WIDEBAND AMPLIFIERS

Wideband Amplifiers

by

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Springer

A C.I.P. Catalogue record for this book is available from the Library of Congress.

ISBN-10 0-387-28340-4 (HB)
ISBN-13 978-0-387-28340-1 (HB)
ISBN-10 0-387-28341-2 (e-book)
ISBN-13 978-0-387-28341-8 (e-book)

Published by Springer,
P.O. Box 17, 3300 AA Dordrecht, The Netherlands.

www.springer.com

Printed on acid-free paper

Printed in the Netherlands.

We dedicate this book to all our friends and colleagues in the art of electronics.

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Acknowledgments

The authors are grateful to ***John Addis***, ***Carl Battjes***, ***Dennis Feucht***, ***Bruce Hoffer*** and ***Bob Ross***, all former employees Tektronix, Inc., for allowing us to use their class notes, ideas, and publications, and for their help when we had run into problems concerning some specific circuits.

We are also thankful to prof. ***Ivan Vidav*** of the Faculty of Mathematics and Physics in Ljubljana for his help and reviewing of Part 1, and to ***Csaba Szathmary***, a former employee of EMG, Budapest, for allowing us to use some of his measurement results in Part 5.

However, if, in spite of meticulously reviewing the text, we have overlooked some errors, this, of course, is our own responsibility alone; we shall be grateful to everyone for bringing such errors to our attention, so that they can be corrected in the next edition. To report the errors please use one of the e-mail addresses below.

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Foreword

With the exception of the tragedy on September 11, the year 2001 was relatively normal and uneventful: remember, this should have been the year of the *Clarke's* and *Kubrick's* Space Odyssey, mission to Juiter; it should have been the year of the HAL-9000 computer.

Today, the Personal Computer is as ubiquitous and omnipresent as was HAL on the Discovery spaceship. And the rate of technology development and market growth in electronics industry still follows the famous 'Moore Law', almost four decades after it has been first formulated: in 1965, *Gordon Moore* of Intel Corporation predicted the doubling of the number of transistors on a chip every 2 years, corrected to 18 months in 1967; at that time, the landing on the Moon was in full preparation.

Curiously enough, today noone cares to go to the Moon again, let alone Jupiter. And, in spite of all the effort in digital engineering, we still do not have anything close to 0.1% of the HAL capacity (fortunately?!). Whilst there are many research labs striving to put artificial intelligence into a computer, there are also rumors that this has already happened (with Windows-95, of course!).

In the early 1990s it was felt that digital electronics will eventually render analog systems obsolete. This never happened. Not only is the analog sector vital as ever, the job market demands are expanding in all fields, from high-speed measurement instrumentation and data acquisition, telecommunications and radio frequency engineering, high-quality audio and video, to grounding and shielding, electromagnetic interference suppression and low-noise printed circuit board design, to name a few. And it looks like this demand will be going on for decades to come.

But whilst the proliferation of digital systems attracted a relatively high number of hardware and software engineers, analog engineers are still rare birds. So, for creative young people, who want to push the envelope, there are lots of opportunities in the analog field.

However, analog electronics did not earn its "Black-Magic Art" attribute in vain. If you have ever experienced the problems and frustrations from circuits found in too many 'cook-books' and 'sure-working schemes' in electronics magazines, and if you became tired of performing exorcism on every circuit you build, then it is probably the time to try a different way: in our own experience, the 'hard' way of doing the correct math first often turns out to be the 'easy' way!

Here is the book **"Wideband Amplifiers"**. The book is intended to serve both as a design manual to more experienced engineers, as well as a good learning guide to beginners. It should help you to improve your analog designs, making better and faster amplifier circuits, especially if time-domain performance is of major concern. We have strieved to provide the complete math for every design stage. And, to make learning a joyful experience, we explain the derivation of important math relations from a design engineer point of view, in an intuitive and self-evident manner (rigorous mathematicians might not like our approach). We have included many practical applications, schematics, performance plots, and a number of computer routines.

However, as it is with any interesting subject, the greatest problem was never what to include, but rather what to leave out!

In the foreword of his popular book “A Brief History of Time”, *Steven Hawking* wrote that his publisher warned him not to include any math, since the number of readers would be halved by each formula. So he included the $E = m\,c^2$ and bravely cut out one half of the world population.

We went further: there are some 220 formulae in Part 1 only. By estimating the current world population to some $6{\times}10^9$, of which 0.01% could be electronics engineers and assuming an average lifetime interest in the subject of, say, 30 years, if the publisher’s rule holds, there ought to be one reader of our book once every:

$$2^{220}/(6 \times 10^9 \times 10^{-4} \times 30 \times 356 \times 24 \times 3600) \approx 3 \times 10^{51} \text{ seconds}$$

or something like $6.6{\times}10^{33} \times$ the total age of the Universe!

Now, whatever you might think of it, this book is **not** about math! It is about getting your design to run right first time! Be warned, though, that it will be not enough to just read the book. To have any value, a theory must be put into practice. Although there is no theoretical substitute for hands-on experience, this book should help you to significantly shorten the trial-and-error phase.

We hope that by studying this book thoroughly you will find yourself at the **beginning of a wonderful journey**!

Peter Starič and Erik Margan,

Ljubljana, June 2003

Important Note:

We would like to reassure the Concerned Environmentalists that during the writing of this book, no animal or plant had suffered any harm whatsoever, either in direct or indirect form (excluding the authors, one computer ‘mouse’ and countless computation ‘bugs’!).

Release Notes

The manuscript of this book appeared first in spring of 1988.

Since then, the text has been revised sveral times, with some minor errors corrected and figures redrawn, in particular in Part 2, where inductive peaking networks are analyzed. Several topics have been updated to reflect the latest developments in the field, mainly in Part 5, dealing with modern high-speed circuits. The Part 6, where a number of computer algorithms are developed, and Part 7, containing several algorithm application examples, were also brought up to date.

This is a release version 3 of the book.

The book also commes in the Adobe Portable Document Format (PDF), readable by the Adobe Acrobat™ Reader program (the latest version can be downloaded free of charge from http://www.adobe.com/products/Acrobat/).

One of the advantages of the book, offered by the PDF format and the Reader program, are numerous links (blue underlined text), which enable easy access to related topics by pointing the 'mouse' cursor on the link and clicking the left 'mouse' button. Returning to the original reading position is possible by clicking the right 'mouse' button and select "Go Back" from the pop-up menu (see the AR HELP menu). There are also numerous highlights (green underlined text) relating to the content within the same page.

The cross-file links (red underlined text) relate to the contents in different PDF files, which open by clicking the link in the same way.

The Internet and World-Wide-Web links are in violet (dark magenta) and are accessed by opening the default browser installed on your computer.

The book was written and edited using 𝔼𝕏ℙ™, the Scientific Word Processor, version 5.0, (made by Simon L. Smith, see http://www.expswp.com/).

The computer algorithms developed and described in Part 6 and 7 are intended as tools for the amplifier design process. Written for Matlab™, the Language of Technical Computing (The MathWorks, Inc., http://www.mathworks.com/), they have all been revised to conform with the newer versions of Matlab (version 5.3 'for Students'), but still retaining downward compatibility (to version 1) as much as possible. The files can be found on the CD in the 'Matlab' folder as '*.M' files, along with the information of how to install them and use within the Matlab program. We have used Matlab to check all the calculations and draw most of the figures. Before importing them into 𝔼𝕏ℙ, the figures were finalized using the Adobe Illustrator, version 8 (see http://www.adobe.com/products/Illustrator/).

