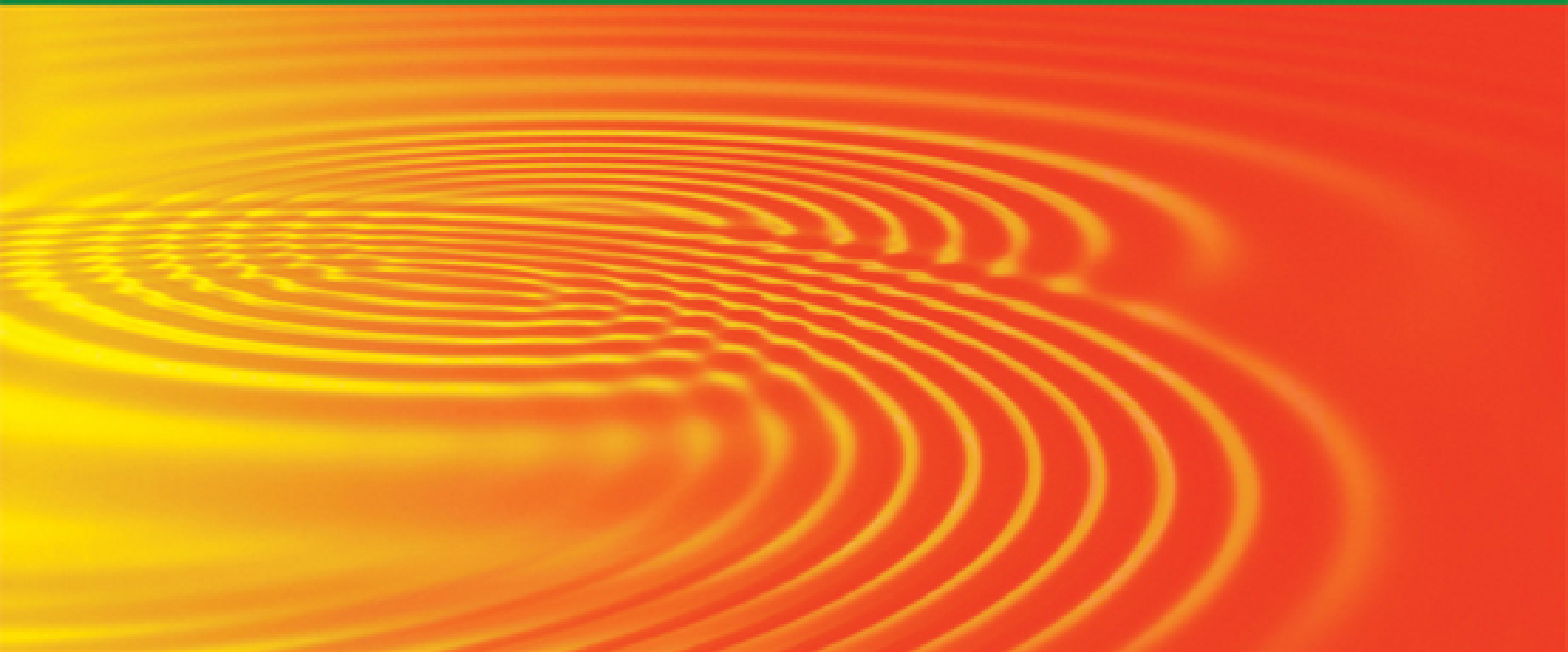

Vibrations and Acoustic Radiation of Thin Structures

Physical basis, theoretical analysis and numerical methods

Paul J.T. Filippi

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To my wife Dominique, who inspires in all ways

Preface

This monograph is the result of lectures given by the author at the *Université d'Aix-Marseille (France)* to students of the *Diplôme d'Etudes Approfondies de Méanique* (which is now the *Master de Mécanique*). It is aimed mainly at postgraduate students, PhD students and practicing acoustical scientists and engineers.

Among the most important sources of noise pollution are transport means, that is, cars, trucks, trains, planes, boats, etc. All these vehicles are essentially composed of thin vibrating structures. This is the reason why the present book is devoted to the vibrations and vibro-acoustics of thin structures only.

The simplest thin structure is the thin plate, then comes the circular cylindrical thin shell and the spherical thin shell. These basic structures provide a set of examples which make it possible to understand the basis of the physical phenomena of vibrations and sound radiation. Of course, most of the practical situations involve more complex structures, but their vibratory and acoustic behaviors are very similar to those of the simple structures described here, and the mathematical and numerical tools necessary to predict their response are much the same as those used for the simple examples.

Another aim of this monograph is to propose a homogenous theoretical approach to plates and shells.

Chapter 1 is devoted to equations which describe a good approximation of the vibrations of thin solids, and more precisely: plates, circular cylindrical shells and spherical shells. Analytical or numerical solutions of the mechanics equations are always based on a variational principle, which is, of course, the mathematical transcription of the conservation of energy principle which governs any phenomenon in physics. Thus, to establish the approximate equations governing the vibrations of thin structures we start from the expressions of the potential and kinetic energies of three-dimensional elastic solids, written in a convenient coordinate system: a Cartesian system for the plate, a cylindrical or spherical system for the shells. The hypothesis “thin structure” makes it possible to expand the three components of the displacement and the six independent components of the stresses as Taylor-like series of the transverse variable, leading to an approximate system of equations. We adopt the simplest approximations which are quite sufficient for a good understanding of the physical phenomena. Nevertheless, the method which is used can easily provide more accurate equations as they are proposed in the basic treatises cited in the bibliography.

Chapter 2 deals with the vibrations of *in vacuo* thin structures. The most important part concerns beams and plates. The classical method, based on the separation of variables, used to solve the vibration equation of simple plates of constant thickness (circular and rectangular) is developed in detail. Then, similar methods are applied to plates with a non-constant thickness. Finally, the Boundary Element Method (BEM) is described in some detail and illustrated by a comparison between numerical predictions and experimental results.

The chapter then continues with the problem of shell vibrations. For circular cylindrical shells, some of the existing analytical methods are proposed which enable us to give the expression of the resonance modes and of the response to a harmonic excitation. The Boundary Element Method is also described. For spherical shells, it seems that no analytical method exists. The main reason is that the equations are not separable. Thus, the presentation is limited to the variational equations which govern the resonance modes and the forced harmonic regime and to a general method for solving them is briefly outlined.

