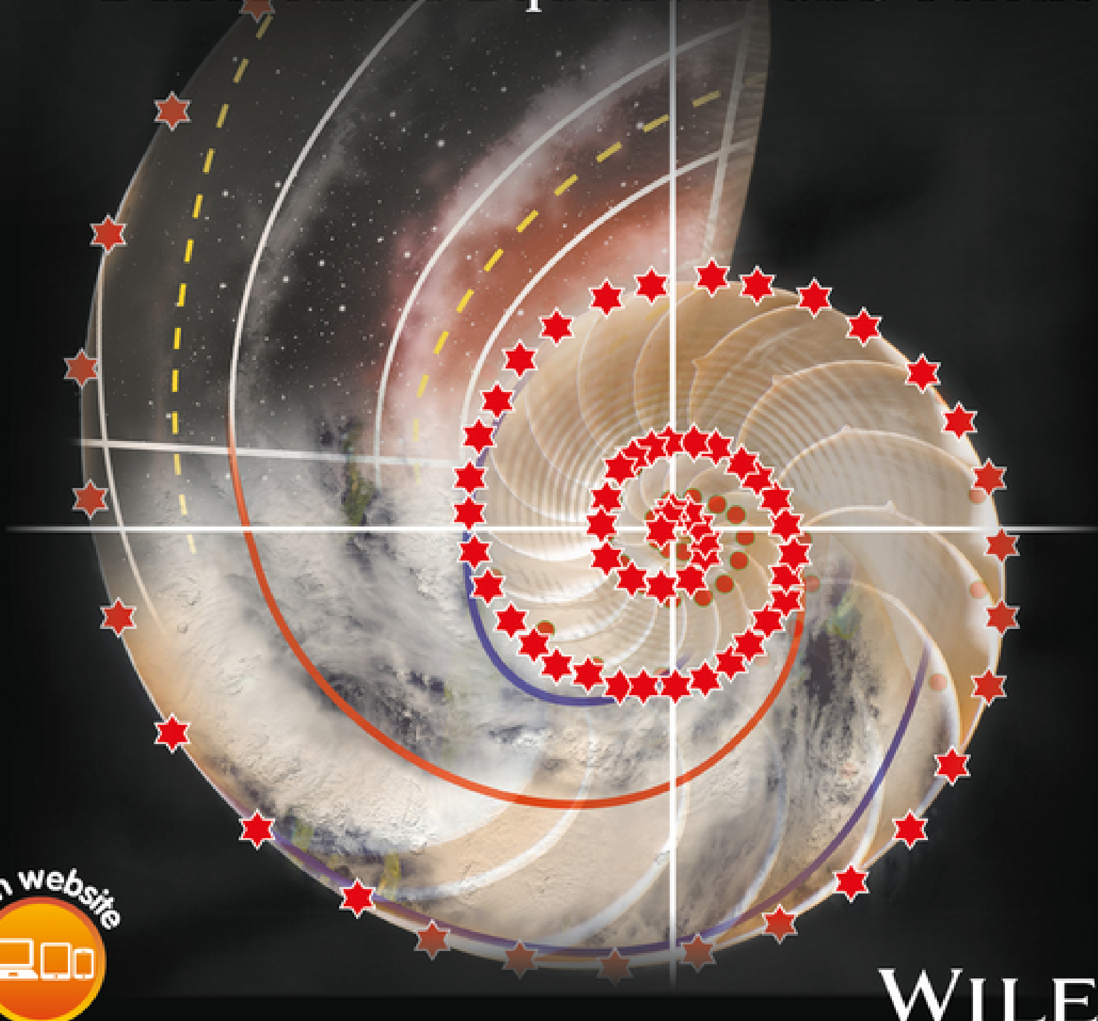
CARL F. LORENZO AND TOM T. HARTLEY

The Fractional Trigonometry

With Applications to Fractional Differential Equations and Science



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Preface

There has been a strong resurgence of interest in the fractional calculus over the last two or three decades. This expansion of the classical calculus to derivatives and integrals of fractional order has given rise to the hope of a new understanding of the behavior of the physical world. The hope is that problems that have resisted solution by the integer-order calculus will yield to this greatly expanded capability. As a result of our work in the fractional calculus, and more particularly, in functions for the solutions of fractional differential equations, an interest was fostered in the behavior of generalized exponential functions for this application. Our work with the fundamental fractional differential equation had developed a function we named the F-function. This function, which had previously been mentioned in a footnote by Oldham and Spanier, acts as the fractional exponential function. It was a natural step from there to an interest in a fractional trigonometry. At that time, only a few pages of work were available in the literature and were based on the Mittag-Leffler function. These are shown in Appendix A.