All circuit designs were checked using Micro-CAP™, the Microcomputer Circuit Analysis Program, v. 5 (Spectrum Software, http://www.spectrum-soft.com/). Some of the circuits described in the book can be found on the CD in the 'MicroCAP' folder as '*.CIR' files, which the readers with access to the MicroCAP program can import and run the simmulations by themselves.

P. Starič, E. Margan

Wideband Amplifiers

Part 1

The Laplace Transform

There is nothing more practical than a good theory!
(William Thompson, Lord Kelvin)

About Transforms

The Laplace transform can be used as a powerful method of solving linear differential equations. By using a time domain integration to obtain the frequency domain transfer function and a frequency domain integration to obtain the time domain response, we are relieved of a few nuisances of differential equations, such as defining boundary conditions, not to speak of the difficulties of solving high order systems of equations.

Although Laplace had already used integrals of exponential functions for this purpose at the beginning of the 19th century, the method we now attribute to him was effectively developed some 100 years later in Heaviside's operational calculus.

The method is applicable to a variety of physical systems (and even some non physical ones, too!) involving trasport of energy, storage and transform, but we are going to use it in a relatively narrow field of calculating the time domain response of amplifier filter systems, starting from a known frequency domain transfer function.

As for any tool, the transform tools, be they Fourier, Laplace, Hilbert, etc., have their limitations. Since the parameters of electronic systems can vary over the widest of ranges, it is important to be aware of these limitations in order to use the transform tool correctly.

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1.0 Introduction

With the advent of television and radar during the Second World War, the behavior of wideband amplifiers in the time domain has become very important [Ref. 1.1]. In today's digital world this is even more the case. It is a paradox that designers and troubleshooters of **digital** equipment still depend on oscilloscopes, which — at least in their fast and low level input part — consist of **analog** wideband amplifiers. So the calculation of the time domain response of wideband amplifiers has become even more important than the frequency, phase, and time delay response.

The emphasis of this book is on the amplifier's time domain response. Therefore a thorough knowledge of time related calculus, explained in Part 1, is a necessary pre-requisite for understanding all other parts of this book where wideband amplifier networks are discussed.

The time domain response of an amplifier can be calculated by two main methods: The first one is based on **differential equations** and the second uses the **inverse Laplace transform ($\mathcal{L}^{-1}$ transform)**. The differential equation method requires the calculation of boundary conditions, which — in case of high order equations — means an unpleasant and time consuming job. Another method, which also uses differential equations, is the so called *state variable* calculation, in which a differential equation of order n is split into n differential equations of the first order, in order to simplify the calculations. The state variable method also allows the calculation of non linear differential equations. We will use neither of these, for the simple reason that the Laplace transform and its inverse are based on the system poles and zeros, which prove so useful for network calculations in the frequency domain in the later parts of the book. So most of the data which are calculated there is used further in the time domain analysis, thus saving a great deal of work. Also the use of the $\mathcal{L}^{-1}$ transform does not require the calculation of boundary conditions, giving the result directly in the time domain.

In using the $\mathcal{L}^{-1}$ transform most engineers depend on tables. Their method consists firstly of splitting the amplifier transfer function into partial fractions and then looking for the corresponding time domain functions in the $\mathcal{L}$ transform tables. The sum of all these functions (as derived from partial fractions) is then the result. The difficulty arises when no corresponding function can be found in the tables, or even at an earlier stage, if the mathematical knowledge available is insufficient to transform the partial fractions into such a form as to correspond to the formulae in the tables.

In our opinion an amplifier designer should be self-sufficient in calculating the time domain response of a wideband amplifier. Fortunately, this can be almost always derived from simple rational functions and it is relatively easy to learn the $\mathcal{L}^{-1}$ transforms for such cases. In Part 1 we show how this is done generally, as well as for a few simple examples. A great deal of effort has been spent on illustrating the less clear relationships by relevant figures. Since engineers seek to obtain a first glance insight of their subject of study, we believe this approach will be helpful.

This part consists of four main sections. In the first, the concept of harmonic (e.g., sinusoidal) functions, expressed by pairs of counter-rotating complex conjugate phasors, is explained. Then the Fourier series of periodic waveforms are discussed to obtain the discrete spectra of periodic waveforms. This is followed by the Fourier integral to obtain continuous spectra of non-repetitive waveforms. The convergence problem of the Fourier integral is solved by introducing the complex frequency variable $s = \sigma + j\omega$, thus coming to direct Laplace transform ($\mathcal{L}$ transform).

The second section shows some examples of the $\mathcal{L}$ transforms. The results are useful when we seek the inverse transforms of simple functions.

The third section deals with the theory of functions of complex variables, but only to the extent that is needed for understanding the inverse Laplace transform. Here the line and contour integrals (Cauchy integrals), the theory of residues, the Laurent series and the $\mathcal{L}^{-1}$ transform of rational functions are discussed. The existence of the $\mathcal{L}^{-1}$ transform for rational functions is proved by means of the Cauchy integral.

Finally, the concluding section deals with some aspects of the $\mathcal{L}^{-1}$ transforms and the convolution integral. Only two standard problems of the $\mathcal{L}^{-1}$ transform are shown, because all the transient response calculations (by means of the contour integration and the theory of residues) of amplifier networks, presented in Parts 2–5, give enough examples and help to acquire the necessary know-how.

It is probably impossible to discuss Laplace transform in a manner which would satisfy both engineers and mathematicians. Professor *Ivan Vidav* said: "*If we mathematicians are satisfied, you engineers would not be, and vice versa*". Here we have tried to achieve the best possible compromise: to satisfy electronics engineers and at the same time not to 'offend' the mathematicians. But, as late colleague, the physicist *Marko Kogoj*, used to say: "*Engineers never know enough of mathematics; only mathematicians know their science to the extent which is satisfactory for an engineer, but they hardly ever know what to do with it!*" Thus successful engineers keep improving their general knowledge of mathematics — far beyond the text presented here.

After studying this part the readers will have enough knowledge to understand all the time domain calculations in the subsequent parts of the book. In addition, the readers will acquire the basic knowledge needed to do the time-domain calculations by themselves and so become independent of $\mathcal{L}$ transform tables. Of course, in order to save time, they will undoubtedly still use the tables occasionally, or even make tables of their own. But they will be using them with much more understanding and self-confidence, in comparison with those who can do $\mathcal{L}^{-1}$ transform only via the partial fraction expansion and the tables of basic functions.

Those readers who have already mastered the Laplace transform **and its inverse**, can skip this part up to Sec. 1.14, where the $\mathcal{L}^{-1}$ transform of a two pole network is dealt with. From there on we discuss the basic examples, which we use later in many parts of the book; the content of Sec. 1.14 should be understood thoroughly. However, if the reader notices any substantial gaps in his/her knowledge, it is better to start at the beginning.

In the last two parts of this book, Part 6 and 7, we derive a set of computer algorithms which reduce the circuit's time domain analysis, performance plotting and pole layout optimization to a pure routine. However attractive this may seem, we nevertheless recommend the study of Part 1: a good engineer must understand the tools he/she is using in order to use them effectively.

1.1 Three Different Ways of Expressing a Sinusoidal Function

We will first show how a sinusoidal function can be expressed in three different ways. The most common way is to express the instantaneous value a of a sinusoid of amplitude A and angular frequency $\omega_1 = 2\pi f_1$, ($f_1 =$ frequency) by the well known formula:

$$a = f(t) = A \sin \omega_1 t \tag{1.1.1}$$

The reason that we have appended the index '1' to ω will become apparent very soon when we will discuss complex signals containing different frequency components. The amplitude vs. time relation of this function is shown in Fig. 1.1.1a. This is the most familiar display seen by using any sine-wave oscillator and an oscilloscope.