The third and last chapter deals with the important problems of acoustical engineering of sound generation by vibrating structures and sound transmission through elastic structures. It starts with a very simple academic one-dimensional example: the transmission of acoustic energy through a spring supported piston in a wave guide and the radiation of sound by such a system. Although this system is not realistic we do not see how an experiment could be conducted its simplicity makes it possible to develop an exhaustive study: the equations which describe the system can be solved analytically, both in the frequency and time domains. The interest of such an example is that it points out clearly the main aspects of the phenomena involved in sound transmission and sound radiation by vibrating structures.

After a short section, in which the basic concepts and equations of acoustics are recalled, several vibro-acoustics problems are examined in some detail. These concern plates and circular cylindrical shells. The important notion of "fluid-loaded resonance modes" is introduced: these modes are characteristics of the structure-fluid system and can be used to predict the response to any excitation (harmonic, transient, random). Numerical methods for computing either the resonance modes of a fluid-loaded structure or its response to an external excitation are described. Numerical results are given and, as far as possible, compared with experiments which have been selected from recent PhD theses.

At the end of the three chapters, a few exercises are proposed as complements of the text itself. At the end of this monograph, the bibliography proposes two kinds of references: basic textbooks in which the reader can find much more detail on the different aspects which are developed; specialized papers on the topics, and particularly those from which numerical and experimental results have been used to illustrate the theoretical developments.

The aim of this monograph is to present the basic concepts and methods necessary for the study of vibro-acoustics phenomena. As such, only classical analytical and numerical methods are described: separation of variables, series expansions in terms of special functions, matched asymptotic expansions, Boundary Element Methods (BEM). Nowadays, much more powerful numerical methods have been developed, for example, Statistical Energy Analysis (SEA), Finite Element Methods (FEM) and mixed methods such as various BEM-FEM methods, medium and high frequency approximations, numerical techniques for improving the performances of BEM and FEM computer programs (in particular the Fast Multi-pole Method), etc. Several specialized books have already been published on these topics. Several pieces of software for acoustics and vibro-acoustics engineering are now available.

Chapter 1

Equations Governing the Vibrations of Thin Structures

1.1. Introduction

1.1.1. *General Considerations on Thin Structures*

Thin structures are commonly called thin plates, thin shells, beams or rings (the term *thin* is generally omitted when there is no ambiguity). A plate or a shell is a solid in which one of its dimensions — called its *thickness* — is small compared to the other two; beams and rings are solids which have two dimensions which are small compared to the third one. The term *small* means that some approximations of the general equations of elasticity are sufficient to describe stresses and strains accurately enough: thus, depending on the accuracy which is required to predict the physical phenomenon, different equations are used. The present chapter deals with the simplest approximation. More elaborated approximations can be found in several textbooks, such as those mentioned in the bibliography [FLU 90, LAN 67, LEI 69, LEI 73, LOV 44].

The geometry of a shell (or a plate) is described by three variables: two of them — say ξ_1 and ξ_2 — are the parametric coordinates of a surface Σ (a plane in the case of a plate); the third one, ξ_3, in the direction normal to Σ and sometimes called the *transverse variable*, is a function of the first two which varies within two bounds, $-h^{-}(\xi_1,\xi_2)$ and $+h^{+}(\xi_1,\xi_2)$, which remain small compared to any characteristic dimension of Σ. Beams and rings are described in a similar way.

Approximate equations governing the vibrations of thin structures are based on several hypotheses. The main one is that the expansions of every mechanical quantity (displacement, forces, momentums, etc.) into a Taylor series of the transverse variable can be truncated at a low order. The second hypothesis is that the two boundaries $\xi_3 = -h^{-}(\xi_1,\xi_2)$ and $\xi_3 = +h^{+}(\xi_1,\xi_2)$ can be considered as free of any constraint: this means that the constraints exerted on these surfaces are small compared to the volume constraints. When vibrations are concerned, it is necessary to assume that wavelengths involved are large compared to the maximum thickness of the structure.

There are essentially two methods to establish approximate equations for the vibrations of thin structures. The most ancient one consists of approximating the forces and momentums exerted of an elementary volume of solid. This leads immediately to a system of partial differential equations and, then, the energy equations can be deduced.

The second method — which we can call the *energy method* — starts with the energy equation of the three-dimensional solid and approximations are made: this leads to the energy equation of the thin structure from which the partial differential equations are deduced. This second approach is adopted here. The main reason is that it leads to a variational form of the problem which is perfectly suitable for numerical computation (expansion of the solution in terms of a set of basis functions or finite element methods).

1.1.2. *Overview of the Energy Method*

Let Σ, with boundary $\partial\Sigma$, be a surface which can be parametrized by a coordinate system -$h^{-}(\xi_{1,}\xi_{2})$. It is assumed that a unit normal vector $\vec{\xi}_3$ exists everywhere on this surface. A point in the neighborhood of Σ can be defined by local coordinates $(\xi_{1,}\xi_{2,}\ \xi_3)$, where $\boldsymbol{\xi}_3$ is counted along the normal vector $\vec{\xi}_3$ For simplicity, it is assumed that this coordinate system is an orthogonal system.

Let us define two regular functions $h^{-}(\xi_{1,}\xi_{2})< 0$ and $h^{+}(\xi_{1,}\xi_{2}) > 0$ with $|h^{-}|$ and h^{+} small compared to the domains of variations of ξ_1 and ξ_2. Space domain Ω defined by $\{(\xi_{1,}\xi_{2}) \in \Sigma\ h^{-}(\xi_{1,}\xi_{2}) \leq \xi_3 \leq h^{+}(\xi_{1,}\xi_{2})\}$ is occupied by an elastic (or visco-elastic) solid. It is assumed that boundaries $\xi_3 = h^{-}(\xi_{1,}\xi_{2})$ and $\xi_3\ h^{+}(\xi_{1,}\xi_{2})$ are free (or submitted to loads which, in a first approximation, are negligible).