This book brings together our research in this area over the past 15 years and adds much new unpublished material.

The classical trigonometry plays a very important role relative to the integer-order calculus; that is, it, together with the common exponential function, provides solutions for linear differential equations. We will find that the fractional trigonometry plays an analogous role relative to the fractional calculus by providing solutions to linear fractional differential equations. The importance of the classical trigonometry goes far beyond the solutions of triangles. Its use in Fourier integrals, Fourier series, signal processing, harmonic analysis, and more provided great motivation for the development of a fractional trigonometry to expand application to the fractional calculus domain.

Because we are engineers, this book has been written in the style of the engineering mathematical books rather than the more rigorous and compact style of definition, theorem, and proof, found in most mathematical texts. We, of course, have made every effort to assure the derivations to be correct and are hopeful that the style has made the material accessible to a larger audience. We are also hopeful that this will not detract the interest of the mathematical community in the area since their skills will be needed to develop this important new area. Most of the materials of this book should be accessible to an undergraduate student with a background in Laplace transforms.

After an introductory chapter, which offers a brief insight into the fractional calculus, the book is organized in two major parts. In Chapters 2–11, the definitions and theory of the fractional exponential and the fractional trigonometry are developed. Chapters 12–19 provide insight into various areas of potential application.

Chapter 2 develops the F-function from consideration of the fundamental fractional differential equation. It generalizes the common exponential function for application in the fractional calculus. The F-function, the fractional eigenfunction, together with its generalization, the

R-function (Chapter 3), will later form the theoretical basis of the fractional trigonometry. Properties of these functions are developed in these two chapters. Their relationship to other functions for the fractional calculus is presented. An important characteristic of the R-function is that it contains the F-function as a special case and also contains its derivatives and integrals. In later chapters, it is shown that many of the newly defined fractional trigonometric functions inherit this important property. Chapter 4 further develops properties of the R-function that expose the character of this fractional exponential function.

In Chapter 5, the R-function, $R_{q,v}(a,t)$, with real arguments for a and t, is used to define the fractional hyperbolic functions. These functions generalize the classical hyperbolic functions. Fractional exponential forms of the hyperbolic functions are derived as well as their Laplace transforms. Furthermore, fractional differintegrals of the functions are determined. An example demonstrates the use of the Laplace transform in the solution of fractional hyperbolic differential equations. The fractional hyperbolic functions are found to be closely related to the R_1-trigonometric functions defined in Chapter 6.

Chapters 6–8 present three fractional trigonometries. We have tried to make each of these chapters as stand-alone developments, at the expense of minor repetition. Chapter 6 develops the R_1-trigonometry. It is based on the R-function with imaginary parameter a, namely $R_{q,v}(ia,t)$. Multiplication of the parameter by i toggles the R_1-hyperbolic functions to the R_1-trigonometric functions, and so on.

A fractional trigonometry, the R_2-trigonometry based on an imaginary time variable, $R_{q,v}(a,it)$, is developed in Chapter 7. It is found that these functions are characterized by their attraction to circles when plotted in phase plane format.

The obvious extension of these two trigonometries, the R_3-trigonometry of Chapter 8, sets both the a parameter and the t variable to be imaginary, $R_{q,v}(ia,it)$. It was thought at the time that this trigonometry would behave as an hyperbolic analog to the R_2-trigonometry. However, such simple relationships between the two were not found.

Chapter 9 presents the heart of the fractional trigonometry, namely the fractional meta-trigonometry. Here, both a and t are allowed to be fully complex, by choosing as the basis $R_{q,v}(i^\alpha a, i^\beta t)$. This chapter generalizes the results of the previous four chapters. Laplace transforms for the generalized functions are determined along with their fractional differintegrals. Fractional exponential forms for the functions are also determined.

