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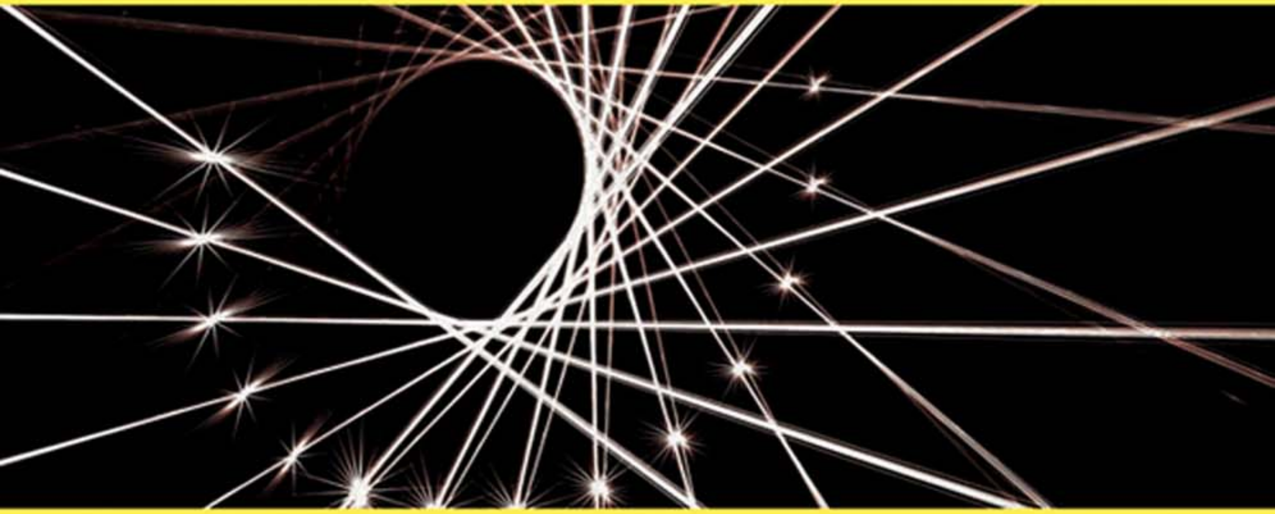

Linear Systems

Henri Bourlès

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Preface

The notion of system

The notion of *system* is the basic concept of *control theory*. What is a system? It is quite difficult to address this question in all its aspects. We can say somewhat loosely that it is an entity consisting of interacting parts; an entity that is itself, most often, also interacting with other systems. The solar system, a computer system, etc. are examples of a system.

The control system analyst is interested in systems which are, at least in part, designed and constructed by man, in order to be utilized – a car engine or the entire automobile; an alternator in a power station or the "power system" in its entirety (consisting of production centers, lines of energy transport and consumption centers); an airplane, such as the A380 Airbus, where control systems play a very important role; a factory production line, etc. We can act upon these systems and these react to actions exerted on them.

Objectives of systems theory

Several objectives exist in control systems theory.

Modeling

The above-mentioned systems are part of the material world and are thus governed by the laws of physics. By putting the system in an equation based on these laws, we obtain a mathematical model which will greatly facilitate its understanding. Therefore, *modeling* is one of the important activities of the control systems analyst. The modeling process does not always start with the laws of physics; it can simply be based on a more or less empirical and qualitative observation of the behavior of the system. Within the framework of this book, we will however limit ourselves to cases where a mathematical description of the system behavior is possible. In general, the model obtained consists of a set of differential equations (sometimes of partial differential equations) or of difference equations.

Identification

The modeling process, as understood, involves determining the structure of the equations which govern the behavior of the system, and also in fixing *a priori* the values of certain system parameters: for example, the lengths, masses, resistance values, capacitance values, etc. But it is often impossible to come to an *a priori* complete and precise understanding of all the parameters of this model. In order to refine and complete this understanding, it becomes necessary to proceed to an *identification* of the system: from the reactions of the latter to known and given stimulations, we can, under certain conditions, identify the yet unknown parameters. Identification is one of the major aspects in control theory.

Analysis

Once the system is modeled and identified, it becomes possible to analyze its behavior. This *analysis* can be very complex. As an example, the analysis of the European and North American electric systems is difficult and requires powerful computing resources and enormous databases because we deal with very large systems. And understanding these systems well is essential for security reasons: points such as the possibility of "black-out" risk? In general, the analysis of a system makes it possible to determine its essential properties.

Control

Thus, systems theory is a "theoretical" science in which one of its objectives is *knowing* systems, through modeling, identification and analysis. But it is also (we may be tempted to say essentially) a "practical" science, a science of "action". We try to understand systems in order to be able to control them, and to regulate them in the best possible way. The last major chapter of systems theory is that of *control*.