Let $\mathcal{D}_{ij}$ be the strain tensor and $\mathcal{S}_{ij}$ the stress tensor, where the subscripts *i* and *j* take the values 1, 2 and 3. The potential energy of the solid is given by the integral:

$$\mathcal{E}_p = \frac{1}{2}\int_{\Sigma}\int_{h^-}^{h^+} \mathcal{S}_{ij}\mathcal{D}_{ij}\,\mathrm{d}\xi_3\,\mathrm{d}\Sigma$$

In this equation, as well as throughout this chapter, the convention of *summation over repeated subscripts* is adopted, that is:

$$\mathcal{S}_{ij}\mathcal{D}_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3}\mathcal{S}_{ij}\mathcal{D}_{ij}$$

Because of the hypothesis that the wavelengths of the vibratory waves are large compared with the thickness of domain Ω, the strain and stress tensors are expected to vary slowly with respect to variable ξ_3. Thus, it is reasonable to expand each of them into a Taylor series of this variable:

$$\mathcal{D}_{ij} = \mathcal{D}_{ij}^0 + \xi_3 \mathcal{D}_{ij}^1 + \cdots \qquad \mathcal{S}_{ij} = \mathcal{S}_{ij}^0 + \xi_3 \mathcal{S}_{ij}^1 + \cdots$$

The hypothesis of free boundaries for $\xi_3 = h^-$ and $\xi_3 = h^+$ is written as:

$$\mathcal{S}_{ij}^0 + h^- \mathcal{S}_{ij}^1 + \cdots = 0\ , \qquad \mathcal{S}_{ij}^0 + h^+ \mathcal{S}_{ij}^1 + \cdots = 0 \qquad \forall(\xi_1, \xi_2)$$

This provides relationships between the terms of the stress tensor expansion, in particular, we obtain:

$$\mathcal{S}_{ij}^0 = 0 \qquad \mathcal{S}_{ij}^1 = 0$$

The stress-strain relationship (here, Hooke's law) is then applied and relationships between the $\mathcal{D}_{ij}^k$ are obtained. All these results are introduced into the expression of the potential energy. The quantity to be integrated is thus a Taylor series with respect to transverse variable ξ_3 and, as a consequence, the integral over this variable can be performed analytically. Finally, the potential energy is expressed by a two-dimensional integral over the mean surface Σ. The same approximation is made to express the kinetic energy.

To obtain the variational form of the approximated equation governing the vibrations of the thin body, the *virtual works theorem* is applied. As is usually done, an integration by parts leads to the corresponding partial differential equations and provides boundary conditions along ∂Σ

1.2. Thin Plates

Let Σ be a domain of the plane.(x_1; x_2), with boundary ∂Σ. It is assumed that there exists almost everywhere a unit vector $\vec{n}$ normal to ∂Σ and pointing outward; there also exists a unit tangent vector $\vec{s}$ which makes an angle $\pi/2$ with $\vec{n}$. Let

Ω be the cylindrical domain with basis Σ and extending from x_3 = ¯h/2 to x_3 = h/2, where h remains small compared to any characteristic dimension of Σ: this means that the thickness of Ω is a few percent of this characteristic length. In section 1.2.1, it is assumed that h is constant; the equations for a plate of variable thickness are given section 1.2.2.

A homogenous isotropic elastic solid occupies Ω: it has a density μ_S, a Young's modulus E and a Poisson's ratio v. The boundaries $x_3 = -h/2$ and x_3 h/2 are free (external forces applied to the plate are zero or negligible). It is assumed that there is no in-plane external force.

As is commonly done in mechanics, in the following, the derivation of a function f with respect to variable x_i is denoted by $f,_i$.

1.2.1. *Plate with Constant Thickness*

Let.(U_1; U_2; U_3) be the components of the displacement of a point of the solid. The strain tensor $\mathcal{D}_{ij}$ is defined by:

$$\mathcal{D}_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i})$$

Let $\mathcal{S}_{ij}$ be the stress tensor. Assuming that Hooke's law is valid, the strain stress relationship is expressed by:

$$\mathcal{S}_{11} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\mathcal{D}_{11} + \nu(\mathcal{D}_{22} + \mathcal{D}_{33})\right]$$

$$\mathcal{S}_{22} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\mathcal{D}_{22} + \nu(\mathcal{D}_{33} + \mathcal{D}_{11})\right]$$

$$\mathcal{S}_{33} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\mathcal{D}_{33} + \nu(\mathcal{D}_{11} + \mathcal{D}_{22})\right]$$

$$\mathcal{S}_{12} = \frac{E}{1+\nu}\mathcal{D}_{12} = \mathcal{S}_{21}\ ,\ \mathcal{S}_{13} = \frac{E}{1+\nu}\mathcal{D}_{13} = \mathcal{S}_{31}$$

$$\mathcal{S}_{23} = \frac{E}{1+\nu}\mathcal{D}_{23} = \mathcal{S}_{32} \qquad (1.1)$$

We look for approximations of the displacement and the stress tensor as truncated Taylor series in x_3, that is:

$$U_i(x_1,x_2,x_3) = U_i^0(x_1,x_2) + x_3 U_i^1(x_1,x_2) + \mathcal{O}({x_3}^2)$$
$$\mathcal{S}_{ij}(x_1,x_2,x_3) = \mathcal{S}_{ij}^0(x_1,x_2) + x_3 \mathcal{S}_{ij}^1(x_1,x_2) + \mathcal{O}({x_3}^2)$$

The free boundary condition at $x_3 = \pm h/2$ is written as:

$$\mathcal{S}_{i3}(x_1,x_2,\pm h/2) = 0 \ \forall (x_1,x_2)$$

This implies the following equalities:

$$\mathcal{S}_{i3}^0(x_1,x_2) = \mathcal{S}_{i3}^1(x_1,x_2) = 0 \ \forall (x_1,x_2) \ \Rightarrow \ \mathcal{S}_{i3}(x_1,x_2,x_3) = \mathcal{O}({x_3}^2)$$

Introducing this result into Hooke's law, it appears that all the components of the displacement can be expressed in terms of component $w = U_3^0$ only; more precisely, we obtain:

$$U_1 \simeq -x_3 w_{,1} \quad , \quad U_2 \simeq -x_3 w_{,2}$$
$$\mathcal{D}_{11} \simeq -x_3 w_{,11} = d_{11} \quad , \mathcal{D}_{22} \simeq -x_3 w_{,22} = d_{22}$$
$$\mathcal{D}_{12} \simeq -x_3 w_{,12} = d_{12} \quad , \quad \mathcal{D}_{13} \simeq 0 = d_{13} \quad , \quad \mathcal{D}_{23} \simeq 0 = d_{23}$$

(1.2)
$$\mathcal{D}_{33} \simeq x_3 \frac{\nu}{1-\nu}\left(w_{,11} + w_{,22}\right) = d_{33}$$

The potential energy of the solid is the integral over Ω of quantity $\mathcal{S}_{ij}\mathcal{D}_{ij}$; it is approximated by the following positive quantity:

$$\mathcal{E}_p = \frac{E}{2(1-\nu^2)} \int_\Sigma \int_{-h/2}^{+h/2} {x_3}^2 \,\mathrm{d}x_3 \Big[\nu\big(w_{,11} + w_{,22}\big)^2 + (1-\nu)\big(w_{,12}^2 + w_{,21}^2 + w_{,11}^2 + w_{,22}^2\big)\Big] \mathrm{d}\Sigma$$

(1.3)
$$= \frac{E}{2(1-\nu^2)} \frac{h^3}{12} \int_\Sigma \Big[\nu\big(w_{,11} + w_{,22}\big)^2 + (1-\nu)\big(w_{,12}^2 + w_{,21}^2 + w_{,11}^2 + w_{,22}^2\big)\Big] \mathrm{d}\Sigma$$

The same approximations of the displacement leads to the following approximation for the kinetic energy:

(1.4)
$$\mathcal{E}_c = \frac{\mu_s h}{2} \int_\Sigma \dot{w}^2 \,\mathrm{d}\Sigma$$

where $\dot{w}$ is the time derivative of w.

