Fatigue of Materials and Structures

Application to Design and Damage

Edited by
Claude Bathias and André Pineau

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Foreword

This book on fatigue, combined with two other recent publications edited by Claude Bathias and André Pineau[1], are the latest in a tradition that traces its origins back to a summer school held at Sherbrooke University in Quebec in the summer of 1978 which was organized by Professors Claude Bathias (then at the University of Technology of Compiegne, France) and Jean Pierre Bailon of Ecole Polytechnique, Montreal, Quebec. This meeting was held under the auspices of a program of cultural and scientific exchanges between France and Quebec. As one of the participants in this meeting, I was struck by the fact that virtually all of the presentations provided a tutorial background and an in-depth review of the fundamental and practical aspects of the field as well as a discussion of recent developments. The success of this summer school led to the decision that it would be of value to make these lectures available in the form of a book which was published in 1980. This broad treatment made the book appealing to a wide audience. Indeed, within a few years, dog-eared copies of “Sherbrooke” could be found on the desks of practicing engineers, students and researchers in France and in French-speaking countries. The original book was followed by an equally successful updated version that was published in 1997 which preserved the broad appeal of the first book. This book represents a part of the continuation of the approach taken in the first two editions while providing an even more in-depth treatment of this crucial but complex subject.

It is also important to draw attention to the highly respected “French School” of fatigue which has been at the forefront in integrating the solid mechanics and materials science aspects of fatigue. This integration led to the development of a deeper fundamental understanding thereby facilitating application of this knowledge to real engineering problems from microelectronics to nuclear reactors. Most of the authors who have contributed to the current edition have worked together over the years on numerous high-profile, critical problems in the nuclear, aerospace, and power generating industries. The informal teaming over the years perfectly reflects the mechanics/materials approach and, in terms of this book, provides a remarkable degree of continuity and coherence to the overall treatment.

The approach and ambiance of the "French School" is very much in evidence in a series of bi-annual international colloquia. These colloquia are organized by a very active "fatigue commission" within the French Society of Metals and Materials (SF2M) and are held in Paris in the spring. Indeed, these meetings have contributed to an environment which fostered the publication of this series.

The first two editions (in French), while extremely well-received and influential in the French-speaking world, were never translated into English. The third edition was recently published (again in French) and has been very well received in France. Many English-speaking engineers and researchers with connections to France strongly encouraged the publication of this third edition in English. The current three books on fatigue were translated from the original four volumes in French[2] in response to that strong encouragement and wide acceptance in France.

In his preface to the second edition, Prof. Francois essentially posed the question (liberally translated), "Why publish a second volume if the first does the job?" A very good question indeed! My answer would be that technological advances place increasingly severe performance demands on fatigue-limited structures.

Consider, as an example, the economic, safety and environmental requirements in the aerospace industry. Improved economic performance derives from increased payloads, greater range and reduced maintenance costs. Improved safety, demanded by the public, requires improved durability and reliability. Reduced environmental impact requires efficient use of materials and reduced emission of pollutants. These requirements translate into higher operating temperatures (to increase efficiency), increased stresses (to allow for lighter structures and greater range), improved materials (to allow for higher loads and temperatures) and improved life prediction methodologies (to set safe inspection intervals). A common thread running through these demands is the necessity to develop a better understanding of fundamental fatigue damage mechanisms and more accurate life prediction methodologies (including, for example, application of advanced statistical concepts). The task of meeting these requirements will never be completed; advances in technology will require continuous improvements in materials and more accurate life prediction schemes. This notion is well illustrated in the rapidly developing field of gigacycle fatigue. The necessity to design against fatigue failure in the regime of 10^9 + cycles in many applications required in-depth research which in turn has called into question the old, comfortable notion of a fatigue limit at 10^7 cycles. New developments and approaches are an important component of this edition and are woven through all of the chapters of the three books.

It is not the purpose of this preface to review all of the chapters in detail. However, some comments about the organization and over-all approach are in order. The first chapter in the first book[3] provides a broad background and historical context and sets the stage for the chapters in the subsequent books. In broad outline, the experimental, physical, analytical and engineering fundamentals of fatigue are developed in this first book. However, the development is done in the context of materials used in engineering applications and numerous practical examples are provided which illustrate the emergence of new fields (e.g. gigacycle fatigue) and evolving methodologies (e.g. sophisticated statistical approaches). In the second[4] and third[5] books, the tools that are developed in the first book are applied to newer classes of materials such as composites and polymers and to fatigue in practical, challenging engineering applications such as high temperature fatigue, cumulative damage and contact fatigue.