In Chapter 10, the ratio and reciprocal functions associated with the generalized fractional sines and cosines of Chapter 9, that is, $Sin_{q,v}(a,\alpha,\beta,k,t)$ and $Cos_{q,v}(a,\alpha,\beta,k,t)$, as well as the generalized parity functions are considered. The parity functions represent a new set of fractional trigonometric functions with no counterpart in the classical trigonometry. Because of the large number of possible ratio and reciprocal functions, the treatment in this chapter is cursory.

Because of the newness of this material, we have tried to be generous with the graphic forms of the functions. In spite of this attitude, we have found that because of different behavior over various domains of the functions and the number of parameters in the functions that complete coverage in this regard to be impossible and the reader is encouraged to program some of the functions and to experiment for themselves.

In Chapter 3, two new functions, the G- and H-functions, are introduced. These functions are generalizations of the R-function with multiple real and complex roots in the denominators of their Laplace transforms. Because of the great generality of these functions, consideration of these functions as the basis for further generalization of the fractional trigonometry is discussed in Chapter 11. In Chapter 12, these functions are needed for the solution of linear fractional differential equations with repeated roots.

Part II of the book is largely dedicated to applications and potential application of the fractional trigonometry.

The most important application is the use of the fractional trigonometry for the solution of linear constant-coefficient commensurate-order fractional differential equations. In Chapter 12, specialized Laplace transforms for the meta-trigonometric functions are developed and applied to the solution of these linear fractional differential equations. Examples showing the solution of fractional differential equations with unrepeated roots and with repeated real and complex roots are given.

Chapter 13 studies the time- and frequency-domain responses for linear fractional systems based on the R-function and the meta-trigonometric functions. The stability of the basic fractional elements is also considered.

Unlike the classical trigonometric functions, the fractional counterparts do not generally share the periodicity property. As a practical result, we are limited to evaluation of the defining infinite series for function evaluation. This presents numerical difficulties as the time and/or order variables increase. Chapter 14 discusses this problem and establishes series convergence.

Phase plane plots of pairs of the fractional trigonometric functions define a new and unique family of spirals, the *fractional* spirals. Chapter 15 studies these spirals and their relationship to some of the classical spirals.

Linear oscillators are often used in the study of ordinary differential equations and in the modeling of physical systems. Chapter 16 identifies those linear fractional trigonometric oscillators that are neutrally stable. This chapter also explores possible application of coupled fractional trigonometric oscillators.

Chapters 17 – 19 study the possible application of the fractional spirals and thus the fractional trigonometry and fractional differential equations. The potential applications include sea shell growth and morphology, mathematical classification of spiral galaxy morphology, and various weather phenomena such as hurricanes and tornados.

Finally, Chapter 20 looks at some of the many remaining challenges and opportunities relative to the fractional exponential function and the fractional trigonometry, in particular, the need for a fractional field equation as it relates to spatial fractional spiral morphology.

For the professional with a background in the fractional calculus, a quick coverage of the essence of the book may be had from Chapters 2, 3, 9, and 12, with selected attention to the applications of interest contained in Chapters 15 – 19.

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Cleveland OH,
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About the Companion Website

This book is accompanied by a companion website:

www.wiley.com/go/Lorenzo/Fractional_Trigonometry

The website includes:

- Figures from the book appearing in color.

1

Introduction

The ongoing development of the fractional calculus and the related development of fractional differential equations have created new opportunities and new challenges. The need for a generalized exponential function applicable to fractional-order differential equations has given rise to new functions. In the traditional integer-order calculus, the role of the exponential function and the trigonometric functions is central to the solution of linear ordinary differential equations. Such a supporting structure is also needed for the fractional calculus and fractional differential equations.

The purpose of this book is the development of the fractional trigonometries and hyperboletries that generalize the traditional trigonometric and hyperbolic functions based on generalizations of the common exponential function. The fundamental idea is that through the development of the fractional calculus, which generalizes the integer-order calculus, generalizations of the exponential function have been developed. The exponential function in the integer-order calculus provides the basis for the solution of linear fractional differential equations. Also, it may be thought of as the basis of the trigonometry.