Open-loop and closed-loop systems

A fundamental difference exists between "open-loop" and "closed-loop" systems. Controlling an open-loop system is doing it blindly, without taking into account the results of the action taken. We know the expression: "I could walk down this path blind-folded". Because I know at which moment I need to turn left, and then right. I know when I need to accelerate, when to slow down,... And it is true that if, as with Laplace's genius, we had a perfect knowledge of the world, we could control, in open-loop, every system belonging to it. However, our knowledge of things is incomplete. Even with a route we know by heart, unpredictable events may occur: a child unexpectedly crossing the street, a spell of rain making the road more slippery than usual, etc. It is thus necessary to note at all times, and to adjust actions, taking into account the reality that appears from moment to moment. Control theory is the art (or science) of making this unceasing adjustment, which we call a *loop* or a *feedback* or a *servo-mechanism*. It is a common concept which nevertheless poses numerous problems.

One of the major difficulties encountered with feedback systems is their possible instability. There are certainly unstable open-loop systems, but they are relatively rare: a helicopter, an alternator connected with a long power line, certain types of fighter jets, etc. On the other hand, nothing is more commonplace than making a system unstable with a poor-performing feedback loop. Let us think of a child striving to take a shower: when the water is cold, he wants to heat it up, turns the hot water tap on too quickly, and gets boiling hot water; then, over-doing it the same way with the cold water tap, he makes the water ice cold, and so on.

The notion of feedback control is thus very powerful, but cannot be applied without proper knowledge.

Presentation

This book is divided into chapters, and sections; for example, section 2.1 is the first section of Chapter 2, and section 2.1.3 is the third subsection of section 2.1.

It contains the basics that a non-specialized engineer must know in control engineering. I had the opportunity to teach this course at Conservatoire National des Arts et Métiers (Paris), several engineering institutes Grandes Ecoles, Ecole Normale Supérieure de Cachan, and, for the most difficult parts, at Paris XI University (Master 2 level). The course contained in this book is progressive, making it accessible to any reader with an L2 level in science, and includes many examples. Nevertheless, to make it coherent, passages (sections or groups of sentences) had to be included in the first chapters that call upon somewhat more difficult notions which may need a second or third reading. *These passages are preceded by asterisks for sections, and situated between two asterisks for groups of words or sentences.*

The purpose of this book is to study *system modeling, identification, analysis and control.* Among these four themes, *modeling* is perhaps the trickiest problem: to know how to model electrical, mechanical, thermal, hydraulic, or other systems – as they are complex – a person has to be an electrical, mechanical, thermal, hydraulic engineer. It is physics in general, all of physics, which is useful for modeling; and modeling, which is the subject matter of Chapter 1, is not exclusive to the control engineer. Some examples will serve as basic reminders of hydraulics and thermodynamics. Reminders relative to electricity are somewhat less succinct. With respect to mechanics, I thought that it would be useful to go into more details. However, the only objective of the presentation is to enable the reader to understand examples and resolve exercises. Obviously, it cannot replace a treatise on mechanics.

From Chapters 2 to 11, the control engineer finds himself/herself in his/her own private domain. These chapters deal with linear systems analysis, control, and identification. I have put together some *mathematical elements* which I deemed essential in the *appendices* included in Chapters 12 and 13. This will spare the reader from constantly referring to the bibliography: elements of analysis first, and then of algebra. Some of them (Smith form of a polynomial matrix, $\Re\mathcal{H}_\infty$ algebra, module theory etc.) are probably new to most readers. But I preferred not to discuss them in the main body of the text, to save the latter, to the extent possible, for what truly comes within the scope of control theory. Likewise, the Laplace transform and the z-transform, which are *mathematical tools* that classically appear high in textbooks on control theory, are included in these appendices. These are organized as a presentation of "mathematics for systems theory" with proofs when they are constructive or simply useful for the reader's understanding. The reader can thus choose to refer to the appendices if needed or to read them entirely as actual chapters. I have decided not to present the theory of measure and integration. Concerning the latter, I refer the reader to the second volume of [35] or to [103]. I achieved this with the help of some mathematical gymnastics here and there.

About 80 exercises are given to the readers to test and refine their understanding of the subject. Solutions to most of the exercises are provided (sometimes in brief) in Chapter 14.

MATLAB and/or SCILAB files to run examples and solve exercises can be found at www.iste.co.uk/bourles/LS.zip.

Course outline

Beginner's course

The beginner at L3 (i.e. third-year undergraduate) level can start with the easiest parts of Chapter 1. The part that might be difficult for the beginner is section 1.2 which is devoted to the modeling of a mechanical system. I advise the reader to approach this part in a pragmatic way, using the examples and exercises which show what results are essential and how to make good use of them.

In Chapter 2, readers should leave aside everything that relates to multi-input multi-output systems. It is better if they focus on "left-forms", "right-forms", and on methods which make it possible to obtain system transfer functions. They will eventually develop particular interest in treated examples and exercises.

The reader should study Chapters 3 to 6 in depth (leaving aside, of course, the starred passages "intended for a second reading"). From my point of view, the reader's course therefore ends with the RST controller ("usual case" section).

Advanced course

The reader at M1 (i.e. first-year graduate) level should begin to go deeper into what he/she only skimmed over when he/she was a beginner, i.e. section 1.2 of Chapter 1, and Chapter 2. Afterwards, the reader will resolutely move on to Chapter 7, which includes complementary material on systems theory, and will then be ready to study

Chapters 8 to 11. I advise the reader to systematically skip the passages related to module theory.