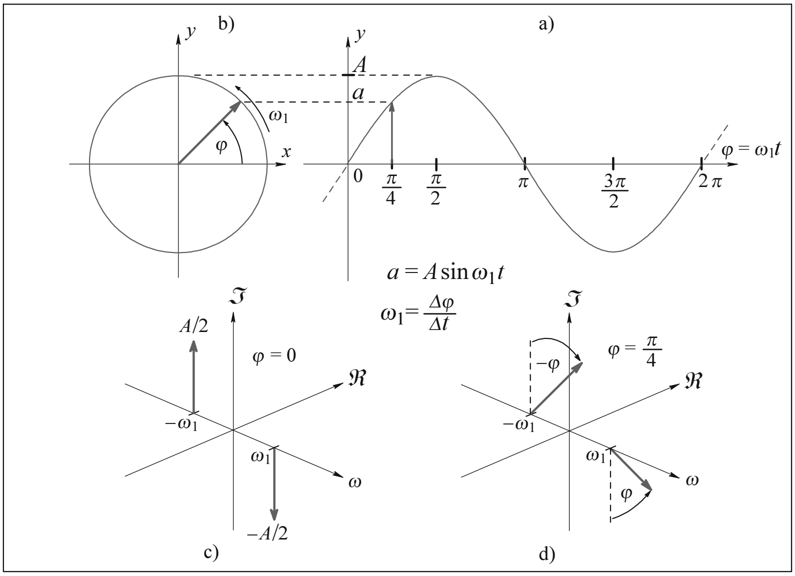


Fig. 1.1.1: Three different presentations of a sine wave: **a)** amplitude in time domain; **b)** a phasor of length A, rotating with angular frequency ω_1; **c)** two complex conjugate phasors of length $A/2$, rotating in opposite directions with angular frequency ω_1, at $\omega_1 t = 0$; **d)** the same as c), except at $\omega_1 t = \pi/4$.

In electrical engineering, another presentation of a sinusoidal function is often used, coming from the vertical axis projection of a rotating phasor A, as displayed in Fig. 1.1.1b, for which the same Eq. 1.1.1 is valid. Here both axes are real, but one of the axes may also be imaginary. In this case the corresponding mathematical presentation is:

$$f(t) = \widehat{A} = A\, \mathrm{e}^{j\omega_1 t} \tag{1.1.2}$$

where $\widehat{A}$ is a complex quantity and $\mathrm{e} = 2.718\,281\ldots$ is the basis of natural logarithms. However, we can also obtain the real quantity a by expressing the sinusoidal function by **two complex conjugate phasors** of length $A/2$ which rotate in opposite directions, as displayed in a three-dimensional presentation in Fig. 1.1.1c. Here both phasors are shown at $\omega t = 0$ (or $\omega t = 2\pi,\ 4\pi, \dots$). The sum of both phasors has the instantaneous value a, which is **always real**. This is ensured because both phasors rotate with the same angular frequency $+\omega_1$ and $-\omega_1$, starting as shown in Fig. 1.1.1c, and therefore they are always complex conjugate at any instant. We express a by the well-known *Euler* formula:

$$\boxed{a = f(t) = A\sin\omega_1 t = \frac{A}{2j}\left(e^{j\omega_1 t} - e^{-j\omega_1 t}\right)} \tag{1.1.3}$$

The j in the denominator means that both phasors are imaginary at $t = 0$. The sum of both rotating phasors is then zero, because:

$$f(0) = \frac{A}{2j}\,e^{j\omega_1 0} - \frac{A}{2j}\,e^{-j\omega_1 0} = 0 \tag{1.1.4}$$

Both phasors in Fig. 1.1.1c and 1.1.1d are placed on the frequency axis at such a distance from the origin as to correspond to the frequency $\pm\omega_1$. Since the phasors rotate with time the Fig. 1.1.1d, which shows them at $\varphi = \omega_1 t = \pi/4$, helps us to acquire the idea of a three-dimensional presentation. The understanding of these simple time-frequency relations, presented in Fig. 1.1.1c and 1.1.1d and expressed by Eq. 1.1.3, is essential for understanding both the Fourier transform and the Laplace transform.

Eq. 1.1.3 can be changed to the **cosine** function if the phasor with $+\omega_1$ is multiplied by $j = e^{j\pi/2}$ and the phasor with $-\omega_1$ by $-j = e^{-j\pi/2}$. The first multiplication means a **counter-clockwise** rotation by 90° and the second a **clockwise** rotation by 90°. This causes both phasors to become real at time $t = 0$, their sum again equaling A:

$$f(t) = \frac{A}{2j}\,e^{j\omega_1 t} + \frac{A}{2j}\,e^{-j\omega_1 t} = A\cos\omega_1 t \tag{1.1.5}$$

In general a sinusoidal function with a non-zero phase angle φ at $t = 0$ is expressed as:

$$A\sin(\omega t + \varphi) = \frac{A}{2j}\left[e^{j(\omega t+\varphi)} - e^{-j(\omega t+\varphi)}\right] \tag{1.1.6}$$

The need to introduce the frequency axis in Fig. 1.1.1c and 1.1.1d will become apparent in the experiment shown in Fig. 1.1.2. Here we have a unity gain amplifier with a poor loop gain, driven by a sinewave source with frequency ω_1 and amplitude A_1, and loaded by the resistor R_L. If the resistor's value is too low and the amplitude of the input signal is high the amplifier reaches its maximum output current level, and the output signal $f(t)$ becomes distorted (we have purposely kept the same notation A as in the previous figure, rather than introducing the sign V for voltage). The distorted output signal contains not just the original signal with the same fundamental frequency ω_1, but also a third harmonic component with the amplitude $A_3 < A_1$ and frequncy $\omega_3 = 3\,\omega_1$:

$$f(t) = A_1\sin\omega_1 t + A_3\sin 3\,\omega_1 t = A_1\sin\omega_1 t + A_3\sin\omega_3 t \tag{1.1.7}$$

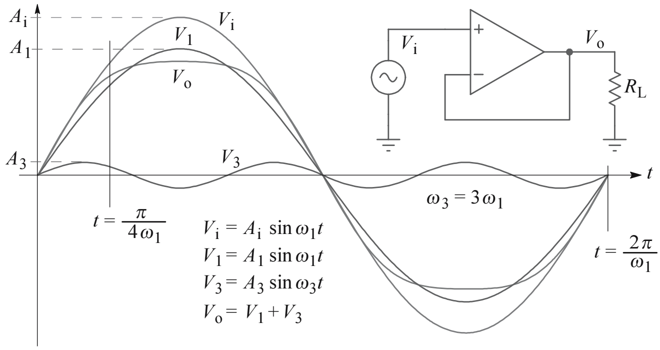


Fig. 1.1.2: The amplifier is slightly overdriven by a pure sinusoidal signal, V_i, with a frequency ω_1 and amplitude A_i. The output signal V_o is distorted, and it can be represented as a sum of two signals, $V_1 + V_3$. The fundamental frequency of V_1 is ω_1 and its amplitude A_1 is somewhat lower. The frequency of V_3 (the third harmonic component) is $\omega_3 = 3\,\omega_1$ and its amplitude is A_3.

Now let us draw the output signal in the same way as we did in Fig. 1.1.1c,d. Here we have two pairs of harmonic components: the first pair of phasors $A_1/2$ rotating with the fundamental frequency $\pm\,\omega_1$, and the second pair $A_3/2$ rotating with the third harmonic frequency $\pm\,\omega_3$, which are three times more distant from the origin than ω_1. This is shown in Fig. 1.1.3a, where all four phasors are drawn at time $t = 0$. Fig. 1.1.3b shows the phasors at time $t = \pi/4\,\omega$. Because the third harmonic phasor pair rotates with an angular frequency three times higher, they rotate up to an angle $\pm\,3\pi/4$ in the same time.