A high-level summary of the flow of the development of the book is given in Figure 1.1. The generalized exponential functions that we use, the F-function and the R-function, are fractional eigenfunctions; that is, they return themselves on fractional differintegration. The F-function is the solution to the fundamental fractional differential equation

$$_0d_t^q x(t) + ax(t) = \delta(t)$$

when driven by a unit impulse. The R-function, $R_{q,\nu}(a,t)$, generalizes the F-function by including its integrals and derivatives as well. First, we show that these functions provide solutions to the fundamental fractional-order differential equation. Then, we explore the properties of the generalized exponential functions and develop some properties of the functions that will aid in the development and understanding of the fractional trigonometries and hyperboletries. This development follows a few mathematical preliminaries.

The R_1, R_2, and R_3 trigonometries along with the R_1 hyperboletry are developed by replacing a and t in the R-function with various combinations of real and purely imaginary variables. Based on the newly defined functions, a variety of basic properties and identities are determined. Furthermore, the Laplace transforms of the functions are determined and the fractional derivatives and fractional integrals of the functions elucidated.

The following chapters develop an overarching fractional trigonometry called the fractional meta-trigonometry that contains all of the fractional trigonometries shown in Figure 1.1 and infinitely many more. This is accomplished by replacing a and t in the R-function with general complex variables. We find that the fractional trigonometric functions lead to a generalization

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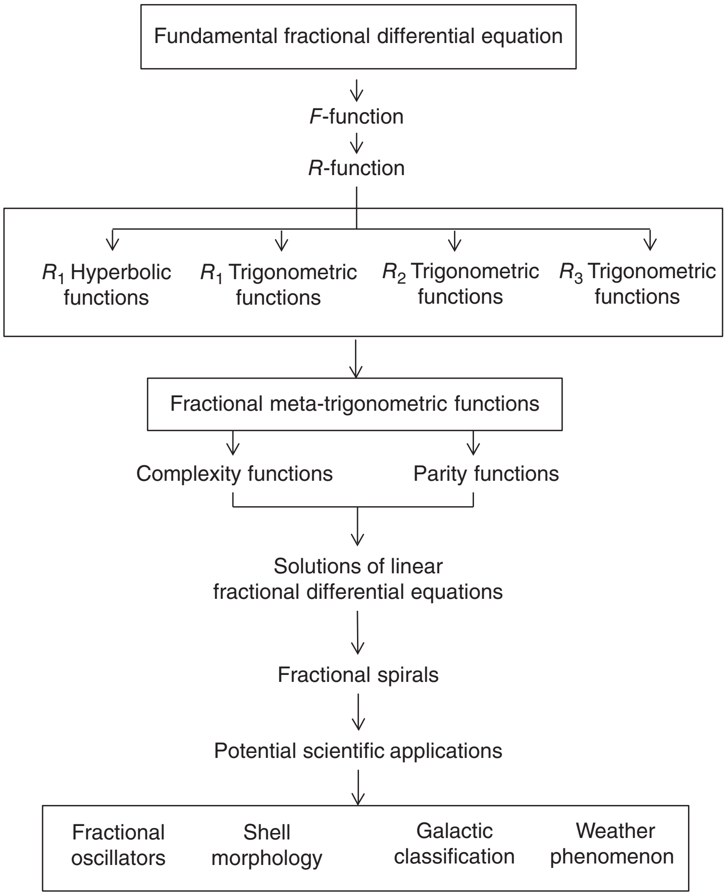

Figure 1.1 Overview of the development of the fractional trigonometry and its applications.

of the circular functions, which we have called the *fractional spiral functions*. These functions appear to model various natural phenomena, and preliminary applications of these functions to the properties of fractional oscillators, sea shells, galaxies, and more are explored. An important aspect of this modeling is that we can infer from the spiral functions the underlying fractional differential equations describing the phenomena, which is demonstrated. More importantly, the new fractional functions provide the solutions to classes of linear fractional differential equations.

1.1 Background

Because of the close association of the fractional calculus and the fractional trigonometry to be developed, we present here a brief introduction to the concepts of the fractional calculus for the reader who is unfamiliar with the area.

Several important textbooks have been written that are extremely helpful to someone entering the field. Perhaps the most useful from the engineering and scientific viewpoint, are the textbooks by Oldham and Spanier, "*The Fractional Calculus*" [104], and by Igor Podluby entitled "*Fractional Differential Equations*" [109]. A more mathematically oriented treatment is given in the book by Miller and Ross [95]. An encyclopedic reference volume written by Samko et al. [114] has also been published. Furthermore, a very good engineering book has been written by Oustaloup [105] and is available in French and Bush [19].