Let us now assume that a force, normal to Σ, with density f is exerted on the plate. The *virtual works theorem* implies that the work of the external force corresponding to a virtual displacement δ*w* obtained within a time interval δ*t* is equal to the variation of the total energy of the solid, that is:

$$\int_\Sigma \left\{ \frac{Eh^3}{12(1-\nu^2)} \Big[\nu (w_{,11} + w_{,22})(\delta w_{,11} + \delta w_{,22}) + (1-\nu)(w_{,11}\,\delta w_{,11} + w_{,22}\,\delta w_{,22} + w_{,12}\,\delta w_{,12} + w_{,21}\,\delta w_{,21}) \Big] + \mu_s h\, \ddot{w}\, \delta w \right\} \mathrm{d}\Sigma = \int_\Sigma f\, \delta w \ \mathrm{d}\Sigma \tag{1.5}$$

or equivalently:

$$\int_\Sigma \left\{ \frac{Eh^3}{12(1-\nu^2)} \Big[(w_{,11} + w_{,22})(\delta w_{,11} + \delta w_{,22}) + (1-\nu)(2w_{,12}\,\delta w_{,12} - w_{,11}\,\delta w_{,22} - w_{,22}\,\delta w_{,11}) \Big] + \mu_s h\, \ddot{w}\, \delta w \right\} \mathrm{d}\Sigma = \int_\Sigma f\, \delta w \ \mathrm{d}\Sigma \tag{1.5′}$$

The variation of the kinetic energy is obtained using the following equality:

$$\delta \dot{w}^2 = 2\dot{w}\, \delta\dot{w} = 2\dot{w}\, \ddot{w}\, \delta t = 2\ddot{w}\, \delta w$$

Integrations by parts lead to:

$$\int_\Sigma \left\{ \frac{Eh^3}{12(1-\nu^2)} \Delta^2 w + \mu_s h\, \ddot{w} \right\} \delta w\, \mathrm{d}\Sigma + \frac{Eh^3}{12(1-\nu^2)} \int_{\partial\Sigma} \Big[\ell_1(w)\ \mathrm{Tr}\, \partial_n \delta w - \mathrm{Tr}\, \partial_n \Delta w\ \mathrm{Tr}\, \delta w + \ell_2(w)\ \mathrm{Tr}\, \partial_s \delta w \Big] \mathrm{d}\bar{s} = \int_\Sigma f\, \delta w \ \mathrm{d}\Sigma \tag{1.6}$$

where $\bar{s}$ is the curvilinear abscissae along ∂Σ. The different operators in (1.6) are defined as follows:

$$\Delta^2 = \frac{\partial^4}{\partial x_1^4} + 2\frac{\partial^4}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4}{\partial x_2^4}$$

$$\mathrm{Tr}\, w(M) = \lim_{P\in\Sigma\to M\in\partial\Sigma} w(P)$$

$$\mathrm{Tr}\,\partial_n w = \lim_{P\in\Sigma\to M\in\partial\Sigma} \vec{n}(M)\cdot\vec{\nabla}_P\, w(P)\ ,\quad \mathrm{Tr}\,\partial_s w = \lim_{P\in\Sigma\to M\in\partial\Sigma} \vec{s}(M)\cdot\vec{\nabla}_P\, w(P)$$

$$\mathrm{Tr}\,\partial_{s^2} w = \lim_{P\in\Sigma\to M\in\partial\Sigma} \vec{s}(M)\cdot\vec{\nabla}_P[\vec{s}(M)\cdot\vec{\nabla}_P\, w(P)]$$

$$\mathrm{Tr}\,\partial_n\partial_s w = \lim_{P\in\Sigma\to M\in\partial\Sigma} \vec{n}(M)\cdot\vec{\nabla}_P[\vec{s}(M)\cdot\vec{\nabla}_P\, w(P)]$$

$$\ell_1(w) = \mathrm{Tr}\,\Delta w - (1-\nu)\ \mathrm{Tr}\,\partial_{s^2} w \quad , \quad \ell_2(w) = (1-\nu)\ \mathrm{Tr}\,\partial_n\partial_s w$$

REMARK.- An elementary calculation shows that Tr $\partial_s w = \partial_{\bar{s}}$ Tr *w*, where $\partial_{\bar{s}}$ is the derivation with respect to the curvilinear abscissae.

PROOF OF EQUATION (1.6).- Let θ be the angle between the axis x_1 and the normal $\vec{n}$ at a point *P* of ∂Σ. The differential operators with respect to (x_1; x_2) and to the directions (*n, s*) are related as follows (see Figure 1.1):

$$\partial_{x_1} = \cos(\theta)\partial_n - \sin(\theta)\partial_s \qquad \partial_n = \cos(\theta)\partial_{x_1} + \sin(\theta)\partial_{x_2}$$

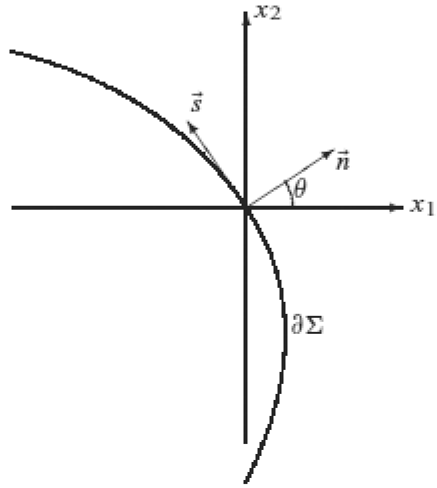
$$\partial_{x_2} = \sin(\theta)\partial_n + \cos(\theta)\partial_s \qquad \partial_s = -\sin(\theta)\partial_{x_1} + \cos(\theta)\partial_{x_2}$$

Let us consider the first integral in equation (1.5′)

$$I_1 = \int_\Sigma (w_{,11} + w_{,22})(\delta w_{,11} + \delta w_{,22})\,\mathrm{d}\Sigma$$