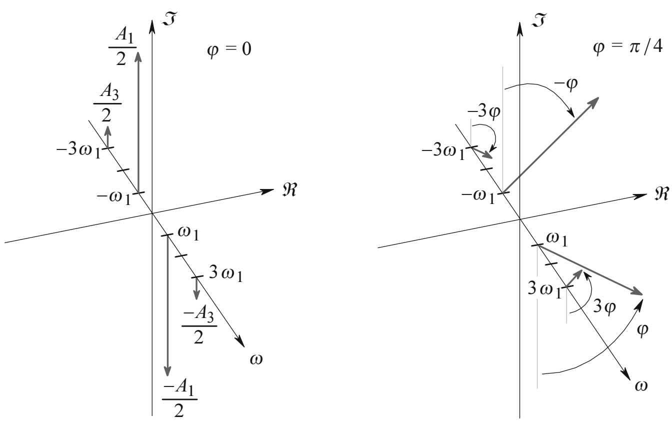


Fig. 1.1.3: The output signal of the amplifier in Fig. 1.1.2, expressed by two pairs of complex conjugate phasors: **a)** at $\omega_1\,t = 0$; **b)** at $\omega_1\,t = \pi/4$.

Mathematically Eq. 1.1.7, according to Fig. 1.1.2 and 1.1.3, can be expressed as:

$$
\begin{aligned}
f(t) &= A_1 \sin\omega_1 t + A_3 \sin\omega_3\, t \\
&= \frac{A_1}{2j}\left(e^{j\omega_1 t} - e^{-j\omega_1 t}\right) + \frac{A_3}{2j}\left(e^{j\omega_3 t} - e^{-j\omega_3 t}\right)
\end{aligned}
\tag{1.1.8}
$$

The amplifier output obviously cannot exceed either its supply voltage or its maximum output current. So if we keep increasing the input amplitude the amplifier will clip the upper and lower peaks of the output waveform (some input protection, as well as some internal signal source resistance must be assumed if we want the amplifier to survive in these conditions), thus generating more harmonics. If the input amplitude is very high and if the amplifier loop gain is high as well, the output voltage $f(t)$ would eventually approach a square wave shape, such as in Fig. 1.2.1b in the following section. A true mathematical square wave has an infinite number of harmonics; since no amplifier has an infinite bandwidth, the number of harmonics in the output voltage of any practical amplifier will always be finite.

In the next section we are going to examine a generalized harmonic analysis.

1.2 The Fourier Series

In the experiment shown in Fig. 1.1.2 we have **composed** the sinusoidal waveforms with the amplitudes A_1 and A_3 to get the output time-function $f(t)$. Now, if we have a square wave, as in Fig. 1.2.1b, we would have to deal with many more discrete frequency components. We intend to calculate the amplitudes of them, assuming that the time function of the square wave is known. This means a **decomposition** of the time function $f(t)$ into the corresponding harmonic frequency components. To do so we will examine the *Fourier* series, following the French mathematician *Jean Baptiste Joseph de Fourier*[1].

The square wave time function is periodic. A function is periodic if it acquires the same value after its characteristic period $T_1 = 2\pi/\omega_1$, at any instant t:

$$f(t) = f(t + T_1) \tag{1.2.1}$$

Consequently the same is true for $f(t) = f(t + n T_1)$, where n is an integer. According to Fourier this square wave can be expressed as a sum of harmonic components with frequencies $f_n = \pm n/T_1$. If $n = 1$ we have the fundamental frequency f_1 with a phasor $A_1/2$, rotating counter-clockwise. The phasor f_{-1} with the same length $A_1/2 = A_{-1}/2$ rotates clockwise and forms a complex conjugate pair with the first one. A true square wave would have an infinite number of odd-order harmonics (all even order harmonics are zero).

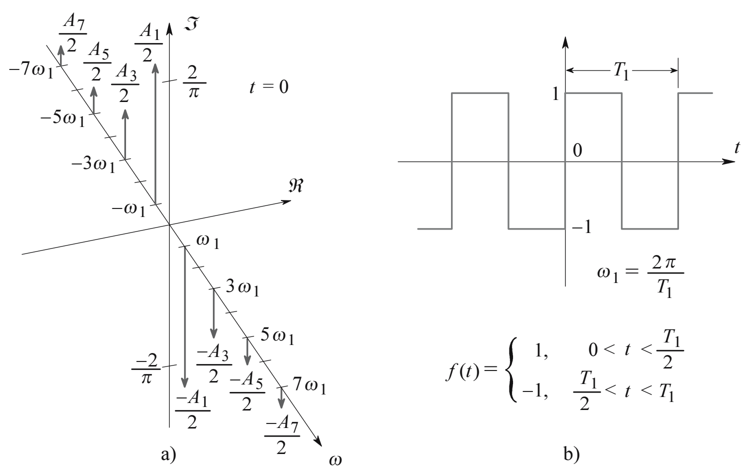


Fig. 1.2.1: A square wave, as shown in **b)**, has an infinite number of odd-order frequency components, of which the first 4 complex-conjugate phasor pairs are drawn in **a)** at time $t = (0 \pm 2n\pi)/\omega_1$, where n is an integer representing the number of the period.

[1] It is interesting that Fourier developed this method in connection with thermal engineering. As a general in the Napoleon's army he was concerned with gun deformation by heat. He supposed that one side of a straight metal bar is heated and then bent, joining the ends, to form a thorus. Then he calculated the temperature distribution along the circle so formed, in such a way that it would be the sum of sinusoidal functions, each having a different amplitude and a different angular frequency.

In Fig. 1.2.1, we have drawn the complex-conjugate phasor pairs of the first 4 harmonics. Because all the phasor pairs are always complex-conjugate, the sum of any pair, as well as their total sum, is always real. The phasors rotate with different speeds and in opposite directions. Fig. 1.2.2a shows them at time $T_1/8$ to help the reader's imagination. Although this figure looks confusing, the phasors shown have an exact inter-relationship. Looking at the positive ω axis, the phasor with the amplitude $A_1/2$ has rotated in the counter-clockwise direction by an angle of $\pi/4$. During the same interval of $T_1/8$ the remaining phasors have rotated: $A_3/2$ by $3\pi/4$; $A_5/2$ by $5\pi/4$; $A_7/2$ by $7\pi/4$; etc. The corresponding complex conjugate phasors on the negative ω axis rotate likewise, but in the opposite (clockwise) direction. The sum of all phasors at any instant t is the instantaneous amplitude of the time domain function. In general, the time function with the fundamental frequency ω_1 is expressed as:

$$
\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} \frac{A_n}{2}\,\mathrm{e}^{jn\omega_1 t} \\
&= \cdots + \frac{A_{-n}}{2}\,\mathrm{e}^{-jn\omega_1 t} + \cdots + \frac{A_{-2}}{2}\,\mathrm{e}^{-j2\omega_1 t} + \frac{A_{-1}}{2}\,\mathrm{e}^{-j\omega_1 t} \\
&\qquad + A_0 + \frac{A_1}{2}\,\mathrm{e}^{j\omega_1 t} + \frac{A_2}{2}\,\mathrm{e}^{j2\omega_1 t} + \cdots + \frac{A_n}{2}\,\mathrm{e}^{jn\omega_1 t} + \cdots
\end{aligned}
\tag{1.2.2}
$$