Advanced graduate level

The reader at M2 (i.e. second-year graduate) level still has many things to discover before launching into specialized courses. These are all the passages that he/she skipped during previous readings, in particular those presenting the theory of systems in the language of modules. It may also be in the interest of the readers to systematically re-read the elements of mathematics in Appendices 1 and 2 (Chapters 12 and 13).

Coherent course and "butterfly" course

This book should be read in a more systematic way to understand the contents. Another approach to this book is also possible for readers who wish to be systematic while not necessarily belonging to the above categories: these readers should read the chapters of the book, without skipping any passage, in the order I have presented them (with the exception of the appendices – Chapters 12 and 13 – which they should read first). I have often started reading a science book in that spirit, and only a lack of time has made me take an opposite turn, which is to "flit about", trying to go as fast as possible (a strategy, besides, which does not always pay off). I do not have any way to propose to the "butterfly" readers, but I have included a detailed index at the end of the book for them.

Choices and necessities

Since this book is about linear time-invariant systems, I have deemed it essential to present this concept correctly. Such a system can no longer be properly defined by its transfer matrix (too poor a representation), or by Kalman's approach, as one of its state realizations, or by Rosenbrock's approach, as a "system matrix" (too contingent a representation). Wonham's "geometric approach", which was proposed in the 1980s, in the end, only reformulates Kalman's approach in a little more modern mathematical language, and does not seem to be a good answer. Following the works of experts in differential algebra, it appeared that, what control engineers call "system" and "linear system", should be defined as an *extension of a differential field* and a *module defined over a ring of differential operators,* respectively. In the early 1990s, this property was revealed by M. Fliess to the community of control engineering (see the synthesis presented in [47]). Such a language may have discouraged a beginner, but it is this conception that I have attempted to make the reader gradually sensitive to, in Chapters 2 and 7. In reality, module theory, at the level used here (finitely generated modules over principal ideal domains) is very simple for a reader somewhat accustomed to abstraction. It does clarify manipulations that can be done with polynomial matrices, as well as the Jordan canonical form. For this reason, I have made it the main theme of "passages intended for a third reading". The reader wishing to go further into systems theory can refer to [47] on nonlinear aspects, and [22] on linear time-varying systems ([15] is a preliminary version of [22]). Module theory is helpful in presenting (in the case of linear systems) the notion of flatness (due to Fliess and his collaborators [48]), which has become essential to efficiently resolve the problem of motion planning (see section 7.5).

I wanted to show that control engineering is not magic but science. On that premise, I have insisted, in Chapter 4, on the *limitations due to certain characteristics of the system to be controlled* (unstable poles and zeros, specifically). For more details on this point, see [51].

The control engineer is permanently confronted with the problem of uncertainties in modeling. Thus, this book is constantly preoccupied with the *robustness of control laws*. However, *this is not a book on robust control*. The theory of robustness has become extremely sophisticated, and its recent development is not covered here. Besides, excellent books on this issue ([107] and [122], among others) do exist.

I have emphasized in this book on control (whether in polynomial or state formalism) that system stabilization is exceptionally the role of the feedback loop; the point is often to compel the output of such a system to follow a reference signal, in spite of various disturbances that it may be subjected to. This is why – in order to design an RST controller as well as a control by state feedback, possibly with an observer – *it is essential to first make the necessary model augmentations* (following what Wonham called the "internal model principle"). Without this prerequisite, "modern methods" only generate nonsense.

It also appeared to me that it was important to show that, in the multi-input multi-output case, *a control law with execrable qualities can correspond to a good pole placement*, because the latter does not determine the former. The worst method (even though it is quite elegant from a strictly theoretical point of view) is probably the one that consists of reverting to a cyclic state matrix through a first loop, and then proceeding as in the single-input single-output case.

I have abstained from presenting the linear-quadratic (LQ) optimal control and its "dual" version, the Kalman filter. For the conception of control by state feedback and observer, I have indeed preferred to propose essentially algebraic methods for several reasons. Once these methods are well assimilated, it becomes easy to implement an efficient "LQG control".[1] The theory of LQG control (which encompasses LQ control and Kalman filter theories) is now classic and, at a basic level, is well covered in treatises such as [1] and [2]. On the other hand, the minimization of a criterion is only a convenient intermediate step for the control engineer to obtain a control law with the desired properties [78]. Another element that kept me from including optimal control

1. "LQG" stands for "Linear Quadratic Gaussian".

is that it would have made this book longer (especially to correctly present continuous-time stochastic optimal control, with, in the background, Ito's differential calculus [68], [118]). I found the volume of this book to be large enough. Nevertheless, except for a few subtleties, the methods proposed in the multi-input multi-output case in Chapters 8 and 9 only differ from LQ control and Kalman filtering in their presentation style (the minimization of a criterion is not presented as a goal – see Remark 245 in section 8.1.4). I encourage the reader to continue the study of this book along with that of "LQG control with frequency-shaped weights" [2], which is a good complement.

