.

This argument shows that in the rest of the proof we may assume that μ is σ-finite. Then for all $x \in X$ the constant function $\mathbf{1}_S \otimes x$ is strongly μ-measurable. Letting $(x_j)_{j\geqslant 1}$ be a dense sequence in X_1, it follows that each of the functions $g_{jk} = \langle f - x_j, x_k^* \rangle$ is μ-measurable. Hence by Proposition 1.1.16 there is a μ-null set $N \in \mathscr{A}$ such that the functions $\mathbf{1}_{\complement N} g_{jk}$ are measurable. Replacing if necessary S by $\complement N$, we may therefore assume that each of the functions g_{jk} is measurable. Then also the functions $\|f - x_j\|$ are measurable.

Let f_n be the simple functions constructed in the proof of (2)⇒(1) of Theorem 1.1.6. These functions converge to f pointwise. If $(S^{(n)})_{n\geqslant 1}$ is an exhaustion for μ, we have $\mathbf{1}_{S^{(n)}} f_n \to f$ pointwise, and each of the functions $\mathbf{1}_{S^{(n)}} f_n$ is μ-simple. □

For completeness we list a number of corollaries which may be proved in the same way as their strongly measurable counterparts.

Corollary 1.1.21. *If $f : S \to X$ is strongly μ-measurable, there exists a sequence of μ-simple functions $(f_n)_{n\geqslant 1}$ such that*

$$\|f_n(x)\| \leqslant \|f(x)\| \quad \text{and} \quad f_n(x) \to f(x) \quad \text{for } \mu\text{-almost all } x \in X.$$

Corollary 1.1.22. *If $f : S \to X$ is strongly μ-measurable and takes values in a closed subspace X_0 of X almost everywhere, then f is strongly μ-measurable as a function with values in X_0.*

Corollary 1.1.23. *The μ-almost everywhere limit $f : S \to X$ of a sequence of strongly μ-measurable functions $f_n : S \to X$ is strongly μ-measurable.*

Corollary 1.1.24. *If $f : S \to X$ is strongly μ-measurable and $\phi : X \to Y$ is measurable, where Y is another Banach space, then $\phi \circ f$ is strongly μ-measurable provided at least one of the following two conditions is satisfied:*

(i) *μ is σ-finite;*
(ii) $\phi(0) = 0$.

Proof. If μ is σ-finite, the result follows by combining Corollary 1.1.11 and Proposition 1.1.16.

If $\phi(0) = 0$ and if $\lim_{n\to\infty} f_n = f$ is an almost everywhere approximation of f by μ-simple functions f_n, then f vanishes μ-almost everywhere outside the union A of the sets supporting the f_n. On A, μ restricts to a σ-finite measure; outside A we have $\phi \circ f = 0$ almost everywhere. Now the result follows by applying the previous case to $f|_A$, viewed as a strongly $\mu|_A$-measurable function on A. □

The conditions (i) and (ii) cannot be omitted, even if $X = Y = \mathbb{K}$. Indeed, suppose that μ is non-σ-finite, let $f \equiv 0$ and $\phi(t) = 1$ for all $t \in \mathbb{K}$. The function $\mathbf{1} = \phi \circ f$ fails to be strongly μ-measurable.

The following result gives a convenient way to reduce proofs of vector-valued equalities to the scalar case. The corresponding version for strongly measurable functions is trivial.

Corollary 1.1.25. *If f and g are strongly μ-measurable X-valued functions which satisfy $\langle f, x^*\rangle = \langle g, x^*\rangle$ μ-almost everywhere for every $x^* \in Y$, where Y is a weak* dense linear subspace of X^*, then $f = g$ μ-almost everywhere.*

Proof. Both f and g take values in a separable closed subspace X_0 μ-almost everywhere, say outside the μ-null set N. Using Proposition B.1.11 we choose a sequence $(x_n^*)_{n\geqslant 1}$ in Y separating the points of X_0. Since $\langle f, x_n^*\rangle = \langle g, x_n^*\rangle$ outside a μ-null set N_n, we conclude that f and g agree outside the union of the μ-nulls set N and $\bigcup_{n\geqslant 1} N_n$. □

The following example illustrates how the results above can be used to check strong measurability.