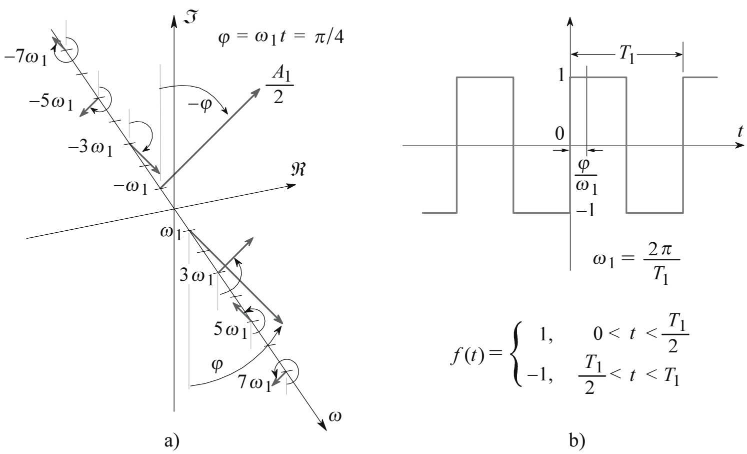


Fig. 1.2.2: As in Fig. 1.2.1, but at an instant $t = (\pi/4 \pm 2n\pi)/\omega_1$; **a)** the spectrum, expressed by complex conjugate phasor pairs, corresponds to the instant $t = \varphi/\omega_1$ in **b)**.

Note that for the square wave all the even frequency components are missing. For other types of waveforms the even coefficients can be non-zero. In general A_i may also be complex, thus containing some non-zero initial phase angle φ_i. In Eq. 1.2.2 we have also introduced A_0, the DC component, which did not exist in our special case. The meaning of A_0 can be understood by examining Fig. 1.2.3a, where the so-called sawtooth waveform is shown, with no DC component. In Fig. 1.2.3b, the waveform has a DC component of magnitude A_0.

Eq. 1.2.2 represents the *complex spectrum* of the function $f(t)$, while Fig. 1.2.1 represents the corresponding most significant part of the complex spectrum of a square wave. The next step is the calculation of the magnitudes of the rotating phasors.

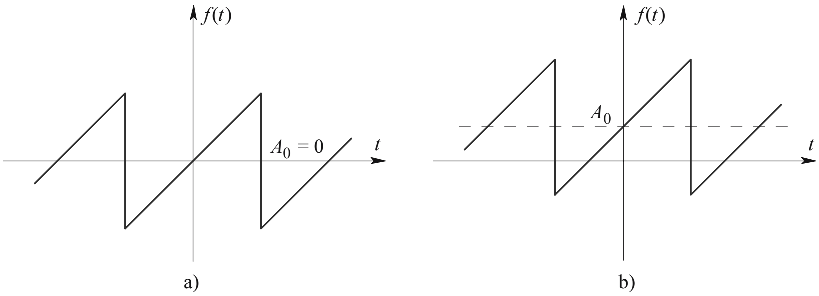


Fig. 1.2.3: a) A waveform without a DC component; **b)** with a DC component A_0.

If we want to measure safely and accurately the diameter of a wheel of a working machine, we must first stop the machine. Something similar can be done with our Eq. 1.2.2, except that here we can mathematically stop the rotation of any single phasor. Suppose we have a phasor $A_\mathrm{k}/2$, rotating counter-clockwise with frequency $\omega_\mathrm{k} = \mathrm{k}\,\omega_1$ with an initial phase angle φ_k (at $t = 0$), which is expressed as:

$$\frac{A_\mathrm{k}}{2}\,\mathrm{e}^{j(\omega_\mathrm{k}\,t+\varphi_\mathrm{k})} = \frac{A_\mathrm{k}}{2}\,\mathrm{e}^{j\,\omega_\mathrm{k}\,t}\,\mathrm{e}^{j\,\varphi_\mathrm{k}} \tag{1.2.3}$$

Now we multiply this expression by a unit amplitude, clockwise rotating phasor $\mathrm{e}^{-j\,\omega_\mathrm{k}}$ (having the same angular frequency ω_k) to cancel the $\mathrm{e}^{j\,\omega_\mathrm{k}}$ term, [Ref. 1.2]:

$$\frac{A_\mathrm{k}}{2}\,\mathrm{e}^{j\,\varphi_\mathrm{k}}\,\mathrm{e}^{j\,\omega_\mathrm{k}\,t}\,\mathrm{e}^{-j\,\omega_\mathrm{k}\,t} = \frac{A_\mathrm{k}}{2}\,\mathrm{e}^{j\,\varphi_\mathrm{k}} \tag{1.2.4}$$

and obtain a non-rotating component which has the magnitude $A_\mathrm{k}/2$ and phase angle φ_k **at any time**. With this in mind let us attack the whole time function $f(t)$. The duration of the multiplication must last exactly one whole period and the corresponding expression is:

$$\frac{A_\mathrm{k}}{2} = \frac{1}{T}\int\limits_{-T/2}^{T/2} f(t)\,\mathrm{e}^{-j\,\omega_\mathrm{k}\,t}\,dt \tag{1.2.5}$$

Since we have integrated over the whole period T in order to get the average value of that harmonic component, the result of the integration must be divided by T, as in Eq. 1.2.5. If there is a DC component (with $\omega = 0$) in the spectrum, the calculation of it is simply:

$$A_0 = \frac{1}{T}\int\limits_{-T/2}^{T/2} f(t)\,dt \tag{1.2.6}$$

To return to Eq. 1.2.5, let us explain the meaning of the integration Eq. 1.2.5 by means of Fig. 1.2.4.

By multiplying the function $f(t)$ by $\mathrm{e}^{-j\omega_\mathrm{k} t}$ we have stopped the rotating phasor $A_\mathrm{k}/2$, while during the time interval of integration all the other phasors have rotated through an angle of $n\,2\,\pi$ (where n is an integer), **including** the DC phasor A_0, because it is now multiplied by $\mathrm{e}^{-j\omega_\mathrm{k} t}$. The result of the integration for all these rotating phasors is zero, as indicated in Fig. 1.2.4a, while the phasor $A_\mathrm{k}/2$ has stopped, integrating eventually to its full amplitude; the integration for this phasor only is shown in Fig. 1.2.4b.

The understanding of the effect described of the multiplication $f(t)\,\mathrm{e}^{-j\omega t}$ is essential to understanding the basic principles of the Fourier series, the Fourier integral and the Laplace transform.

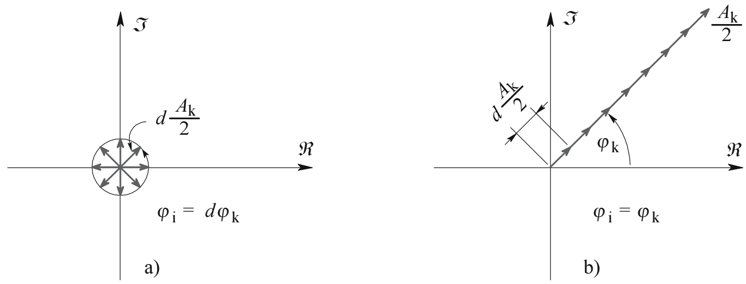
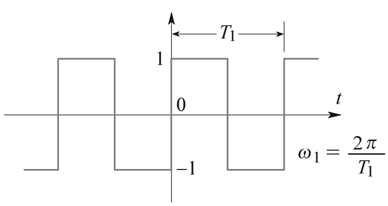


Fig. 1.2.4: a) The integral over the full period T of a rotating phasor is zero; **b)** the integral over a full period T of a non-rotating phasor $d\,A_\mathrm{k}/2$, gives its amplitude, $A_\mathrm{k}/2$, (the symbol d stands for dt/T — in this figures $dt \rightarrow \Delta t$ such that $\Delta t\,\omega_\mathrm{k} = \pi/4$). Note that a stationary phasor retains its initial angle φ_k.