Only Chapter 10 deals with discrete-time systems and control. All previous chapters (including the one on RST controller) are thus presented in the context of continuous-time, which is a bit unusual. Despite the ideas spread by some people, I have experienced that continuous-time formalism offers much more flexibility for the design of control laws, especially with respect to the choice of poles and the question of "roll-off", which are essential to robustness. Once the synthesis of continuous-time control laws is mastered, it is very simple to switch to the synthesis of discrete-time control laws – without any approximation (this point is of course discussed in detail in Chapter 10).

Chapter 11 discusses parametric identification of discrete-time systems by minimization of the l_2 norm of the prediction error. The presentation is relatively classic as far as open-loop identification is concerned; as for closed-loop identification, it essentially includes Ljung and Forsell's contribution [82] (as a complement, the reader can consult [75] and Chapter 10 of [110]).

Many other topics would have been worth presenting: for example, anti-windup methods in the presence of saturations (which is clearly expounded in [4]) or gain scheduling (see survey papers [104] and [80]), not to mention nonlinear control (for a general presentation of this theme, see [108] and [62], and begin with the first reference because it is simple; *robust* nonlinear control is discussed in [49]; see also [48] which deals with *flatness* from both theoretical and practical viewpoints, in addition to section 7.5 of this book). We can also cite the following subjects: adaptive control (which is the subject of a lot of literature, but [57], [3] and [108] can be recommended as a good initiation on this theme); fuzzy or neural control (the reader can find a *genuinely scientific presentation* of the last two types of control in [40]); the control of systems governed by delayed differential equations or partial differential equations (on this subject, see [55] and [33] in the linear context, as well as [58] and [59] where nonlinear problems are discussed). As already mentioned, the extension of many methods presented in this book to linear time-varying systems can be found in [22].

Note on the English edition

This English edition has given me the opportunity to correct many errors which, despite proof-reading, had been left in the original French edition, and also include additional information which might be useful. Furthermore, it has given me the opportunity to complete Chapter 7 with a section on flatness and to entirely revise Chapter 11 on identification: the mathematical bases included therein are given in Appendix 1, and a section on closed-loop identification has also been added. Finally, this edition contains additional exercises and one of them presents the basics of what needs to be known about the "delta transform".

Acknowledgments

I would first like to thank my "elders", especially E. Irving, I.D. Landau and P. de Larminat, who have shared their experience with me during informal, but always passionate, discussions. I also thank M. Fliess who has opened my eyes to the algebraic theory of systems – which I hope this book has benefited from – and P. Chantre for our exchanges on identification. I thank my son Nicolas who has helped me finalize figures. I also thank my translator G.K. Kwan for his efforts and his kindness. Last but not the least, I thank my wife Corinne, whose patience has often been severely tested during this work which has demanded a lot of time and energy from me.

Chapter 1

Physical Models

1.1. Electric system

Electric circuits can be modeled by applying Kirchhoff's two laws: Kirchhoff's Current Law and Kirchhoff's Voltage Law. They are also known as the nodal rule and the mesh rule. Following are the two examples in which these two laws have been applied.

1.1.1. *Mesh rule*

Consider the "RLC" circuit shown in Figure 1.1.

This circuit constitutes a mesh. *The mesh rule* states that between any two points on a mesh, points A and B in this circuit for example, the potential difference is independent of the path that is taken (which, in physics, is a fundamental property of a *potential*, in this case the electric potential). Therefore, we have $V_A - V_B = V = V_R + V_L + V_C$, where V_R, V_L, and V_C are the voltages across the resistance R, the inductance L, and the capacitance C, respectively. We have $V_R = Ri, V_L = L\frac{di}{dt}, V_C = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$; therefore we obtain the integro-differential equation $V = Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$. Differentiating this equation we get the following linear differential equation:

$$LC\frac{d^2 i}{dt^2} + RC\frac{di}{dt} + i - C\frac{dV}{dt} = 0. \tag{1.1}$$

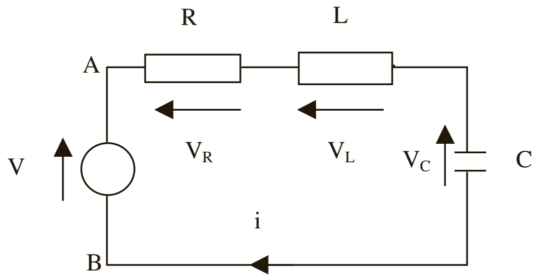

Figure 1.1. *RLC circuit*

1.1.2. *Nodal rule*

Consider the electric circuit given in Figure 1.2, which is a "pi circuit" used to represent a transmission line in the domain of electric networks.