Example 1.1.26. Suppose X and Y are Banach spaces with X separable, and let $T : X \to Y$ be an injective bounded linear operator. If $f : S \to X$ is a function with the property that $T \circ f$ is strongly μ-measurable, then f is strongly μ-measurable. Indeed, f is separably valued by assumption, and for all $y^* \in Y^*$ the function $\langle f, T^* y^*\rangle$ is μ-measurable. The injectivity of T implies that T^* has weak* dense range, and therefore the result follows from the Pettis measurability theorem.

The example $S = (0,1)$, $X = L^\infty(0,1)$, $Y = L^2(0,1)$, T the natural injection $f \mapsto f$, and $f(t) = \mathbf{1}_{(0,t)}$ shows that the separability assumption on X cannot be omitted.

1.1.c Operator-valued functions

Throughout these volumes there will be occasions to study properties of operator-valued functions. With respect to the uniform operator topology, the Banach space $\mathscr{L}(X,Y)$ is in general non-separable and because of this, few functions $f : S \to \mathscr{L}(X,Y)$ will be strongly measurable. To get a grasp of the situation, just consider the mapping $T : \mathbb{R} \to \mathscr{L}(L^2(\mathbb{R}))$, $t \mapsto T_t$, defined by

$$T_t f(u) = f(u+t), \quad u \in \mathbb{R}.$$

To see that this function fails to be strongly measurable with respect to the uniform operator topology of $\mathscr{L}(L^2(\mathbb{R}))$ we may argue as follows. For any two $s \neq t$ in $\mathbb{R}$ we note that

$$\|T_s - T_t\| = 2.$$

As a result, no matter how we choose the null set $N \subseteq \mathbb{R}$, the set $\{T_t : t \in \mathbb{R}\backslash N\}$ cannot be contained in a separable closed subspace of $\mathscr{L}(L^2(\mathbb{R}))$. Hence, by the Pettis measurability theorem, $t \mapsto T_t$ fails to be strongly measurable as an $\mathscr{L}(L^2(\mathbb{R}))$-valued function.

On the other hand, the orbits $t \mapsto T_t x$ enjoy many good properties, such as being continuous with respect to the norm of $L^2(\mathbb{R})$. This suggests the following definition.

Definition 1.1.27. *A function $f : S \to \mathscr{L}(X,Y)$ is called* strongly measurable *(respectively,* strongly μ-measurable*) if for all $x \in X$ the Y-valued function $fx : s \mapsto f(s)x$ is strongly measurable (respectively, strongly μ-measurable).*

It would be more accurate to refer to such functions as being *strongly (μ-)measurable with respect to the strong operator topology*, as opposed to those functions which are strongly measurable with respect to the uniform operator topology (for the definitions of these topologies we refer to Appendix B). The reader will agree that this terminology would be unnecessarily cumbersome. The slight ambiguity in our terminology is therefore taken for granted.

Proposition 1.1.28. *Let $(S,\mathscr{A})$ be a measurable space (respectively, $(S,\mathscr{A},\mu)$ a measure space) and let X and Y be Banach spaces. If $f : S \to X$ and $g : S \to \mathscr{L}(X,Y)$ are strongly (μ-)measurable, then $gf : S \to Y$ is strongly (μ-)measurable.*

Proof. By assumption there exists a sequence $(f_n)_{n\geqslant 1}$ of (μ-)simple functions converging pointwise to f (μ-almost everywhere). The functions gf_n are strongly (μ-)measurable and satisfy $\lim_{n\to\infty} gf_n \to gf$ pointwise (μ-almost everywhere). Corollary 1.1.9 (Corollary 1.1.23) now implies the strong (μ-)-measurability of gf. □