These three books cover the most important fundamental and practical aspects of fatigue in a clear and logical manner and provide a sound basis that should make them as attractive to English-speaking students, practicing engineers, and researchers as they have proved to be to our French colleagues.

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December 2010

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C. BATHIAS, A. PINEAU (eds.), *Fatigue of Materials and Structures: Application to Damage*, ISTE, London and John Wiley & Sons, New York, 2011.

2. C. BATHIAS, A. PINEAU (eds.), *Fatigue des matériaux et des structures*, Volumes 1, 2, 3 and 4, Hermes, Paris, 2009.

3. C. BATHIAS, A. PINEAU (eds.), *Fatigue of Materials and Structures: Fundamentals*, ISTE, London and John Wiley & Sons, New York, 2010.

4. C. BATHIAS, A. PINEAU (eds.), *Fatigue of Materials and Structures: Application to Damage*, ISTE, London and John Wiley & Sons, New York, 2011.

5. This book.

Chapter 1

Multiaxial Fatigue [1]

1.1. Introduction

Nowadays, everybody agrees on the fact that good multiaxial constitutive equations are needed in order to study the stress-strain response of materials. After many studies, a number of models have been developed, and the "quality/cost" ratio of the different existing approaches is well defined in the literature. Things are very different in the case of the characterization of multiaxial fatigue. In this domain, as in others related to the study of damage and failure phenomena, the phase of "settling" which leads to the classification of the different approaches has not been carried out yet, which can explain why so many different models are available. These models are not only different because of the different types of equations they present, but also because of their critical criteria. The main reason is that fatigue phenomena involve some local mechanisms, which are thus controlled by some local physical variables, and which are thus much more sensitive to the microstructure of the material rather than to the behavior laws which only give a global response. It is then difficult to present in a single chapter the entire variety of the existing fatigue criteria for the endurance as well as for the low cycle fatigue domains.

Nevertheless, right at the design phase, the improvements of the methods and the tools of numerical simulation, along with the growth supported by any available calculation power, can provide some historical data stress and strain to the engineer in charge of the study. The multiaxiality of both stresses and strains is a fundamental aspect for a high number of safety components: rolling issues, contact—friction problems, anisothermal multiaxial fatigue issues, etc. Multiaxial fatigue can be observed within many structures which are used in every day life (suspension hooks, subway gates, automotive suspensions). In addition to these observations, researchers and engineers regularly pay much attention to some important and common applications: fatigue of railroads involving some complex phenomena, where the macroscopic analysis is not always sufficient due to metallurgical modifications within the contact layer. Friction can also be a critical phenomenon at any scale, from the industrial component to micromachines. The thermo-mechanical aspects are also fundamental within the hot parts of automotive engines, of nuclear power stations, of aeronautical engines, but also in any section of the hydrogen industry for instance. The effects of fatigue then have to be evaluated using adapted models, which consider some specific mechanisms. This chapter presents a general overview of the situation, stressing the necessity of defending some rough models which can be clearly applied to some random loadings rather than a simple smoothing effect related to a given experiment, which does not lead to any interesting general use.

Brown and Miller, in a classification released in 1979 [BRO 79], distinguish four different phases in the fatigue phenomenon: (i) nucleation — or microinitiation — of the crack; (ii) growth of the crack depending on a maximum shearing plane; (iii) propagation normal to the traction strain; (iv) failure of the specimen. The germination and growth steps usually occur within a grain located at the surface of the material. The growth of the crack begins with a step, which is called “short crack”, during which the geometry of the crack is not clearly defined. Its propagation direction is initially related to the geometry and to the crystalline orientations of the grains, and is sometimes called micropropagation. The microscopic initiation, from the engineer's point of view, will also be the one which will get most attention from the mechanical engineer because of its volume element: it perfectly matches the moment where the size of the crack becomes large enough for it to impose its own stress field, which is then much more important than the microstructural aspects. At this scale (usually several times the size of the grains), it is possible to give a geometric sense to the crack, and to specifically treat the problem within the domain of failure mechanics, whereas the first ones are mainly due to the fatigue phenomenon itself. This chapter gathers the models which can be used by the engineer and which lead to the definition of microscopic initiation.