For us the Fourier series represents only a transitional station on the journey towards the Laplace transform. So we will drive through it with a moderate speed "via the Main Street", without investigating some interesting things in the side streets. Nevertheless, it is useful to make a practical example. Since we have started with a square wave, shown in Fig. 1.2.5, let us calculate its complex spectrum components $A_n/2$, assuming that the square wave amplitude is $A = 1$.



$$f(t) = \begin{cases} 1, & 0 < t < \frac{T_1}{2} \\ -1, & \frac{T_1}{2} < t < T_1 \end{cases}$$

Fig. 1.2.5: A square wave signal.

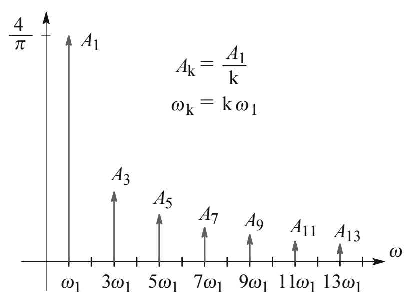


Fig. 1.2.6: The frequency spectrum of a square wave, expressed by **real** values (magnitudes) only.

For a **single period** the corresponding mathematical expression for this function is:

$$f(t) = \begin{cases} -1 & \text{for} \quad -T/2 < t < 0 \\ +1 & \text{for} \quad 0 < t < T/2 \end{cases}$$

According to Eq. 1.2.5 we calculate:

$$\frac{A_n}{2} = \frac{1}{T}\left[\int_{-T/2}^{0}(-1)\,\mathrm{e}^{-j2\pi n t/T}\,dt + \int_{0}^{T/2}(+1)\,\mathrm{e}^{-j2\pi n t/T}\,dt\right]$$

$$= \frac{1}{T}\left(\frac{T}{j2\pi n}\,\mathrm{e}^{-j2\pi n t/T}\Big|_{-T/2}^{0} + \frac{T}{-j2\pi n}\,\mathrm{e}^{-j2\pi n t/T}\Big|_{0}^{T/2}\right)$$

$$= \frac{1}{j2\pi n}\left(1 - \mathrm{e}^{j\pi n} - \mathrm{e}^{-j\pi n} + 1\right) = \frac{-1}{j\pi n}\left(\frac{\mathrm{e}^{j\pi n} + \mathrm{e}^{-j\pi n}}{2} - 1\right)$$

$$= \frac{-1}{j\pi n}\left(\cos\pi n - 1\right) \tag{1.2.7}$$

The result is zero for $n = 0$ (the DC component A_0) and for any even n. For any odd n the value of $\cos\pi n = -1$, and for such cases the result is:

$$\frac{A_n}{2} = \frac{2}{j\pi n} = \frac{-2j}{\pi n} \tag{1.2.8}$$

The factor $-j$ in the numerator means that for any positive n (and for $t = 0$, 2π, 4π, $6\pi, \dots$) the phasor is negative and imaginary, whilst for negative n it is positive and imaginary. This is evident from Fig. 1.2.1a.

Let us calculate the first eight phasors by using Eq. 1.2.8. The lengths of phasors in Fig. 1.2.1a and 1.2.2.b correspond to the values reported in Table 1.2.1. All the phasors form complex conjugate pairs and their total sum **always gives a real value**.

Table 1.2.1: The first few harmonics of a square wave

$\pm n$	0	1	3	5	7	9	11	13
$\mp A_n/2$	0	$2j/\pi$	$2j/3\pi$	$2j/5\pi$	$2j/7\pi$	$2j/9\pi$	$2j/11\pi$	$2j/13\pi$

However, a spectrum can also be shown with real values only, e.g., as it appears on the cathode ray tube screen of a spectrum analyzer. To obtain this, we simply sum the corresponding complex conjugate phasor pairs (e.g., $|A_n/2| + |A_{-n}/2| = A_n$) and place them on the abscissa of a two-dimensional coordinate system, as shown in Fig. 1.2.6. Such a non-rotating spectrum has only the positive frequency axis. Although such a presentation of spectra is very useful in the analysis of signals containing several (or many) frequency components, we will continue calculating with the complex spectra, because the phase information is also important. And, of course, the Laplace transform, which is our main goal, is based on a complex variable.

Now let us recompose the waveform using only the harmonic frequency components from Table 1.2.1, as shown in Fig. 1.2.7a. The waveform resembles the square wave but it has an exaggerated overshoot $\delta \simeq 18\,\%$ of the nominal amplitude.

The reason for the overshoot δ is that we have abruptly cut off the higher harmonic components from a certain frequency upwards. Would this overshoot be lower if we take more harmonics? In Fig. 1.2.7b we have increased the number of harmonic components three times, but the overshoot remained the same. No matter how many, yet for any **finite** number of harmonic components, used to recompose the waveform, the overshoot would stay the same (only its duration becomes shorter if the number of harmonic components is increased, as is evident from Fig. 1.2.7a and 1.2.7b).

This is the *Gibbs*' phenomenon. It tells us that we should not cut off the frequency response of an amplifier abruptly if we do not wish to add an undesirably high overshoot to the amplified pulse. Fortunately, real amplifiers can not have an infinitely steep high frequency roll off, so a gradual decay of high frequency response is always ensured. However, as we will explain in Part 2 and 4, the overshoot may increase as a result of other effects.

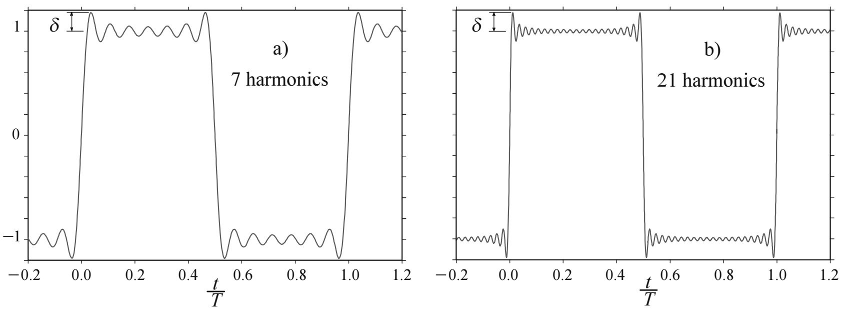


Fig. 1.2.7: The Gibbs' phenomenon; **a)** A signal composed of the first seven harmonics of a square wave spectrum from Table 1.2.1. The overshoot is $\delta \simeq 18\,\%$ of the nominal amplitude; **b)** Even if we take three times more harmonics the overshoot δ is nearly equal in both cases.

In a similar way to that for the square wave, **any periodic signal of finite amplitude and with a finite number of discontinuities within one period, can be decomposed into its frequency components.** As an example the waveform in Fig. 1.2.8 could also be decomposed, but we will not do it here. Instead in the following section we will analyze another waveform which will allow us to generalize the method of frequency analysis.