There are a growing number of physical systems whose behavior can be compactly described using fractional differential equations theory. Areas include long lines, electrochemical processes, diffusion, dielectric polarization, noise, viscoelasticity, chaos, creep, rheology, capacitors, batteries, heat conduction, percolation, cylindrical waves, cylindrical diffusion, water through a weir notch, Boussinesq shallow water waves, financial systems, biological systems, semiconductors, control systems, electrical machinery, and more.

1.2 The Fractional Integral and Derivative

The first question we need to address is "just what is a fractional derivative?" There are two separate but equivalent definitions of the fractional differintegral (Oldham and Spanier [104]), known as the Grünwald definition and the Riemann–Liouville definition. We present the Grünwald definition first, as it most recognizably generalizes the standard calculus. We then follow with the Riemann–Liouville definition as it is most easily used in practice.

1.2.1 Grünwald Definition

The Grünwald definition of the fractional-order differintegral is essentially a generalization of the derivative definition that most of us learned in introductory calculus, namely

$$\left.\frac{d^q f(t)}{[d(t-a)]^q}\right|_{GRUN} \equiv \lim_{N\to\infty} \frac{\left(\frac{t-a}{N}\right)^{-q}}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f\left(t - j\left(\frac{t-a}{N}\right)\right), \tag{1.1}$$

or in a slightly more familiar form

$$\left.\frac{d^q f(t)}{[d(t-a)]^q}\right|_{GRUN} \equiv \lim_{N\to\infty} \sum_{j=0}^{N-1} b_j(q) \frac{f(t - j\Delta t_N)}{(\Delta t_N)^q}, \tag{1.2}$$

where

$$\Delta t_N = \frac{t-a}{N}, \quad b_j(q) = \frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}.$$

In this definition, q is not limited to the integers and may be any real or complex number, and a is the starting time of the fractional differintegration, not to be confused with a in the differential equation in the introduction. Also, $q > 0$ defines differentiation, and $q < 0$ integration. Furthermore, $\Gamma(\circ)$ is the gamma function, or the generalized factorial function. It basically acts as a calibration constant here to properly interpolate the operators for values of q between the integers. In terms of notation, Oldham and Spanier [104] provide a development of equation (1.2) and generalize the fractional differintegral as

$$\frac{d^q x(t)}{[d(t-a)]^q}, \tag{1.3}$$

where it should be noticed that the differential in the denominator starts at some time a, and ends at a final time t. Thus, we see that the fractional derivative is defined on an interval and is no longer a local operator except for integer orders. Interestingly, the gamma functions force the series to terminate with a finite number of terms whenever q is any integer greater than or equal to zero, which represent the usual integer-order derivatives. When q is a negative integer, this series also contains the single and multiple integrals as well (which have always had infinite memory). The important aspect to be recognized is that there exists an uncountable infinity of fractional derivatives and integrals between the integers. The Grünwald definition is also equivalent to the more often used Riemann – Liouville definition, which is discussed in the following section.

1.2.2 Riemann – Liouville Definition

The Riemann – Liouville definition is easiest to present for fractional integrals first, and then generalize that to the fractional derivatives. The qth-order integral is defined as (see, e.g., Oldham and Spanier [104], Podlubny [109])

$$ {}_a d_t^{-q} x(t) \equiv \frac{d^{-q} x(t)}{[d(t-a)]^{-q}} \equiv \int_a^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau, \quad t \geq a, \tag{1.4} $$

It is important to note that this is the *key definition* of the fractional integral and is used by most investigators. Miller and Ross [95] provide four separate developments of this important equation. It can be shown that whenever q is a positive integer, this equation becomes a standard integer-order multiple integral. The Riemann – Liouville fractional derivative is defined as the integer-order derivative of a fractional integral

$$ {}_a d_t^{q} x(t) \equiv \frac{d^m}{dt^m} \left({}_a d_t^{q-m} x(t) \right), \quad t \geq a, \tag{1.5} $$

where m is typically chosen as the smallest integer such that $q-m$ is negative, and the integer-order derivatives are those as defined in the traditional calculus. These equations define the *uninitialized* fractional integral and derivative.