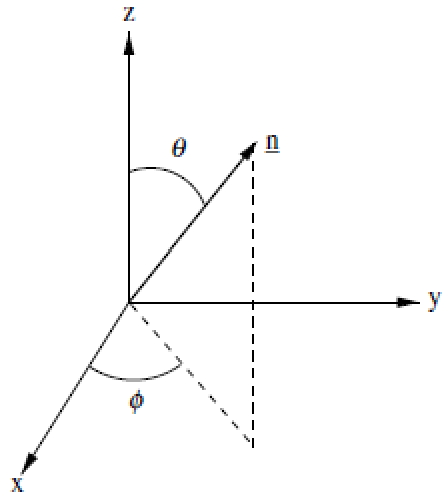
Section 1.2 of this chapter presents the different ingredients which are necessary to the modeling of multiaxial fatigue and introduces some techniques useful for the implementation of any calculation process in this domain, especially regarding the characterization of multiaxial fatigue cycles. Section 1.3 briefly deals with the main experimental results in the domain of endurance which will lead to the design of new models. Section 1.4 tries then to give a general idea of the endurance criteria under multiaxial loading, starting with the most common ones and presenting some more recent models. Finally, section 1.5 introduces the domain of low cycle multiaxial fatigue. Once again, some choices had to be made regarding the presented criteria, and we decided to focus on the diversity of the existing approaches, without pretending to be exhaustive.

1.1.1. *Variables in a plane*

Some of the fatigue criteria that will be presented below — which are of the critical plane type — can involve two different types of variables: the variables related to the stresses and the strains normal to a given plane, and the variables related to the strains or stresses that are tangential to this same plane.

From a geometric point of view, a plane can be observed from its normal line $\underline{n}$. The criteria involving an integration in every plane usually do so by analyzing the planes thanks to two different angles, θ and ϕ, which can respectively vary between 0 and π, and 0 and 2π in order to analyze all the different planes (see Figure 1.1).

Figure 1.1. *Spotting the normal to a plane via both angles θ and ϕ within a Cartesian reference frame*



In the case of a normal plane $\underline{n}$ and in the case of a stress state $\underset{\sim}{\sigma}$, the stress vector normal to the plane, $\underline{T}$, is given by:

[1.1] $\underline{T} = \underset{\sim}{\sigma}.\underline{n}$

where represents the results of the matrix-vector multiplication, with a contraction on the index. This stress vector can be split into a normal stress defined by the scalar variable σ_n and a tangential stress vector $\underline{\tau}$, which can be written as:

[1.2] $\sigma_n = \underline{n}.\underline{T} = \underline{n}.\underset{\sim}{\sigma}.\underline{n}$

[1.3] $\underline{\tau} = \underline{T} - \sigma_n \underline{n}$

The normal value of the vector $\underline{\tau}$ is equal to:

[1.4] $\|\underline{\tau}\| = \left(\|\underline{T}\|^2 - \sigma_n^2\right)^{1/2}$

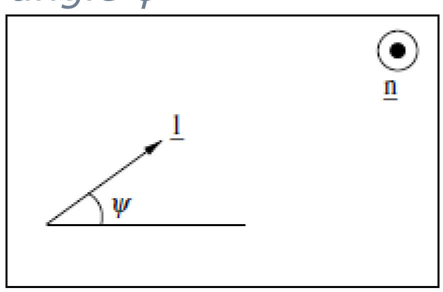
Some attention can also be paid to the resolved shear stress vector, which corresponds to the projection of the tangential stress vector towards a single direction $\underline{l}$ given by normal plane $\underline{n}$, which is then equal to:

[1.5] $\tau(\underline{l}) = \underline{l}.\underline{\tau} = \underline{l}.\underline{T} = \underline{l}.\underline{\sigma}.\underline{n} = \underline{\sigma} : \underline{m}$

where $\underline{m}$ is the orientation tensor, symmetrical section of the product of $\underline{l}$ and $\underline{n}$, and where stands for the product which is contracted two times between two symmetrical tensors of second order, $(\underline{l} \otimes \underline{n} + \underline{n} \otimes \underline{l})/2.$.

In this case, a specific direction will be spotted by an angle ψ and it will be then possible to integrate in any direction for a normal plane $\underline{n}$ when ψ varies between 0 and 2π (see Figure 1.2).