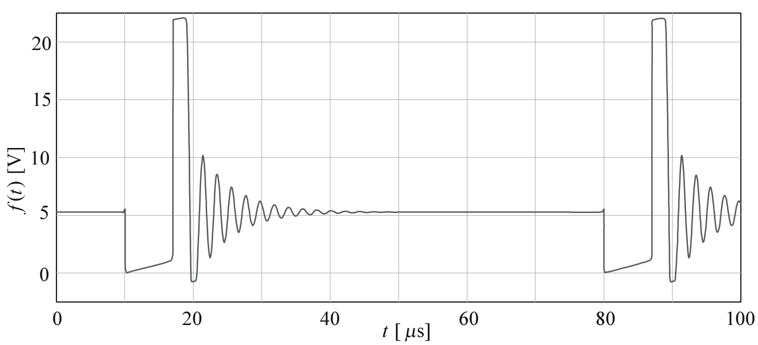


Fig. 1.2.8: An example of a periodic waveform (a typical flyback switching power supply), having a finite number of discontinuities within one period. Its frequency spectrum can also be calculated using the Fourier transform, if needed (e.g., to analyze the possibility of electromagnetic interference at various frequencies), in the same way as we did for the square wave.

1.3 The Fourier Integral

Suppose we have a function $f(t)$ composed of square waves with the duration τ and repeating with a period T, as shown in Fig. 1.3.1. For this function we can also calculate the Fourier series (the corresponding spectrum is shown in Fig. 1.3.2) in the same way as for the continuous square wave case in the previous section.

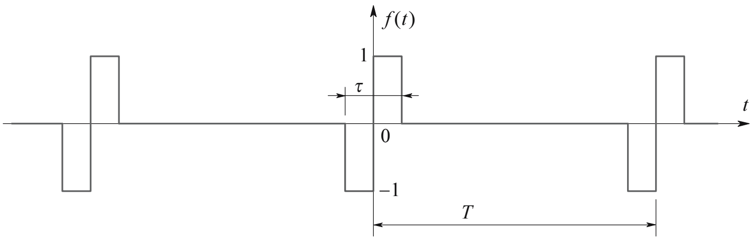


Fig. 1.3.1: A square wave with duration τ and period $T = 5\tau$.

The difference between the continuous square wave spectrum and the spaced square wave in Fig. 1.3.1 is that the integral of this function can be broken into two parts, one comprising the length of the pulse, τ, and the zero-valued part between two pulses of a length $T - \tau$. The reader can do this integration for himself, because it is fairly simple. We will only write the result:

$$\frac{A_n}{2} = -j\,\tau\,\frac{\sin^2\left[\,n\,\omega_1(\tau/4)\,\right]}{n\,\omega_1(\tau/4)} \tag{1.3.1}$$

where $\omega_1 = 2\pi\,/T$, assuming that the pulse amplitude is 1 (if the amplitude were A it would simply multiply the right hand side of the equation). For the conditions in Fig. 1.3.1, where $T = 5\tau$ and $A = 1$, the spectrum has the form shown in Fig. 1.3.2, with $\omega_\tau = 2\pi/\tau$.

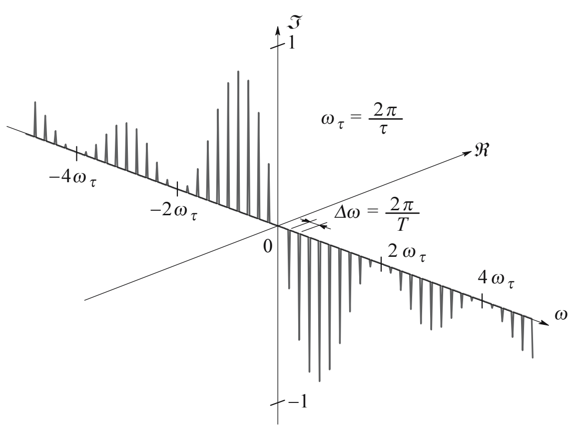


Fig. 1.3.2: Complex spectrum of the waveform in Fig. 1.3.1.

A very interesting question is that of what would happen to the spectrum if we let the period $T \to \infty$? In general a function $f(t)$ can be recomposed by adding all its harmonic components:

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{A_n}{2} \, \mathrm{e}^{j n \omega_1 t} \tag{1.3.2}$$

where A_n may also be complex, thus containing the initial phase angle φ_{i}. Again, as in the previous section, each discrete harmonic component can be calculated with the integral:

$$\frac{A_n}{2} = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \, \mathrm{e}^{-j n \omega_1 t} \, dt \tag{1.3.3}$$

For the case in Fig. 1.3.1 the integration should start at $t = 0$ and the integral has the form:

$$\frac{A_n}{2} = \frac{1}{T} \int_0^T f(t) \, \mathrm{e}^{-j n \omega_1 t} \, dt \tag{1.3.4}$$

Insert this into Eq. 1.3.2:

$$f(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} \int_0^T f(\tau) \, \mathrm{e}^{-j n \omega_1 \tau} \, d\tau \right] \mathrm{e}^{j n \omega_1 t} \tag{1.3.5}$$

Here we have introduced a dummy variable τ in the integral, in order to distinguish it from the variable t outside the brackets. Now we express the integral inside the brackets as:

$$\int_0^T f(\tau) \, \mathrm{e}^{-j n \omega_1 \tau} \, d\tau = \int_0^T f(\tau) \, \mathrm{e}^{-j 2 \pi n \tau / T} \, d\tau = F\left(\frac{2 n \pi}{T} \right) = F(n \omega_1) \tag{1.3.6}$$

Thus:

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} \frac{1}{T} \, F(n \omega_1) \, \mathrm{e}^{j n \omega_1 t} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{2\pi}{T} \, F(n \omega_1) \, \mathrm{e}^{j \omega_1 t} \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \omega_1 F(n \omega_1) \, \mathrm{e}^{j \omega_1 t} \end{aligned} \tag{1.3.7}$$

where $2\pi/T = \omega_1$. If we let $T \to \infty$ then ω_1 becomes infinitesimal, and we call it $d\omega$. Also $n\omega_1$ becomes a continuous variable ω. So in Eq. 1.3.7 the following changes take place:

$$\sum_{n=-\infty}^{\infty} \Rightarrow \int_{-\infty}^{\infty} \qquad\qquad \omega_1 \Rightarrow d\omega \qquad\qquad n \omega_1 \Rightarrow \omega$$

With all these changes Eq. 1.3.7 is transformed into Eq. 1.3.8:

$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \, \mathrm{e}^{j \omega t} \, d\omega} \tag{1.3.8}$$

Consequently Eq. 1.3.6 also changes, obtaining the form:

$$\boxed{F(\omega) = \int_0^{\infty} f(t)\, \mathrm{e}^{-j\omega t}\, dt} \tag{1.3.9}$$

In Eq. 1.3.9 $F(\omega)$ has no discrete frequency components but it forms a **continuous** spectrum. Since $T \to \infty$ the DC part vanishes (as it would for **any** pulse shape, not just symmetrical shapes), according to Eq. 1.2.6:

$$A_0 = \lim_{T\to\infty} \frac{1}{T} \int_0^{T} f(t)\, dt = 0 \tag{1.3.10}$$

Eq. 1.3.8 and 1.3.9 are called *Fourier integrals*. Under certain (usually rather limited) conditions, which we will discuss later, it is possible to use them for the calculation of transient phenomena. The second integral (Eq. 1.3.9) is called the *direct Fourier transform*, which we express in a shorter way:

$$\mathcal{F}\{f(t)\} = F(\omega) \tag{1.3.11}$$

The first integral (Eq. 1.3.8) represents the *inverse Fourier transform* and it is usually written as:

$$\mathcal{F}^{-1}\{F(\omega)\} = f(t) \tag{1.3.12}$$

In Eq. 1.3.9, $F(\omega)$ means a **firm** spectrum and the factor $\mathrm{e}^{j\omega t}$ means the rotation of each of the corresponding infinite spectrum components contained in $F(\omega)$ with its angular frequency ω, which is a continuous variable. In Eq. 1.3.8 $f(t)$ means the complete time function, containing an infinite number of **rotating** phasors and the factor $\mathrm{e}^{-j\omega t}$ means the rotation 'in the opposite direction' to stop the rotation of the corresponding rotating phasor $\mathrm{e}^{j\omega t}$ contained in $f(t)$, at its particular frequency ω.