According to *the mesh rule* described above, we have

$$V_A = \frac{1}{C}\int_{-\infty}^{t} i_1(\tau)d\tau, V_B = \frac{1}{C}\int_{-\infty}^{t} i_2(\tau)d\tau, V_A - V_B = Ri + L\frac{di}{dt}.$$

The nodal rule is based on the conservation of charge. According to this rule, at any node of a circuit, which can be any point within the circuit, the sum of current entering a node is equal to the sum of current leaving that node. Hence, we have the following two equations: $i_A = i + i_1$ and $i = i_B + i_2$. Eliminating the variable i, we can now re-arrange the equations as follows:

$$\left\{\begin{array}{c} C\frac{dV_A}{dt} - i_1 = 0, \\ C\frac{dV_B}{dt} - i_2 = 0, \\ V_A - V_B - R(i_A - i_1) - L\frac{d}{dt}(i_A - i_1) = 0, \\ i_A - i_1 - (i_B + i_2) = 0. \end{array}\right. \qquad (1.2)$$

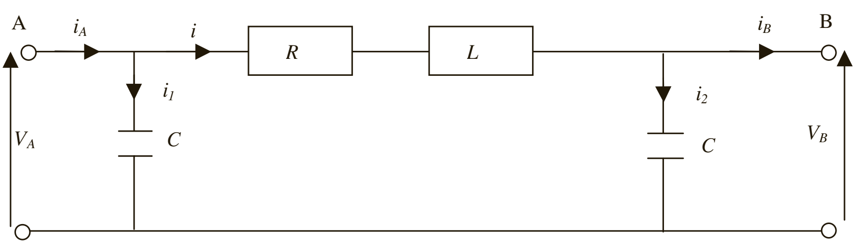

Figure 1.2. *Pi circuit*

1.2. Mechanical system

There are two main approaches to model a mechanical system. The first one uses the fundamental principle of dynamics, and is based on the *analysis of forces and moments* applied to the various elements of the system (particularly, internal forces, i.e. forces applied by subsystems on one another). The second approach uses the Lagrangian formalism and is based on the *energy of the system* (kinetic and potential energy). The analysis of internal forces is then unnecessary. These two approaches are succinctly expounded and applied to two examples.

1.2.1. *Fundamental principle of dynamics*

Torsor

A *torsor* is a field of antisymmetric vectors in a three-dimensional affine space. Let $\overrightarrow{M_\bullet}$ be a field of such vectors; this is a function $A \mapsto \overrightarrow{M_A}$, and $\overrightarrow{M_A}$ is called the *moment* of the torsor $\overrightarrow{M_\bullet}$ at point A. There exists a vector $\overrightarrow{V}$, called the *characteristic vector*, such that for any points A and B, we have the relation $\overrightarrow{M_A} = \overrightarrow{M_B} + \overrightarrow{AB} \wedge \overrightarrow{V}$. The torsor is thus entirely determined by $\overrightarrow{V}$ and the moment $\overrightarrow{M_A}$, for any point A under consideration. We are agreed on the notation $\left\{\overrightarrow{V}\right\}_A$ for the pair $\left(\overrightarrow{V}, \overrightarrow{M_A}\right)$ and $\left\{\overrightarrow{V}\right\}$ for the torsor $\overrightarrow{M_\bullet}$. The two vectors $\overrightarrow{V}$ and $\overrightarrow{M_A}$ are called the elements of reduction of torsor $\left\{\overrightarrow{V}\right\}$ at point A.

The *comoment of* $\left\{\overrightarrow{V}\right\}_A = \left(\overrightarrow{V}, \overrightarrow{M_A}\right)$ and $\left\{\overrightarrow{V'}\right\}_A = \left(\overrightarrow{V'}, \overrightarrow{M'_A}\right)$ is the scalar $\overrightarrow{V}.\overrightarrow{M'_A} + \overrightarrow{V'}.\overrightarrow{M_A}$. This comoment is invariant in the sense that it is independent of the point A considered (as can easily be verified by the reader). Therefore, the comoment of the *torsors* $\left\{\overrightarrow{V}\right\}$ and $\left\{\overrightarrow{V'}\right\}$ is well defined and is denoted by $\left\{\overrightarrow{V}\right\}.\left\{\overrightarrow{V'}\right\}$.

Kinematic torsor

An example of a torsor is the field of velocities of a *rigid body* S, called the *kinetic torsor* of S; this torsor is defined relative to an orthonormal frame of reference $\mathcal{R}$ (we also say the torsor is defined *in this frame of reference*). This is to say that the translational velocities and angular velocities are measured relative to the origin and along the axes of $\mathcal{R}$. Its characteristic vector is the vector of “instantaneous rotation” $\overrightarrow{\omega}$ (this torsor is thus denoted by $\{\overrightarrow{\omega}\}$). Indeed, as easily shown, the velocities at two points A and B on this rigid body are related by

$$\boxed{\overrightarrow{v_A} = \overrightarrow{v_B} + \overrightarrow{AB} \wedge \overrightarrow{\omega}}. \tag{1.3}$$

Now, let us consider a frame of reference $\mathcal{R}'$ attached to the rigid body S. This consists of an origin A and three basis vectors $\overrightarrow{e_i}, i = 1, 2, 3$. Put $\overrightarrow{e_i} = \overrightarrow{AA_i}$, where