Its first component is integrated by parts with respect to x_1 and we obtain:

Figure 1.1. *Orientations of the normal and tangent unit vectors with respect to the coordinate axes*



$$I_{11} = \int_{\Sigma} (w_{,11} + w_{,22})\delta w_{,11}\, \mathrm{d}\Sigma = \int \mathrm{d}x_2 \int_{x_1^-(x_2)}^{x_1^+(x_2)} (w_{,11} + w_{,22})\delta w_{,11}\, \mathrm{d}x_1$$

$$= \int_{\Sigma} (w_{,1111} + w_{,2211})\delta w\, \mathrm{d}\Sigma$$

$$+ \int (w_{,11} + w_{,22})\delta w_{,1}\Big|_{x_1^-(x_2)}^{x_1^+(x_2)} \mathrm{d}x_2 - \int (w_{,11} + w_{,22})_{,1}\, \delta w\Big|_{x_1^-(x_2)}^{x_1^+(x_2)} \mathrm{d}x_2$$

$$= \int_{\Sigma} (w_{,1111} + w_{,2211})\delta w\, \mathrm{d}\Sigma$$

$$+ \int_{\partial\Sigma} \left[\mathrm{Tr}\, \Delta w\, \mathrm{Tr}\, \delta w_{,1} \cos(\theta) - \mathrm{Tr}\, \Delta w_{,1} \cos(\theta)\, \mathrm{Tr}\, \delta w \right] \mathrm{d}\bar{s}$$

In the same way, we obtain:

$$I_{12} = \int_{\Sigma} (w_{,11} + w_{,22})\delta w_{,22}\, \mathrm{d}\Sigma$$

$$= \int_{\Sigma} (w_{,1122} + w_{,2222})\delta w\, \mathrm{d}\Sigma + \int_{\partial\Sigma} \left[\mathrm{Tr}\, \Delta w\, \mathrm{Tr}\, \delta w_{,2} \sin(\theta) - \mathrm{Tr}\, \Delta w_{,2} \sin(\theta)\, \mathrm{Tr}\, \delta w \right] \mathrm{d}\bar{s}$$

Gathering these results, we have that the first integral in (1.5′) becomes:

$$I_1 = \int_{\Sigma} \Delta^2 w\, \mathrm{d}\Sigma + \int_{\partial\Sigma} \left[\mathrm{Tr}\, \Delta w\, \mathrm{Tr}\, \partial_n \delta w - \mathrm{Tr}\, \partial_n \Delta w\, \mathrm{Tr}\, \delta w \right] \mathrm{d}\bar{s}$$

Let us now consider the terms with a factor (1 − *v*). The calculation method being the same, we give the results only. In order to preserve the symmetric roles played by the variables x_1, and x_2, the first integral is split into two equal terms: the integrations by parts are performed with respect to x_1, and, then, to x_2, on the first term, and in the reverse order on the second term. The result is:

$$I_2 = 2\int_\Sigma w_{,12}\,\delta w_{,12}\;\mathrm{d}\Sigma = \int_\Sigma w_{,12}\,\delta w_{,12}\;\mathrm{d}\Sigma + \int_\Sigma w_{,21}\,\delta w_{,21}\;\mathrm{d}\Sigma$$
$$= 2\int_\Sigma w_{,1122}\,\delta w\,\mathrm{d}\Sigma + \int_{\partial\Sigma}\left[\,\mathrm{Tr}\,w_{,12}\;\mathrm{Tr}\,\delta w_{,1}\sin\theta - \;\mathrm{Tr}\,w_{,122}\;\mathrm{Tr}\,\delta w\cos\theta\right]\mathrm{d}\bar{s}$$
$$+ \int_{\partial\Sigma}\left[\,\mathrm{Tr}\,w_{,12}\;\mathrm{Tr}\,\delta w_{,2}\cos\theta - \;\mathrm{Tr}\,w_{,112}\;\mathrm{Tr}\,\delta w\sin\theta\right]\mathrm{d}\bar{s}$$

The third and fourth terms become:

$$I_3 = -\int_\Sigma w_{,11}\,\delta w_{,22}\;\mathrm{d}\Sigma$$
$$= -\int_\Sigma w_{,1122}\,\delta w\,\mathrm{d}\Sigma - \int_{\partial\Sigma}\left[\,\mathrm{Tr}\,w_{,11}\;\mathrm{Tr}\,\delta w_{,2}\sin(\theta) - \;\mathrm{Tr}\,w_{,112}\sin(\theta)\;\mathrm{Tr}\,\delta w\right]\mathrm{d}\bar{s}$$
$$I_4 = -\int_\Sigma w_{,22}\,\delta w_{,11}\;\mathrm{d}\Sigma$$
$$= -\int_\Sigma w_{,1122}\,\delta w\,\mathrm{d}\Sigma - \int_{\partial\Sigma}\left[\,\mathrm{Tr}\,w_{,22}\;\mathrm{Tr}\,\delta w_{,1}\cos(\theta) - \;\mathrm{Tr}\,w_{,122}\cos(\theta)\;\mathrm{Tr}\,\delta w\right]\mathrm{d}\bar{s}$$

Summing up these results leads to:

$$I_2 + I_3 + I_4 = \int_{\partial\Sigma}\Big[\cos(\theta)\big(-\;\mathrm{Tr}\,w_{,22}\;\mathrm{Tr}\,\delta w_{,1} + \mathrm{Tr}\,w_{,12}\;\mathrm{Tr}\,\delta w_{,2}\big) +$$
$$\sin(\theta)\big(-\,\mathrm{Tr}\,w_{,11}\;\mathrm{Tr}\,\delta w_{,2} + \mathrm{Tr}\,w_{,12}\;\mathrm{Tr}\,\delta w_{,1}\,\big)\Big]\,\mathrm{d}\bar{s}$$

To end the proof, the derivation operators with respect to variables (x_1, x_2) are expressed in terms of the derivation operators with respect to (*n, s*). This is a simple, but nevertheless, tedious calculation which is left to the reader.

The boundary integral in expression (1.6) represents the work of the different forces and moments that the plate exerts on its support. The physical meaning of the various terms is, thus, easy:

⬦– $Eh^3/12.(1 - \nu^2)$ Tr $\partial_n \Delta w$ is the factor of δw: it represents the density of shearing forces that the plate boundary exerts on its support.