Figure 1.2. *Spotting of a direction $\underline{l}$ within a plane of normal $\underline{n}$ thanks to angle ψ*



Thus, the integral of a variable *f* in any direction within any plane can be written as:

[1.6] $\iiint f(\theta, \phi, \psi) \mathrm{d}\psi \sin\theta \mathrm{d}\theta \mathrm{d}\phi$

1.1.1.1. *Normal stress*

As the normal stress σ_n is a scalar variable, it can be used as is in the fatigue criteria. Thus, a cycle can be defined with:

– $\sigma_{n_{min}}$, the minimum normal stress;

– $\sigma_{n_{max}}$, the maximum normal stress;

– $\sigma_{n_a} = (\sigma_{n_{max}} - \sigma_{n_{min}})/2$, the amplitude of the normal stress;

– $\sigma_{n_m} = (\sigma_{n_{max}} + \sigma_{n_{min}})/2$, the average normal stress;

– $\sigma_{n_a}(t) = \sigma_n(t) - \sigma_{n_m}$, the alternate part of the normal stress at time t.

1.1.1.2. *Tangential stress*

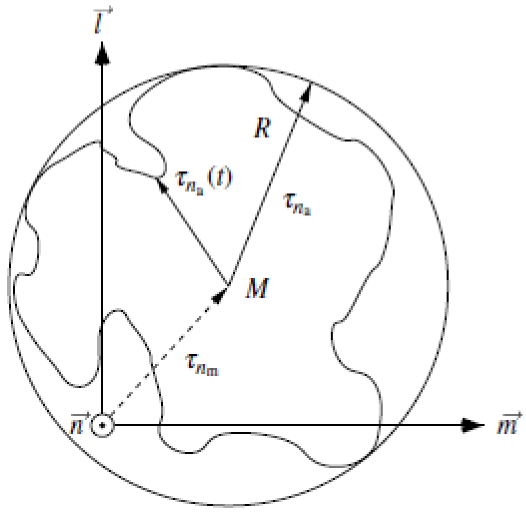
As the tangential stress is a vector variable, it will have to undergo some additional treatment in order to get the scalar variables at the scale of a cycle. To do so, the smallest circle circumscribed to the path of the tangential stress will be used, which has a radius R and a center M (see Figure 1.3). The following variables are thus defined:

– τ_{n_a}, the tangential stress amplitude, equal to radius R of the circle circumscribed to the path defined by the end of the tangential stress vector within the plane of the facet;

– τ_{n_m}, the average tangential stress, equal to the distance *OM,* from the origin of the reference frame to the center of the circumscribed circle;

– $\tau_{n_a}(t) = \|\underline{\tau}(t) - \underline{\tau_{n_m}}\|$, the alternate tangential stress.

Figure 1.3. *Smallest circle circumscribed to the tangential stress*



1.1.1.3. *Determination of the smallest circle circumscribed to the path of the tangential stress*

Several methods were proposed to determine the smallest circle circumscribed to the tangential stress [BER 05], like:

– the algorithm of points combination proposed by Papadopoulos (however, this method should not be applied to high numbers of points, as the calculation time becomes far too long — this algorithm is given as $O(n^4)$ [BER 05]);

– an algorithm proposed by Weber *et al*. [WEB 99b], written as $O(n^2)$ [BER 05], which relies on the Papadopoulos approach but also leads to a significant reduction of the calculating times as it does not calculate all the possible circles as is usually done in the case of the Papadopoulos method;

– the incremental method proposed by Dang Van *et al*. [DAN 89], whose cost cannot be that easily and theoretically evaluated, and which seems to be given as $O(n)$, might not converge straight away in some cases [WEB 99b];

– some optimization algorithms of minimax types, whose performances depend on the tolerance of the initial choice;

– some algorithms said to be “random” [WEL 91, BER 98], given as $O(n)$, which seem to be the most efficient [BER 05].

A random algorithm is briefly presented in this section ([WEL 91, BER 98]). This algorithm was strongly optimized by Gärtner [GÄR 99]. This algorithm consists of reading the list of the considered points and adding them one-by-one to a temporary list if they are found to belong to the current circumscribed circle. If the new point to be inserted is not within the circle, a new circle must be found, and then one or two subroutines have to be used in order to build the new circle circumscribed to the points which have already been added into the list. The pseudo-code of this algorithm is given later on.