A_i's are points on S. We have $\frac{d\overrightarrow{e_i}}{dt} = \overrightarrow{v_{A_i}} - \overrightarrow{v_A} = \overrightarrow{A_i A} \wedge \overrightarrow{\omega}$ according to (1.3), and thus $\frac{d\overrightarrow{e_i}}{dt} = \overrightarrow{\omega} \wedge \overrightarrow{e_i}$. Let $\overrightarrow{Y}$ be any vector; its coordinates within $\mathcal{R}'$ are those Y_i's such that $\overrightarrow{Y} = \sum_{i=1}^{3} Y_i \, \overrightarrow{e_i}$. Differentiating this expression, we get

$$\frac{d\overrightarrow{Y}}{dt} = \sum_{i=1}^{3} \frac{dY_i}{dt} \, \overrightarrow{e_i} + \sum_{i=1}^{3} Y_i \, \frac{d\overrightarrow{e_i}}{dt} = \sum_{i=1}^{3} \frac{dY_i}{dt} \, \overrightarrow{e_i} + \overrightarrow{\omega} \wedge \overrightarrow{Y}.$$

This quantity, denoting it by $\frac{d\overrightarrow{Y}}{dt/\mathcal{R}}$, is the derivative of $\overrightarrow{Y}$ in the reference frame $\mathcal{R}$. The vector $\sum_{i=1}^{3} \frac{dY_i}{dt} \, \overrightarrow{e_i}$ is the derivative of $\overrightarrow{Y}$ in this moving reference frame $\mathcal{R}'$ and is denoted by $\frac{d\overrightarrow{Y}}{dt/\mathcal{R}'}$. We have thus obtained the formula establishing the relation between the differentiation in a moving reference frame and the differentiation in a fixed reference frame as follows:

$$\boxed{\frac{d\overrightarrow{Y}}{dt/\mathcal{R}} = \frac{d\overrightarrow{Y}}{dt/\mathcal{R}'} + \overrightarrow{\omega} \wedge \overrightarrow{Y}}. \tag{1.4}$$

Kinetic torsor

Kinetic moment

The kinetic moment of a *material system* S (i.e. a collection of material points and rigid bodies – a material point can be considered a punctual rigid body) with respect to *any point* A in reference frame $\mathcal{R}$ is

$$\boxed{\overrightarrow{\sigma_A} \triangleq \int_S \overrightarrow{AM} \wedge \overrightarrow{v_M} \, dm}$$

where dm is the density of mass at point M; this density is assumed to be constant over time.[1]

Elements of reduction

The field of vectors $\overrightarrow{\sigma_\bullet}$ is a torsor, as will be shown below, and it is called the *kinetic torsor*.

Indeed, if B is any other point, we have

$$\overrightarrow{\sigma_A} = \int_S (\overrightarrow{AB} + \overrightarrow{BM}) \wedge \overrightarrow{v_M} \, dm = \overrightarrow{AB} \wedge \int_S \overrightarrow{v_M} \, dm + \int_S \overrightarrow{BM} \wedge \overrightarrow{v_M} \, dm.$$

Now let O be the origin of $\mathcal{R}$; we then get $\overrightarrow{v_M} = \frac{d\overrightarrow{OM}}{dt}$, thus $\int_S \overrightarrow{v_M} \, dm = \frac{d}{dt} \int_S \overrightarrow{OM} \, dm$. By definition of the *center of mass* G of S, we have

$$\int_S \overrightarrow{OM} \, dm = m \, \overrightarrow{OG}$$

1. The symbol $\triangleq$ means "equals by definition".

where m is the total mass of S. Therefore,

$$\boxed{\overrightarrow{\sigma_A} = \overrightarrow{\sigma_B} + \overrightarrow{AB} \wedge \overrightarrow{p}} \tag{1.5}$$

where $\overrightarrow{p}$ is the *momentum* of S, defined as

$$\boxed{\overrightarrow{p} \triangleq m\,\overrightarrow{v_G}}.$$

According to (1.5), $\overrightarrow{\sigma_\bullet}$ is actually a torsor $\{\overrightarrow{p}\}$ with characteristic vector $\overrightarrow{p}$.

Case of a rigid body

Suppose S is a *rigid body* and A is *any* point on this rigid body; according to (1.3) we get $\overrightarrow{v_M} = \overrightarrow{v_A} + \overrightarrow{MA} \wedge \overrightarrow{\omega} = \overrightarrow{v_A} + \overrightarrow{\omega} \wedge \overrightarrow{AM}$, thus

$$\begin{aligned}\overrightarrow{\sigma_A} &= \int_S \overrightarrow{AM} \wedge \left(\overrightarrow{v_A} + \overrightarrow{\omega} \wedge \overrightarrow{AM}\right) dm \\ &= \int_S \overrightarrow{AM}\, dm \wedge \overrightarrow{v_A} + \int_S \overrightarrow{AM} \wedge \overrightarrow{\omega} \wedge \overrightarrow{AM}\, dm.\end{aligned}$$