Let us now select a suitable time function $f(t)$ and calculate its continuous spectrum. Since we have already calculated the spectrum of a periodic square wave, it would be interesting to display the spectrum of a single square wave as shown in Fig. 1.3.3b. We use Eq. 1.3.9:

$$F(\omega) = \int_{-\tau/2}^{\infty} f(t)\, \mathrm{e}^{-j\omega t}\, dt = \int_{-\tau/2}^{0} (-1)\, \mathrm{e}^{-j\omega t}\, dt + \int_0^{\tau/2} (+1)\, \mathrm{e}^{-j\omega t}\, dt \tag{1.3.13}$$

Here we have a single square wave with a 'period' T from $t = -\tau/2$ to $+\infty$. However, we need to integrate only from $t = -\tau/2$ to $t = \tau/2$, because $f(t)$ is zero outside this interval. It is important to note that at the discontinuity where $t = 0$, we have started the second integral. For a function with more discontinuities, between each of them we must write a separate integral. Thus it is obvious that the function $f(t)$ must have a **finite** number of discontinuities for it to be possible to calculate its spectrum.

The result of the above integration is:

$$F(\omega) = \frac{1}{-j\omega}\left(-1 + \mathrm{e}^{j\omega\tau/2} + \mathrm{e}^{-j\omega\tau/2} - 1\right) = \frac{2}{j\omega}\left(1 - \frac{\mathrm{e}^{j\omega\tau/2} + \mathrm{e}^{-j\omega\tau/2}}{2}\right)$$

$$= \frac{-2j}{\omega}\left(1 - \cos\frac{\omega\tau}{2}\right) = \frac{-2j}{\omega}\left(2\sin^2\frac{\omega\tau}{4}\right) = \frac{-4j}{\omega}\sin^2\frac{\omega\tau}{4}$$

$$= -j\tau\,\frac{\sin^2\dfrac{\omega\tau}{4}}{\dfrac{\omega\tau}{4}} \tag{1.3.14}$$

A three-dimensional display of a spectrum, corresponding to this result, is shown in Fig. 1.3.3a. Here the frequency scale has been altered with respect to Fig. 1.2.1a in order to display the spectrum better.

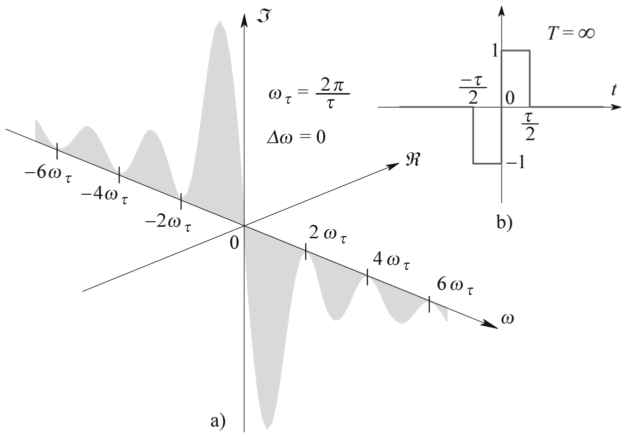


Fig. 1.3.3: a) The frequency spectrum of a single square wave is expressed by complex conjugate phasors. Since the phasors are infinitely many, they merge in a continuous planar form. Also the spectrum extends to $\omega = \pm\infty$. The corresponding waveform is shown in **b)**. Note that all the even frequency components $2\pi n/T$ are missing (n is an integer).

By comparing Fig. 1.2.1a and 1.3.3a we may draw the following conclusions:

1. Both spectra contain no even frequency components, e.g., at $\pm 2\omega_\tau$, $\pm 4\omega_\tau$, etc., where $\omega_\tau = 2\pi/\tau$;

2. In both spectra there is no DC component A_0;

3. By comparing Fig. 1.3.2 and 1.3.3 we note that the envelope of both spectra can be expressed by Eq. 1.3.14;

4. By comparing Eq. 1.3.1 and 1.3.14 we note that the discrete frequency $n\,\omega_1$ from the first equation is replaced by the continuous variable ω in the second equation. Everything else has remained the same.

In the above example we have decomposed an aperiodic waveform (also called a *transient*), expressed as $f(t)$, into a **continuous complex** spectrum $F(\omega)$. Before discussing the functions which are suitable for the application of the Fourier integral let us see some common periodic and non-periodic signals. A sustained tone from a trumpet we consider to be a periodic signal, whilst a beat on a drum is a non-periodic signal (in a strict mathematical sense, both signals are non-periodic, because the first one also started out of silence). The transition from silence to sound we call the *transient*. In accordance with this definition, of the waveforms in Fig. 1.3.4 only a) and b) show a periodic waveform, whilst c) and d) display transients.

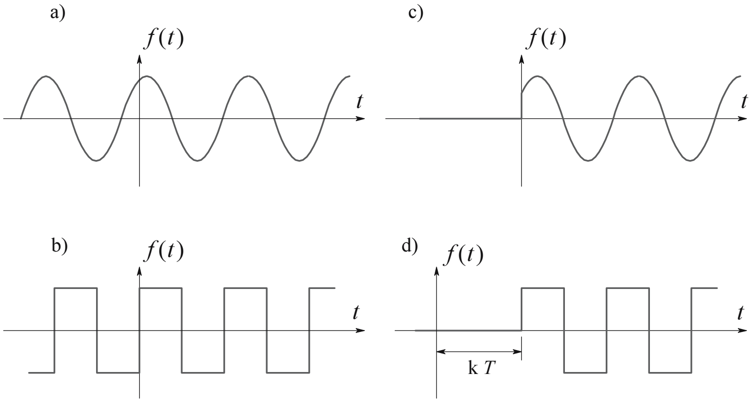


Fig. 1.3.4: a) and **b)** periodic functions, **c)** and **d)** aperiodic functions.

The question arises of whether it is possible to calculate the spectra of the transients in Fig. 1.3.4c and 1.3.4d by means of the Fourier integral using Eq. 1.3.8?

The answer is **no**, because the integral in Eq. 1.3.8 does not converge for any of these two functions. The integral is also non-convergent for the most simple step signal, which we intend to use extensively for the calculation of the step response of amplifier networks.

This inconvenience can be avoided if we multiply the function $f(t)$ by a suitable convergence factor, e.g., e^{-ct}, where $c > 0$ and its magnitude is selected so that the integral in Eq. 1.3.2 remais finite when $t \rightarrow \infty$. In this way, the problem is solved for $t \geq 0$. In doing so, however, the integral becomes divergent for $t < 0$, because for negative time the factor e^{-ct} has a positive exponent, causing a rapid increase to infinity. But this, too, can be avoided, if we assume that the function $f(t)$ is zero for $t < 0$. In electrical engineering and electronics we can always assume that a circuit is dead until we switch the power on or we apply a step voltage signal to its input and thus generate a transient. The transform where $f(t)$ must be zero for $t < 0$ is called a *unilateral transform*.

For functions which are suitable for the unilateral Fourier transform the following relation must hold [Ref. 1.3]:

$$\lim_{T \rightarrow \infty} \int_0^T |f(t)|\, \mathrm{e}^{-ct} dt \; < \; \infty \tag{1.3.15}$$

where $f(t)$ is a single-valued function of t and c is positive and real.