In this case, $\mathcal{P}$ stands for the list of the *n* points whose smallest circumscribed circle has to be determined, P_i are the points belonging to this list and $\mathcal{P}_t$ are the lists of points which were considered. The function $CIRCLE(P_1, P_2(, P_3))$ gives a circle defined by a diameter (for a two-point input) or by three-points (for a three point input) when all the points have already been added to $\mathcal{P}_t$.

main

$\mathcal{C} \leftarrow CIRCLE(P_1, P_2)$

$\mathcal{P}_t \leftarrow P_1, P_2$

for $i \leftarrow 3$ **to** n

$$\textbf{do} \begin{cases} \textbf{if } P_i \in \mathcal{C} \\ \textbf{then } \left\{ \mathcal{P}_t \leftarrow P_i \right. \\ \\ \textbf{else } \begin{cases} \mathcal{C} \leftarrow \mathrm{NVC2}(\mathcal{P}_t, P_i) \\ \mathcal{P}_t \leftarrow P_i \end{cases} \end{cases}$$

return $(\mathcal{C})$

```
procedure NVC2(𝒫, P)
𝒞 ← CIRCLE(P, P1)
𝒫t ← P1
for j ← 2 to SizeOf(𝒫)
do { if Pj ∈ 𝒞
     then { 𝒫t ← Pj
     else { 𝒞 ← NVC3(𝒫t, P, Pj)
            𝒫t ← Pj
return (𝒞)
```

```
procedure NVC3(𝒫, P1, P2)
𝒞 ← CIRCLE(P1, P2, P1)
𝒫t ← P1
for k ← 2 to SizeOf(𝒫)
do { if Pk ∈ 𝒞
     then { 𝒫t ← Pk
     else { 𝒞 ← CIRCLE(𝒫t, P, Pk)
            𝒫t ← Pk
return (𝒞)
```

1.1.1.4. *Notations regarding the strains occurring within a plane*

For some models, especially those focusing on the low cycle fatigue domain, some strains also have to be formulated, corresponding to the stress variables which have already been defined, namely:

– $\varepsilon^n(t)$: strain normal to the critical plane at time *t;*

– γ^n_{mean}: average value, within a cycle, of the shearing effect on the critical plane with a normal variable $\underline{n}$;

– γ^n_{am}: amplitude, within a cycle, of the shearing effect on the critical plane with a normal variable $\underline{n}$;

– $\gamma^n(t)$: shearing effect at time *t;*

– $\gamma^n_{\text{am}}(t)$: amplitude of the shearing effect at time *t*.

1.1.2. ***Invariants***

1.1.2.1. *Definition of useful invariants*

Some criteria can be written as functions of the invariants of the stress (or strain) tensor. Most of the criteria which deal with the endurance domain are given as stresses, as they describe some situations where mechanical parts are mainly elastically strained.

The tensor of the stresses $\underline{\underline{\sigma}}$ can be split into a hydrostatic part, which is written as:

[1.7] $I_1 = \mathrm{tr}\underset{\sim}{\sigma} = \sigma_{ii}$ with $p = \frac{I_1}{3}$

(tr represents the trace, I_1 gives the first invariant of the stress tensor, and *p*, the hydrostatic pressure), and into a deviatoric part, written as $\underset{\sim}{s}$, defined by:

[1.8] $\underset{\sim}{s} = \underset{\sim}{\sigma} - \frac{I_1}{3}\underset{\sim}{1}$ or as components: $s_{ij} = \sigma_{ij} - \frac{\sigma_{ii}}{3}\delta_{ij}$

To be more specific, the second invariant J_2 of the stress deviator can be written as:

[1.9] $J_2 = \frac{\mathrm{tr}(\underset{\sim}{s} : \underset{\sim}{s})}{2} = \frac{s_{ij}s_{ji}}{2}$

It will also be convenient to use the invariant *J* instead of J_2, as it is reduced to $/\sigma_{11}/$ for a uniaxial tension with only one component which is not equal to zero σ_{11}:

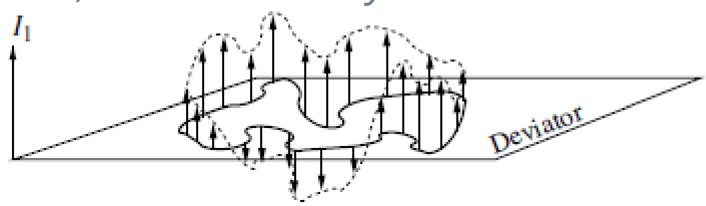
[1.10] $J = \left(\frac{3}{2}s_{ij}s_{ji}\right)^{1/2}$

The previous value has to be analyzed with that of the octahedral shearing stresses, which is the shearing stress occurring on the plane which has a similar angle with the three main directions of the stress tensor (σ_1, σ_2, σ_3). It can be written as:

[1.11] $\tau_{\mathrm{oct}} = \frac{\sqrt{2}}{3}J$

J defines the shearing effect within the octahedral planes, which will be some of the favored planes to represent the fatigue criteria, as all the stress states, which are only different by a hydrostatic tensor, are perpendicularly dropped, within these planes, on the same point. Within the main stress space, *J* characterizes the radius of the von Mises cylinder, which corresponds to the distance of the operating point from the (111) axis. The effects of the hydrostatic pressure are then isolated from the pure shearing effects in the equations of the models. Figure 1.4 illustrates the evolution of the first invariant and of the deviator of the stress tensor at a specific point during a cycle.