⬦– $\ell_1.(w) = -Eh^3/12.(1 - \nu^2)$ [Tr.$\Delta w - (1 - \nu)$ Tr $\partial_{s^2} w$] is the factor of Tr $\partial_n \delta w$: it represents the density of bending moments (rotation around the tangential direction).

⬦– $\ell_2.(w) = -(1 - \nu)Eh^3/12(1 - \nu^2)$ Tr $\partial_n \partial_s w$ is the factor of Tr $\partial_s \delta w$: it represents the density of twisting moments (rotation around the normal direction).

Finally, in the case of a regular boundary $\partial\Sigma$, that is, a boundary without angular points, the term $\ell_2(w)$ in equation (1.6) is continuous and an integration by parts of the term ℓ_2(w) Tr $\partial_s \delta w = \ell_2(w) \partial_{\bar{s}} \mathrm{Tr}\, \delta w$ can be performed without any caution. We obtain:

$$\begin{aligned} &\int_\Sigma \left\{ \frac{Eh^3}{12(1-\nu^2)} \Delta^2 w + \mu_s h\, \ddot{w} \right\} \delta w \, \mathrm{d}\Sigma \\ &\qquad + \frac{Eh^3}{12(1-\nu^2)} \int_{\partial\Sigma} \Big\{ \left[\mathrm{Tr}\, \Delta w - (1-\nu)\ \mathrm{Tr}\, \partial_{s^2} w \right] \mathrm{Tr}\, \partial_n \delta w \\ &\qquad - \left[(1-\nu)\, \partial_{\bar{s}}\ \mathrm{Tr}\, \partial_n \partial_s w + \ \mathrm{Tr}\, \partial_n \Delta w \right]\ \mathrm{Tr}\, \delta w \Big\} \mathrm{d}\bar{s} = \int_\Sigma f\ \delta w\ \mathrm{d}\Sigma \end{aligned} \tag{1.7}$$

The term $-(1-\nu)Eh^3/12(1-\nu^2)\ \partial_s$Tr $\partial_n \partial_s w$ is the tangential derivative of the twisting moment density. The coefficient of Tr δw is called the *Kelvin-Kirchhoff edge reaction*.

Integral relationship (1.7) must be satisfied for any virtual displacement δw; thus, the integrals over Σ and over $\partial\Sigma$ must cancel separately. The cancellation of the integral over Σ leads to the well-known *thin plate equation*:

$$\begin{aligned} &\left(D\Delta^2 + \mu_s h \frac{\partial^2}{\partial t^2} \right) w = f \\ &\text{with :}\quad D = \frac{Eh^3}{12(1-\nu^2)} \end{aligned} \tag{1.8}$$

where D is called the *plate flexural rigidity*. If a harmonic time dependence of the form $e^{-\imath wt}$ is assumed, this equation becomes:

(1.9) $$\left(\Delta^2 - \lambda^4\right) w = \frac{f}{D}\,, \quad \text{with}\ \lambda^4 = \frac{\mu_s h \omega^2}{D}$$

(because no confusion can occur, we have used the same symbols f and w for the amplitudes of the harmonic excitation and the corresponding displacement: this avoids needless heavy notations).

The cancellation of the boundary integrals provides what is called *the natural boundary conditions* (which, of course, are a mathematical idealization of the physical conditions which can be imposed geometrically or mechanically); their expressions are the same for a transient or a harmonic excitation:

◇ Clamped boundary:

$$\mathrm{Tr}\, w = 0 \quad , \quad \mathrm{Tr}\, \partial_n w = 0$$

◇ Free boundary:

$$\mathrm{Tr}\, \Delta w - (1-\nu)\, \mathrm{Tr}\, \partial_{s^2} w = 0$$
$$\mathrm{Tr}\, \partial_n \Delta w + (1-\nu) \partial_{\bar{s}} \mathrm{Tr}\, \partial_n \partial_s w = 0$$

The second of these two conditions is known as *Kirchhoff's condition* (Kirchhoff's contribution to the plates theory has been essential).

◇ Simply supported boundary:

$$\mathrm{Tr}\, w = 0 \quad , \quad \mathrm{Tr}\, \Delta w - (1-\nu)\ \mathrm{Tr}\, \partial_{s^2} w = 0$$

These boundary conditions imply that there is no energy loss across the plate boundaries. For that reason they are called *conservative boundary conditions*.

To conclude this section, let us mention that the plate equation obtained here is the simplest one. Many authors have developed more accurate equations which are valid for plates whose thickness is not very small; equally, equations for plates made of non-isotropic material and for sandwich plates can be found in the literature.

1.2.2. *Plate with Variable Thickness*

Accounting for a thickness variation does not present any extra difficulty. Following exactly the same steps as in the preceding section, we obtain:

$$\int_\Sigma \Big\{\Delta(D\Delta w) + (1-\nu)\big[2(Dw_{,12})_{,12} - (Dw_{,11})_{,22} - (Dw_{,22})_{,11}\big] + \mu_s h\,\ddot{w}\Big\}\,\delta w\,\mathrm{d}\Sigma + \int_{\partial\Sigma}\big[\ell_1^{\mathrm{v}}(w)\ \mathrm{Tr}\,\partial_n\delta w + \ell_2^{\mathrm{v}}(w)\,\partial_{\bar{s}}\mathrm{Tr}\,\delta w - \ell_3^{\mathrm{v}}(w)\ \mathrm{Tr}\,\delta w\big]\,\mathrm{d}s = \int_\Sigma f\,\delta w\ \mathrm{d}\Sigma \quad (1.10)$$

with

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

$$\ell_1^{\mathrm{v}}(w) = \mathrm{Tr}\,(D\Delta w) - (1-\nu)\,\mathrm{Tr}\,(D\partial_{s^2}w)$$

$$\ell_2^{\mathrm{v}}(w) = (1-\nu)\,\mathrm{Tr}\,(D\partial_n\partial_s w)$$

$$\ell_3^{\mathrm{v}}(w) = \mathrm{Tr}\,\partial_n(D\Delta w) - (1-\nu)\big[\,\mathrm{Tr}\,\partial_s(D\partial_n\partial_s w) - \mathrm{Tr}\,\partial_n(D\partial_{s^2}w)\big]$$

These terms are related to the physical efforts exerted by the plate boundary on its support:

- ◇ $-\ell_3^{\mathrm{v}}(w)$ = density of shearing forces;
- ◇ $-\ell_1^{\mathrm{v}}(w)$ = density of bending moments;
- ◇ $-\ell_2^{\mathrm{v}}(w)$ = density of twisting moments.