For most engineering problems, system components, by virtue of their histories, are placed into some initial configuration or are initialized. Using mechanical systems as an example, the initial conditions are often mass positions and velocities at time zero. Fractional-order components, however, require a time-varying initialization Lorenzo [77] and Hartley [85], as they inherently have a fading infinite memory. Considering the aforementioned fractional-order integral, we assume that the fractional-order integration was started in the past, beginning at some time a, while the given problem begins at time $c > a$, where c is usually taken to be zero. Consider two fractional integrals of the same order acting on $x(t)$, where $x(t)$ and all of its derivatives are zero for all $t < a$. If the integral starting at c is to continue the integral starting at a, we must add an initialization ψ, thus

$$ \begin{aligned} {}_a d_t^{-q} x(t) = {}_c d_t^{-q} x(t) + \psi \Rightarrow \int_a^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau \\ = \int_c^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau + \psi, \quad t \geq c, \quad q > 0, \end{aligned} \tag{1.6} $$

therefore

$$ \psi = \int_a^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau - \int_c^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau = \int_a^c \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau, \quad t \geq c,\ q > 0, \tag{1.7} $$

clearly a time-varying function. This term represents the historical effect (Lorenzo and Hartley [68, 71]) or the initialization required for the fractional integral. The *initialized* fractional-order integration operator then is defined as

$$ {}_cD_t^{-q}x(t) \equiv {}_cd_t^{-q}x(t) + \psi(x_i, -q, a, c, t), \quad t \geq c, \tag{1.8}$$

where

$$\psi(x_i, -q, a, c, t) \equiv \int_a^c \frac{(t-\tau)^{q-1}}{\Gamma(q)} x_i(\tau)\, d\tau, \quad t \geq c. \tag{1.9}$$

$\psi(x_i, -q, a, c, t)$ is called the initialization function and is generally a time-varying function that must be added to the fractional-order operator to account for the effects of the past. This is a generalization of the constant of integration that is usually added to the normal order-one integral. The subscript i is appended to x to indicate that x_i is not necessarily the same as x. Clearly then, ${}_cD_t^{-q}x(t) = {}_ad_t^{-q}x(t), \quad t \geq c$. The initialization function is a time-varying function and is required to properly bring the historical effects of the fractional-order integral into the future. Similar considerations also apply for fractional-order derivatives [68, 71], that is, for any real value of q. Again, for convenience, $c = 0$ is typically chosen.

1.2.3 The Nature of the Fractional-Order Operator

The important properties of integer-order integration and differentiation have been shown to hold for initialized fractional-order operators (Lorenzo and Hartley [68] and [71]), including linearity and the index law. Physical insight into the nature of the fractional operators may be found in Hartley and Lorenzo [44, 47]. The fractional differintegral operator is a linear operator, and all the properties associated with linear operators hold for them. Also of considerable importance is the index law; that is, ${}_ad_t^{u+v}x(t) = {}_ad_t^u\, {}_ad_t^v x(t)$. The index law essentially allows us to state, for example, that the half-derivative of the half-derivative of a function is the same as the first-derivative of that function.

Laplace transforms are standard tools for integer-order operators and can still be readily used for fractional-order operators. In this regard, the Laplace transform of the initialized fractional-order differintegral is shown in Lorenzo and Hartley [68, 71] to be

$$L\left\{{}_0D_t^q x(t)\right\} = L\left\{{}_0d_t^q x(t) + \psi(x, q, a, 0, t)\right\} = s^q X(s) + L\left\{\psi(x, q, a, 0, t)\right\} \quad \text{for all real } q. \tag{1.10}$$

It is important to note that $L\{{}_0d_t^q x(t)\} = s^q X(s)$, for all q, as this is the generalization of the derivative rule for the integer-order situation. Also, note that $L^{-1}\{s^{-q}\} = t^{q-1}/\Gamma(q), q > 0$, which leads directly to the Riemann–Liouville definition via convolution

$${}_0d_t^{-q}x(t) \Leftrightarrow s^{-q}X(s) \;\Leftrightarrow\; \int_0^t \frac{(t)^{q-1}}{\Gamma(q)} x(t-\tau)\, d\tau = \int_0^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau. \tag{1.11}$$