Figure 1.4. *Variation of the first invariant of the stress tensor and of its deviator during a cycle (the “plane” is the deviatoric space, which actually has a dimension of 5)*



In order to characterize a uniaxial mechanical cycle, two different variables have to be used. For instance, the maximum stress and the average stress can be used. In the case of multiaxial conditions, an amplitude as well as an average value will be used, respectively, for the deviatoric component and for the hydrostatic component. The hydrostatic component comes to a scalar variable, so the calculation of its average value and of its maximum during a cycle does not lead to any specific issue.

The case of the deviatoric component is much more complex, as it is a tensorial variable which can be represented within a 5 dimensional space (deviatoric space).

An octahedral shearing amplitude $\Delta\tau_{oct}/2$ can also be defined, and within the main reference frame of the stress tensor, when it does not vary, is written as:

$$\frac{\Delta\tau_{oct}}{2} = \frac{1}{\sqrt{2}}\left[(a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_1)^2\right]^{1/2} \quad [1.12]$$

with $a_i = \Delta\sigma_i/2$. However, this equation cannot be used in the general case as the orientation of the main reference frame of the stress tensor varies, and some other approaches thus have to be used. Usually, the radius and the center of the smallest hyper-sphere (which is more commonly called the smallest circle) circumscribed to the loading path (compared to a fixed reference frame) respectively correspond to the average value $(\underset{\sim}{X}_i)$ and to the amplitude ($\Delta\tau_{oct}/2$) of the deviatoric component. This issue is actually a 5D extension of the 2D case which has been presented above in the determination of the smallest circle circumscribed to the path of the tangential stress.

1.1.2.2. *Determination of the smallest circle circumscribed to the octahedral stress*

The *double maximization method* [DAN 84] consists of calculating, for every couple at time t_i, t_j of a cycle, the invariant of the variation of the corresponding stress, which has to be maximized:

[1.13] $$\frac{\Delta\tau_{oct}}{2} = \frac{1}{2}\underset{t_i,t_j}{\text{Max}}\left[\tau_{oct}(\underset{\sim}{\sigma}(t_i) - \underset{\sim}{\sigma}(t_j))\right]$$

which then provides for each period of time the amplitude (or the radius of the sphere) and the center $(\underset{\sim}{X}_i = (\underset{\sim}{\sigma}(t_i) + \underset{\sim}{\sigma}(t_j))/2)$ of the postulated sphere. Nevertheless, this method can be questioned. It geometrically considers that the circumscribed sphere is defined by its diameter, which is itself defined by the most distant two points of the cycle. In the general case, by considering a 2D example, the circle circumscribed is defined by either the most distant points, which then form its diameter, or by three non-collinear points. There is no way this situation can be predicted, and an algorithm considering both possibilities then has to be used, which is not the case with the double maximization method. This observation can also be applied to the case of the spheres whose dimension in space is higher than 2 (more than three points are then needed in order to define the circumscribed sphere in the general case).

Another approach, the *progressive memorization* procedure [NIC 99] is based on the notion of memorization due to plasticity [CHA 79]. Therefore, the loading path has to be studied — several times if needed — until it becomes entirely contained within the sphere. The algorithm can then be written as:

Let $\underset{\sim}{X}_i$ and R_i be the center and the radius of the sphere at time i.

When $t = 0, \underset{\sim}{X}_0 = \underset{\sim}{\sigma}_0$ and $R_0 = 0$

In addition $E_i = J(\underset{\sim}{\sigma}_i - \underset{\sim}{X}_{i-1}) - R_{i-1}$