If ∂Σ is a regular curve (no angular point), an integration by parts of the third term of the boundary integral can be performed, which leads to:

$$\int_\Sigma \Big\{\Delta(D\Delta w) + (1-\nu)\big[2(Dw_{,12})_{,12} - (Dw_{,11})_{,22} - (Dw_{,22})_{,11}\big] + \mu_s h\,\ddot{w}\Big\}\,\delta w\,\mathrm{d}\Sigma + \int_{\partial\Sigma}\Big\{\ell_1^{\mathrm{v}}(w)\ \mathrm{Tr}\,\partial_n\delta w - \big[\partial_{\bar{s}}\ell_2^{\mathrm{v}}(w) + \ell_3^{\mathrm{v}}(w)\big]\,\mathrm{Tr}\,\delta w\Big\}\,\mathrm{d}s = \int_\Sigma f\,\delta w\ \mathrm{d}\Sigma \quad (1.11)$$

This integral relationship must be satisfied for any virtual displacement, so the surface integral and the boundary integral must cancel separately. Thus, the plate displacement *w* satisfies the following partial-differential equation:

$$\Delta(D\Delta w) + (1-\nu)\big[2(Dw_{,12})_{,12} - (Dw_{,11})_{,22} - (Dw_{,22})_{,11}\big] + \mu_s h\,\ddot{w} = f \tag{1.12}$$

The cancellation of the boundary integral is obtained by the boundary conditions satisfied by *w*. As for the plate with constant thickness, there are three classical boundary conditions:

◇ Clamped boundary:

$$\mathrm{Tr}\,w = 0 \quad , \quad \mathrm{Tr}\,\partial_n w = 0$$

◇ Free boundary:

$$\mathrm{Tr}\left(D\Delta w\right) - (1-\nu)\ \mathrm{Tr}\left(D\partial_{s^2} w\right) = 0$$

$$\mathrm{Tr}\,\partial_n\left(D\Delta w\right) + (1-\nu)\big[\ \mathrm{Tr}\,\partial_n\left(D\partial_{s^2} w\right) -\ \mathrm{Tr}\,\partial_s\left(D\partial_n\partial_s w\right) + \partial_{\bar{s}}\mathrm{Tr}\left(D\partial_n\partial_s w\right)\big] = 0$$

◇ Simply supported boundary:

$$\mathrm{Tr}\,w = 0 \quad , \quad \mathrm{Tr}\left(D\Delta w\right) - (1-\nu)\ \mathrm{Tr}\left(D\partial_{s^2} w\right) = 0$$

The “clamped boundary” condition, which is purely geometrical, is identical to the result obtained for a plate of constant thickness. Conversely, the “free boundary” and the “simply supported boundary” conditions, which are essentially mechanical conditions, involve the variations in plate rigidity. These boundary conditions are often called *natural boundary conditions* because they appear naturally when the partial differential equation of the plate is established. However, in practice, the engineer is often faced with more complex boundary conditions which are more difficult to describe mathematically.

1.2.3. *Boundary with an Angular Point*

In the last two sections, equations (1.6) and (1.10) present no difficulty for performing an integration by parts because we are assured that the terms $\ell_2(w)$ and $\ell_2^{\mathfrak{v}}(w)$ are continuous. If the boundary has an angular point, these terms are not *a priori* continuous: indeed, they involve derivatives with respect to normal and tangent vectors. At an angular point, there are two normal vectors and two tangent vectors (see Figure 1.2). Assume that $\partial\Sigma$ has an angular point Q. Let $(\vec{n}_1, \vec{s}_1)$ and $(\vec{n}_2, \vec{s}_2)$ be the two sets of normal and tangent unit vectors. Let Q_1 − resp. Q_2 — be the limit of a point P belonging to the arc (1) — resp. (2) — tending to Q: of course Q_1 and Q_2 are geometrically the same point (they coincide with Q), but the sets of normal and tangent unit vectors are different. The integrals involving $\ell_2(w)$ and $\ell_2^{\mathfrak{v}}(w)$ are taken along the curve $\partial\Sigma$, in the trigonometric sense, starting from the point with normal and tangent vectors $(\vec{n}_1, \vec{s}_1)$ to the point with normal and tangent vectors $(\vec{n}_2, \vec{s}_2)$. Thus, they take the following forms:

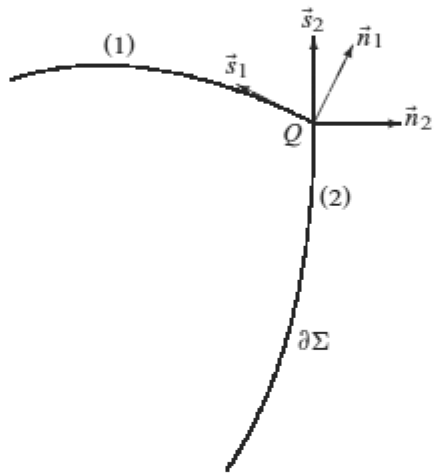
$$\int_{\partial\Sigma} \ell_2(w)\partial_s \mathrm{Tr}\,\delta w\,\mathrm{d}\bar{s} = \int_{Q_1}^{Q_2} \ell_2(w)\partial_s \mathrm{Tr}\,\delta w\,\mathrm{d}\bar{s}$$

$$= -\int_{\partial\Sigma} \partial_s \ell_2(w)\,\mathrm{Tr}\,\delta w\,\mathrm{d}\bar{s} + \left[\ell_2(w)(Q_2) - \ell_2(w)(Q_1)\right]\mathrm{Tr}\,\delta w(Q)$$

$$\int_{\partial\Sigma} \ell_2^{\mathfrak{v}}(w)\partial_s \mathrm{Tr}\,\delta w\,\mathrm{d}\bar{s} = \int_{Q_1}^{Q_2} \ell_2^{\mathfrak{v}}(w)\partial_s \mathrm{Tr}\,\delta w\,\mathrm{d}\bar{s}$$

$$= -\int_{\partial\Sigma} \partial_{\bar{s}} \ell_2^{\mathfrak{v}}(w)\,\mathrm{Tr}\,\delta w\,\mathrm{d}\bar{s} + \left[\ell_2^{\mathfrak{v}}(w)(Q_2) - \ell_2^{\mathfrak{v}}(w)(Q_1)\right]\mathrm{Tr}\,\delta w(Q)$$

Figure 1.2. *The two sets of normal and tangent unit vectors at an angular point*



This result does not change the "clamped" and "simply supported" boundary conditions, but the "free boundary" condition must be modified as follows:

◇ constant thickness:

$$\mathrm{Tr}\,\Delta w - (1-\nu)\ \mathrm{Tr}\,\partial_{s^2} w = 0$$

$$\mathrm{Tr}\,\partial_n \Delta w + (1-\nu)\ \partial_{\bar{s}} \mathrm{Tr}\,\partial_n \partial_s w = 0$$

$$\mathrm{Tr}\,\partial_n \partial_s w(Q_1) = \ \mathrm{Tr}\,\partial_n \partial_s w(Q_2)$$

◇ variable thickness:

$$\mathrm{Tr}\left(D\Delta w\right) - (1-\nu)\ \mathrm{Tr}\left(D\partial_{s^2} w\right) = 0$$

$$\mathrm{Tr}\,\partial_n\left(D\Delta w\right) + (1-\nu)\left[\ \mathrm{Tr}\,\partial_n\left(D\partial_{s^2} w\right)\right.$$

$$\left. - \mathrm{Tr}\,\partial_s\left(D\partial_n\partial_s w\right) + \partial_{\bar{s}}\mathrm{Tr}\left(D\partial_n\partial_s w\right)\right] = 0$$

$$\mathrm{Tr}\left(D\partial_n\partial_s w\right)(Q_1) = \ \mathrm{Tr}\left(D\partial_n\partial_s w\right)(Q_2)$$

The additional condition expresses that the twisting moments (rotation around the normal direction) must remain continuous even if the normal direction has a jump.

1.3. Beams

Let Ω be a space domain defined by: $\Omega = [-L/2 \leq x_1 \leq L/2,\ (x_2;\ x_3) \in \sigma\ (x_1)]$, the dimensions of the cross-section σ being small compared to L. An elastic solid occupying such a "long" and "linear" domain is called a beam. The aim of this section is to establish the equations governing the pure bending of a beam with a constant rectangular cross-section, that is a $\sigma \equiv (0 < |x_2| < d/2,\ 0 < |x_3| < h/2)$. By *pure bending*, we mean that all the points of the solid which, at rest, are in a given plane parallel to the $(x_1;\ x_2)$-plane remain in this plane. Furthermore, it is assumed that the strains are small. Nevertheless, the displacement of a point can be rather large compared to d and h.

Beams with less simple geometries and submitted to more complicated deformations are commonly studied in many classical text-books (see, for example [LAN 67]): circular, elliptical cross-sections, combination of bending in two different directions, twisting around the axis, etc. The interest of the beam here is purely mathematical: indeed, the beam resonance modes provide a good basis for the computation of the resonance modes for an *in vacuo* or fluid-loaded rectangular plate. It is thus sufficient to pay attention to beams with a constant rectangular cross-section, made of a homogenous, isotropic and purely elastic material.

Considering a beam as a narrow rectangular plate, its displacement satisfies equation (1.5′). The hypothesis of pure flexion implies that the displacement is independent of the variable x_2 and an integration with respect to this variable leads to:

$$(1.13)\quad \int_{-L/2}^{L/2} \left[\frac{Eh^3 d}{12(1-\nu^2)} w_{,11}\, \delta w_{,11} + \mu_s h d\, \ddot{w} \delta w\right] \mathrm{d}x_1 = \int_{-L/2}^{L/2} F \delta w$$

with $F = fd$ = force per unit length

After integrations by parts, we obtain:

$$\int_{-L/2}^{L/2}\Big[\frac{Eh^3d}{12(1-\nu^2)}w_{,1111}+\mu_s hd\,\ddot{w}\Big]\delta w\,\mathrm{d}x_1 + \frac{Eh^3d}{12(1-\nu^2)}\Big[w_{,11}\,\delta w_{,1}(L/2)-w_{,11}\,\delta w_{,1}(-L/2) - w_{,111}\,\delta w(L/2)+w_{,111}\,\delta w(-L/2)\Big] = \int_{-L/2}^{L/2} F\delta w \tag{1.14}$$

Classically, the factor ν^2, which is always small (less than 0.12) is neglected. Thus, the partial differential equation which governs the beam flexion is:

$$EI\frac{\partial^4 w}{\partial x_1^4}+\mu_s hd\frac{\partial^2 w}{\partial t_2} = F \quad \text{with} \quad I = \frac{h^3 d}{12} \tag{1.15}$$

The quantity I is called the *inertia momentum* of the cross-section with respect to a line parallel to $x_3 = 0$ and $D = EI$ is called the *flexural rigidity* of the beam. This equation remains valid for any constant cross-section which is symmetric with respect to the plane x_3 and the inertia moment is defined by:

$$I = \int_\sigma x_3^2\,\mathrm{d}x_1\,\mathrm{d}x_2$$

Three boundary conditions at $x_1 = L/2$ and $x_1 = -L/2$ are commonly used:

- ◇ clamped boundaries: $w = 0$; $\partial w/\partial x_1 = 0$;
- ◇ free boundaries: $\partial^2 w/\partial x_1^2 = 0$, $\partial^3 w/\partial x_1^3 = 0$
- ◇ simply supported boundaries: $w = 0$; $\partial^2 w/\partial x_1^2 = 0$

Final Comment

To establish the plate equation, it is generally assumed that there exists a *neutral surface*, that is, a surface along which the distance between two points remains constant: this is of course the case for a constant thickness plate and for a plate with a thickness which varies symmetrically with respect to the x_3-plane. The method used here does not require such a hypothesis.

Nevertheless, the result of the integration over variable x_3 which is performed to obtain equations (1.6) or (1.11) depends on the location of the plane $x_3 = 0$. Thus, the more accurate approximation is certainly obtained if this plane coincides with the neutral surface. The same remark applies equally to the beam equation.

1.4. Circular Cylindrical Shells

Let us consider a circular cylindrical surface Σ and a three-dimensional domain Ω defined in cylindrical coordinates by

$$\Sigma \equiv \{\rho = R \; , \; 0 \leq \varphi < 2\pi \; , \; -L/2 < z < +L/2\}$$

$$\Omega \equiv \{\rho = R + r \text{ with } -h/2 < r < +h/2 \; , \; 0 \leq \varphi < 2\pi \; , \; -L/2 < z < +L/2\}$$

where h, the thickness of Ω, is small compared to both R and $L/2$ (see Figure 1.3). An elastic, homogenous and isotropic solid — characterized by a density μ_S, a Young's modulus E and a Poisson's ratio v — occupies Ω. The two boundaries $r = -h/2$ and $r = +h/2$ of the elastic solid are free (zero or negligible forces). The displacement of a point of the solid is denoted by $U\vec{e}_z + V\vec{e}_\varphi + W\vec{e}_r$

Figure 1.3. *The coordinate system of the cylindrical shell*