The linear mapping $\mathcal{I}_A\ : \overrightarrow{\omega} \mapsto \int_S \overrightarrow{AM} \wedge \overrightarrow{\omega} \wedge \overrightarrow{AM}\, dm$ is called the *inertia tensor* of S with respect to A, and hence we have

$$\boxed{\int_S \overrightarrow{AM} \wedge \overrightarrow{\omega} \wedge \overrightarrow{AM}\, dm = \mathcal{I}_A\, \overrightarrow{\omega}}. \tag{1.6}$$

We can express this tensor in terms of its coordinates in an orthonormal system of reference $\mathcal{R}'$. Let (x, y, z) be the components of $\overrightarrow{AM}$ in $\mathcal{R}'$. Developing expression (1.6), the linear mapping $\mathcal{I}_A$ is identified[2] with the *inertia matrix* of S with respect to A in $\mathcal{R}'$, given by

$$\mathcal{I}_A = \begin{bmatrix} \int_S \left(y^2 + z^2\right) dm & -\int_S xy\, dm & -\int_S xz\, dm \\ -\int_S xy\, dm & \int_S \left(x^2 + z^2\right) dm & -\int_S yz\, dm \\ -\int_S xz\, dm & -\int_S yz\, dm & \int_S \left(x^2 + y^2\right) dm \end{bmatrix}.$$

We thus obtain

$$\boxed{\overrightarrow{\sigma_A} = m\,\overrightarrow{AG} \wedge \overrightarrow{v_A} + \mathcal{I}_A\, \overrightarrow{\omega}}. \tag{1.7}$$

This expression can be simplified when $A = G$ or when A is a *fixed* point on S (if S rotates around this point); then (1.7) reduces to

$$\boxed{\overrightarrow{\sigma_A} = \mathcal{I}_A\, \overrightarrow{\omega}}. \tag{1.8}$$

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2. Once the choice of bases is made, a linear mapping (also called a homomorphism) is represented by a matrix (see section 13.3.2). Abusing the language, we can identify this linear mapping with this matrix. And this is what we do here. "Abuse of language" (in the sense widely used in mathematics) and "abuse of notation" are considered to be synonymous in this book.

Inertia matrix

Matrix $\mathcal{I}_A$ is only constant when the reference frame $\mathcal{R}'$ is rigidly linked to rigid body S and it is therefore our interest to look at things in the context of this case. Of course, this reference frame is in motion, and the derivative of (1.8) is obtained by applying (1.4). On the other hand, the matrix $\mathcal{I}_A$ is symmetric real, and thus can be diagonalized in an orthonormal reference frame (see section 13.5.6), and whose axes are by definition the *principal axes of inertia* of S. The diagonal elements obtained in such a matrix are called the *principal moments of inertia* of S (with respect to A and relative to the principal axes in question).

Torque

In the case where $\overrightarrow{p} = 0$, the kinetic moment is independent of the point being considered, according to (1.5): for any points A and B, $\overrightarrow{\sigma_A} = \overrightarrow{\sigma_B}$. Such a kinetic moment is called a *torque*, which we shall denote by $\overrightarrow{C}$.

Kinetic energy

The kinetic energy T of a material system S is equal to the comoment $\{\overrightarrow{\omega}\} . \{\overrightarrow{p}\}$. In case of a rigid body, we obtain

$$\boxed{T = \frac{1}{2}\left(m\, v_G^2 + \overrightarrow{\omega}.\mathcal{I}_G\, \overrightarrow{\omega}\right)}. \tag{1.9}$$

Force torsor

Now consider a set of external forces $\overrightarrow{f_1}, ..., \overrightarrow{f_n}$ being applied to points $A_1, ..., A_n$ of a material system S. The resultant force is given by $\overrightarrow{f} = \sum_{k=1}^{n} \overrightarrow{f_k}$, while the resultant moment of these forces with respect to a point O, arbitrary at this time, is $\overrightarrow{\mathcal{M}_O} = \sum_{k=1}^{n} \overrightarrow{OA_k} \wedge \overrightarrow{f_k}$. If A is another arbitrary point, we immediately obtain the equality

$$\boxed{\overrightarrow{\mathcal{M}_A} = \overrightarrow{\mathcal{M}_O} + \overrightarrow{AO} \wedge \overrightarrow{f}} \tag{1.10}$$

which shows that the field $\overrightarrow{\mathcal{M}_\bullet}$ defines a torsor with characteristic vector $\overrightarrow{f}$; this torsor $\left\{\overrightarrow{f}\right\}$ is called the *force torsor* (or, more precisely, the *torsor of external forces*).

Fundamental principle of dynamics (Newton's law)

Expression in a Galilean reference frame

Let O be a *fixed point* in a *Galilean reference frame* (also known as *inertial reference frame*) $\mathcal{R}$, and S be a material system. The *fundamental principle of* *dynamics* or *Newton's law* is written as

$$\boxed{\left\{\overrightarrow{f}\right\}_O = \tfrac{d}{dt}\left\{\overrightarrow{p}\right\}_O}. \tag{1.11}$$

This equality between torsors shows that we have the following two relations

$$\overrightarrow{f} = \frac{d\overrightarrow{p}}{dt} \tag{1.12}$$

$$\overrightarrow{\mathcal{M}_O} = \frac{d\overrightarrow{\sigma_O}}{dt}. \tag{1.13}$$

It is reasonable to emphasize the fact that (1.11) is valid in the reference frame $\mathcal{R}$ (or in any other Galilean reference frame). In a non-Galilean frame of reference, it is necessary to take into account the torsor of inertial forces as well as Coriolis forces [76] – notions that can be derived from (1.4).