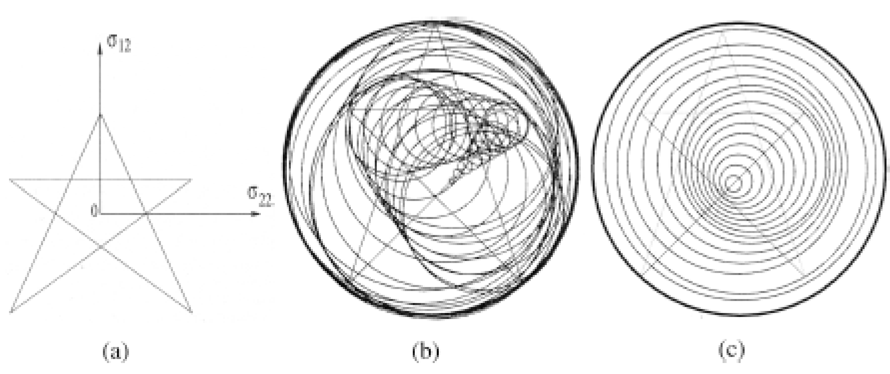
– when $E_i \leq 0$ the point is located within the sphere, and the increment is then equal to $i + 1$;

– when $E_i > 0$ the point does not belong to the sphere. The center is then displaced following the normal variable $\underset{\sim}{n} = \frac{\underset{\sim}{\sigma}_i - \underset{\sim}{\sigma}_{i-1}}{J(\underset{\sim}{\sigma}_i - \underset{\sim}{\sigma}_{i-1})}$ and radius R_i is then increased. Both calculations are weighted by a coefficient α which gives the memorization degree. *(*$\alpha = 0$, no memorization effect; $\alpha = 1$, total memorization effect):

$$R_i = \alpha E_i + R_{i-1}$$

$$\underset{\sim}{X}_i = (1 - \alpha) E_i \underset{\sim}{n} + X_{i-1}$$

Figure 1.5. *Illustration of the progressive memorization method in the case of the determination of the smallest circle circumscribed to the octahedral shear stress. a) Path of the stress within the plane ($\sigma_{22} - \sigma_{12}$). b) Circles successively analyzed by the algorithm for a memorization coefficient α= 0, 2. c) like in b), but with α=0.7 according to [NIC 99]*



This method leads to convergence towards the smallest circle circumscribed to the path of the octahedral stress. The value of the memorization parameter α has to be judiciously adjusted. Indeed, too low a value leads to a high number of iterations, whereas too high a value leads to an over-estimation of the radius, as shown in Figure 1.5.

$\Delta J/2$ stands for the final value of the radius R_i, which is also called the amplitude of the von Mises equivalent stress. This quantity can also be simply calculated by the following equation:

[1.14] $\Delta J = J(\underset{\sim}{\sigma}_{max} - \underset{\sim}{\sigma}_{min})$

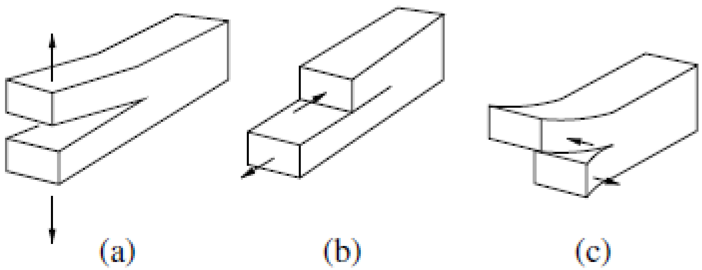
if and only if the extreme points of the loading are already known.

Finally, the algorithm of Welzl [WEL 91, GÄR 99], presented above, can also be applied to an arbitrary number of dimensions and can lead to the determination of the circle circumscribed to the octahedral stress with an optimum time, as Bernasconi and Papadopoulos noticed [BER 05].

1.1.3. *Classification of the cracking modes*

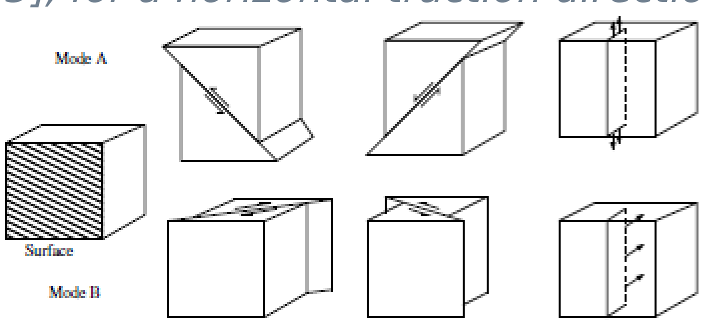
The type of cracking which occurs strongly influences the type of fatigue criterion to be used. Two main classifications should be mentioned in this section. The first one, proposed by Irwin, is now commonly used in failure mechanics and can be split into three different failure modes: modes I, II and III (see Figure 1.6). Mode I is mainly connected to the traction states, whereas modes II and III correspond to the loading conditions of shearing type.