The Laplace transform for the fractional integral is given [78] as

$$\begin{aligned} L\left\{{}_0D_t^{-q}h(t)\right\} &= L\left\{{}_0d_t^{-q}f(t)\right\} + L\left\{\psi\,(f_i, -q, -a, 0, t)\right\} \\ &= \frac{1}{s^q}L\{f(t)\} + \frac{1}{s^q\Gamma(q)}\int_{-a}^{0} e^{-\tau s}\Gamma(q+1, -\tau s) f_i(\tau)d\tau. \quad q \geq 0, \end{aligned} \tag{1.12}$$

where $q \geq 0$ and

$$h(t) = \begin{cases} f_i(t) & -a < t \leq 0, \\ f(t) & 0 < t, \end{cases}$$

and where f_i may differ from f during the initialization period. More detailed forms are presented in Ref. [78].

The transform for the fractional derivative of order u, where $u = n - q$, is given by

$$L\left\{{}_0D_t^u f(t)\right\} = s^{n-q} L\left\{f(t)\right\} - \sum_{j=0}^{n-1} s^{n-1-j} \psi^{(j)}(f_i, -q, -a, 0, t)|_{t=0+} + s^n L\left\{\psi(f_i, -q, -a, 0, t)\right\}, \tag{1.13}$$

where $u = n - q \geq 0$, $n = 1, 2, 3, \ldots$, $q \geq 0$, $f_i(t) = 0, \forall t < -a$, and

$$s^n L\{\psi(f_i, -q, -a, 0, t)\} = \frac{s^{n-q-1}}{\Gamma(q+1)} \left[e^{as}\Gamma(q+1, as) f_i(-a) - \Gamma(q+1) f_i(0) + \int_{-a}^{0} e^{-\tau s}\, \Gamma(q+1, -\tau s) f_i'(\tau) d\tau \right]. \tag{1.14}$$

In this relationship, $\psi(f_i, -q, -a, 0, t)$ is the initialization function for the fractional integral part of the operator. An alternative form of equation (1.14) where the integration is based on $f_i(t)$ rather than $f_i'(t)$ is given by

$$L\left\{{}_0D_t^u f(t)\right\} = s^{n-q} L\{f(t)\} - \sum_{j=0}^{n-1} s^{n-1-j} \psi^{(j)}(f_i, -q, -a, 0, t)|_{t=0+} + \frac{s^{n-q}}{\Gamma(q)} \int_{-a}^{0} e^{-\tau s} \Gamma(q, -\tau s) f_i(\tau) d\tau, \tag{1.15}$$

where $u = n - q \geq 0, \quad n = 1, 2, 3, \ldots, \quad q \geq 0, f_i(t) = 0, \quad \forall t < -a.$

These forms simplify for constant initialization [78], that is, when f_i = constant = b,

$$L\{{}_0D_t^u f(t)\} = s^{n-q} L\{f(t)\} + b s^{n-q-1} \left[\frac{e^{as}\Gamma(q-n+1, as)}{\Gamma(q-n+1)} - 1 \right],$$
$$q \text{ not integer}, \quad 0 \leq u = (n-q) \leq n, \quad n = 1, 2, 3, \ldots. \tag{1.16}$$

1.3 The Traditional Trigonometry

The application of the traditional integer-order trigonometry to analysis as well as engineering and science goes well beyond the calculation of triangles and triangulation. The applications include Fourier analysis, spectral analysis, solutions to ordinary and partial differential equations, and more. The trigonometric functions are found in nearly every branch of mathematics. The traditional trigonometry was originated for the solution of plane triangles. However, an additional way of interpreting the integer-order trigonometry is based on its relationship to the exponential function. The connections between the trigonometric functions and the exponential functions are very close. These relationships center on the Euler equation; that is, for $z = x + iy$

$$e^z = e^x e^{iy} = e^x(\cos y + i \sin y), \tag{1.17}$$

as well as the definitions

$$\cos(t) \equiv \frac{e^{it} + e^{-it}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \tag{1.18}$$

and

$$\sin(t) \equiv \frac{e^{it} - e^{-it}}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \tag{1.19}$$