Expression in a moving frame of reference firmly fixed in the center of mass

Nevertheless, let us consider the center of mass G of the material system S; from (1.5) we get

$$\begin{aligned}\frac{d\overrightarrow{\sigma_G}}{dt} &= \frac{d\overrightarrow{\sigma_O}}{dt} + \frac{d}{dt}\left(\overrightarrow{GO}\wedge\overrightarrow{p}\right) = \overrightarrow{\mathcal{M}_O} - \overrightarrow{v_G}\wedge\overrightarrow{p} + \overrightarrow{GO}\wedge\frac{d\overrightarrow{p}}{dt}\\ &= \overrightarrow{\mathcal{M}_O} + \overrightarrow{f}\end{aligned}$$

according to (1.12). Therefore,

$$\frac{d\overrightarrow{\sigma_G}}{dt} = \overrightarrow{\mathcal{M}_G}$$

which shows that

$$\boxed{\left\{\overrightarrow{f}\right\}_G = \tfrac{d}{dt}\left\{\overrightarrow{p}\right\}_G}.$$

Expression in an arbitrary moving frame of reference

Let A be any *arbitrary* point. We have, according to (1.8), $\overrightarrow{\sigma_O} = \overrightarrow{\sigma_A} + \overrightarrow{OA}\wedge\overrightarrow{p}$, and thus

$$\frac{d\overrightarrow{\sigma_O}}{dt} = \frac{d\overrightarrow{\sigma_A}}{dt} + \overrightarrow{v_A}\wedge\overrightarrow{p} + \overrightarrow{OA}\wedge\frac{d\overrightarrow{p}}{dt}.$$

From (1.10) and (1.12) we also have

$$\overrightarrow{\mathcal{M}_A} = \overrightarrow{\mathcal{M}_O} + \overrightarrow{AO}\wedge\frac{d\overrightarrow{p}}{dt}. \tag{1.14}$$

From (1.13) we obtain

$$\boxed{\overrightarrow{\mathcal{M}_A} = \frac{d\overrightarrow{\sigma_A}}{dt} + \overrightarrow{v_A} \wedge \overrightarrow{p}}$$

which is a generalization of (1.13).

Moving frame of reference: case of a rigid body

Let A be a point on *rigid body* S. The kinetic moment $\overrightarrow{\sigma_A}$ is given by expression (1.7), therefore

$$\frac{d\overrightarrow{\sigma_A}}{dt} = m\,\overrightarrow{AG} \wedge \frac{d\overrightarrow{v_A}}{dt} + m\,\overrightarrow{v_G} \wedge \overrightarrow{v_A} + \mathcal{I}_A\,\frac{d\overrightarrow{\omega}}{dt} + \overrightarrow{\omega} \wedge \mathcal{I}_A\,\overrightarrow{\omega}\,.$$

According to (1.14) we obtain the following:

$$\boxed{\overrightarrow{\mathcal{M}_A} = \mathcal{I}_A\,\frac{d\overrightarrow{\omega}}{dt} + \overrightarrow{AG} \wedge m\frac{d\overrightarrow{v_A}}{dt} + \overrightarrow{\omega} \wedge \mathcal{I}_A\,\overrightarrow{\omega}}. \tag{1.15}$$

This expression can be simplified in the following cases:

i) A is fixed and S rotates around A, or $A = G$ (the second term on the right-hand side drops out);

ii) rotation is around one of the principal axes of inertia (the third term drops out).

Case of a rigid body rotating around an axis

Consider the case where S is a rigid body rotating around one of the principal axes of inertia, denoted by $\overrightarrow{Oz}$ for example. Let $\overrightarrow{C}$ be the resultant torque exerted on S (directed along $\overrightarrow{Oz}$) and write $\overrightarrow{C} = C\,\overrightarrow{k}$, where $\overrightarrow{k}$ is the unit vector of $\overrightarrow{Oz}$. The rotation vector of S is $\overrightarrow{\omega} = \omega\,\overrightarrow{k}$. We have $\mathcal{I}_O\,\overrightarrow{\omega} = J\,\overrightarrow{\omega} = J\,\omega\,\overrightarrow{k}$, where J is the principal moment of inertia of Σ with respect to the axis $\overrightarrow{Oz}$, i.e.

$$\boxed{J = \int_\Sigma \left(x^2 + y^2\right)\,dm}. \tag{1.16}$$

Equation (1.15) reduces to

$$\boxed{J\,\frac{d\omega}{dt} = C}. \tag{1.17}$$

Principle of action and reaction

In the case of a material system S that consists of N interacting rigid bodies S_i, the global equation (1.11) is not sufficient to determine the motion of each of the rigid bodies. We therefore are led to decompose the system into each of its elements S_i and to take into account the forces and moments these rigid bodies exert on one another.