Figure 1.6. *The three failure modes defined by Irwin: (a) mode I (b) mode II (c) mode III*



The initiation step involves some cracks stressed under shearing conditions. As Brown and Miller observed in 1973 ([BRO 73]), the location of these cracks compared to the surface is not random (Figure 1.7). In the case of cracks of type A, the normal line at the propagation front is perpendicular to the surface, and the trace is oriented with an angle of 45° with regards to the traction direction. On the other hand, cracks of type B propagate within the maximum shearing plane, leading to a trace on the external surface which is perpendicular to the traction direction — this type of crack tends to jump more easily to another grain.

Figure 1.7. *Both modes (A and B) observed by Brown and Miller [BRO 73], for a horizontal traction direction*



1.2. Experimental aspects

Section 1.2.1 briefly presents the experimental techniques of multiaxial fatigue, and then the key experimental results which rule the design of multiaxial models are introduced in section 1.2.2.

1.2.1. *Multiaxial fatigue experiments*

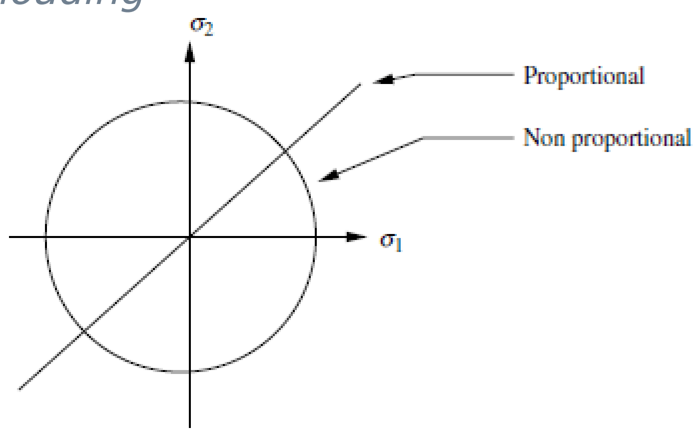
Multiaxial fatigue tests can be performed following different procedures, which allows various stress states, such as some biaxial traction states, torsion—traction states, etc. to be tested. The tests can be successively carried out following these different loading modes (for instance, traction and then torsion) or they can also be simultaneously carried out (traction—torsion).

Tests simultaneously involving several loading modes are said to be in-phase if the main components of the stress or strain tensor simultaneously and respectively reach their maximum and minimum and if their directions remain constant. If it is not the case, they are said to be off-phase. For instance, in the case of a biaxial traction test, the direction of the main stresses does not change, but if their amplitudes do not vary at the same time, the maximum shearing plane will be subjected to a rotation during a cycle (see Figure 1.8). During the in-phase tests, the shearing plane(s) will remain the same during the entire cycle.

1.2.2. *Main results*

Within materials as well as on the surface, a population of defects of any kind (inclusions, scratches, etc.) can be found and will trigger the initiation of microcracks due to a cyclic loading, or even due to a population of small cracks which were formed during the manufacturing cycle. It is also possible that the initiation step occurs without any defects (more details can be found in [RAB 10, PIN 10]). While the damage due to fatigue occurs, the cracks undergo two different steps in a first approximation. At first, they are a similar or smaller size compared to the typical lengths of the microcrack, and their propagation direction then strongly depends on the local fields. Starting from the surface, they can spread following the directions that Brown and Miller observed (see Figure 1.7, modes A and B) or following any other intermediate direction, or based on some more complex schemes, as the real cracks were not exactly planar at the microscopic scale. Then, one or several microcracks develop until they reach a large enough size so that their stress field becomes independent of the microstructure. The fatigue limit then appears as being the stress state and the microcracks cannot reach the second stage. These two stages respectively correspond to the initiation stage and the propagation stage of the fatigue crack [JAC 83]. Also, as will be presented later on, some fatigue criteria which bear two damage indicators can be observed. The first criterion deals with the initiation phase and the other one with the propagation phase [ROB 91].

Figure 1.8. *Example of some proportional and non-proportional loadings within the plane of the main stresses (σ_1, σ_2). For a proportional loading, the path of the stresses comes to a line segment, but not in the case of a non-proportional loading*



A large consensus has to be considered. As in the case of ductile materials, the propagation of the crack mainly occurs on the maximum shearing plane and depends on both modes II and III. Therefore, the criteria dealing with ductile materials will usually rely on some variables related