

APPLIED STOCHASTIC METHODS SERIES

Stochastic Methods for Pension Funds

**Pierre Devolder
Jacques Janssen
Raimondo Manca**



ISTE

 WILEY

Stochastic Methods for Pension Funds

Stochastic Methods for Pension Funds

Pierre Devolder
Jacques Janssen
Raimondo Manca

ISTE

 WILEY

First published 2012 in Great Britain and the United States by ISTE Ltd and John Wiley & Sons, Inc.

Apart from any fair dealing for the purposes of research or private study, or criticism or review, as permitted under the Copyright, Designs and Patents Act 1988, this publication may only be reproduced, stored or transmitted, in any form or by any means, with the prior permission in writing of the publishers, or in the case of reprographic reproduction in accordance with the terms and licenses issued by the CLA. Enquiries concerning reproduction outside these terms should be sent to the publishers at the undermentioned address:

ISTE Ltd
27-37 St George's Road
London SW19 4EU
UK

www.iste.co.uk

John Wiley & Sons, Inc.
111 River Street
Hoboken, NJ 07030
USA

www.wiley.com

© ISTE Ltd 2012

The rights of Pierre Devolder, Jacques Janssen and Raimondo Mancato be identified as the author of this work have been asserted by them in accordance with the Copyright, Designs and Patents Act 1988.

Library of Congress Cataloging-in-Publication Data

Devolder, Pierre.

Stochastic methods for pension funds / Pierre Devolder, Jacques Janssen, Raimondo Manca.

p. cm.

Includes bibliographical references and index.

ISBN 978-1-84821-204-6

1. Pension trusts--Management. 2. Pension trusts--Mathematics. 3. Financial risk management--Mathematical models. 4. Stochastic models. I. Janssen, Jacques, 1939- II. Manca, Raimondo. III. Title.

HD7105.4.D48 2011

332.67'2540151923--dc23

2011048482

British Library Cataloguing-in-Publication Data

A CIP record for this book is available from the British Library

ISBN: 978-1-84821-204-6

Printed and bound in Great Britain by CPI Group (UK) Ltd., Croydon, Surrey CR0 4YY



Table of Contents

Preface	xiii
Chapter 1. Introduction: Pensions in Perspective	1
1.1. Pension issues	1
1.1.1. The challenge	1
1.1.2. Some figures	2
1.2. Pension scheme	7
1.2.1. Definition	7
1.2.2. The four dimensions of a pension scheme	8
1.3. Pension and risks	11
1.3.1. Demographic risks	11
1.3.2. Financial risks	12
1.3.3. Impact of the risks on various kinds of pension schemes	12
1.3.4. The time horizon of a pension scheme	13
1.4. The multi-pillar philosophy	14
Chapter 2. Classical Actuarial Theory of Pension Funding	15
2.1. General equilibrium equation of a pension scheme	15
2.1.1. Principles	15
2.1.2. The retrospective reserve	16
2.1.3. The prospective reserve	18
2.1.4. Equilibrated pension funding	18
2.1.5. Decomposition of the reserve	20
2.1.6. Classification of the methods	20
2.2. General principles of funding mechanisms for DB Schemes	21
2.3. Particular funding methods	22
2.3.1. Unit credit cost methods	23

2.3.2. Level premium methods	25
2.3.3. Aggregate cost methods	28
Chapter 3. Deterministic and Stochastic Optimal Control	31
3.1. Introduction.	31
3.2. Deterministic optimal control	31
3.2.1. Formulation of the optimal control problem	31
3.3. Necessary conditions for optimality	33
3.3.1. Bellman function	33
3.3.2. Bellman optimality equation	34
3.3.3. Hamilton-Jacobi equation	37
3.3.4. The synthesis function	38
3.3.5. Other types of optimal controls.	39
3.3.6. Example: the classical quadratic/linear control problem	41
3.4. The maximum principle	42
3.4.1. The maximum principle from the dynamic programming approach	42
3.5. Extension to the one-dimensional stochastic optimal control	45
3.5.1. Formulation of the one-dimensional stochastic optimal control problem.	46
3.5.2. Necessary conditions for one-dimensional stochastic optimality	46
3.5.3. Extension to the multi-dimensional stochastic optimal control	48
3.5.4. Dynamic programming principle.	50
3.5.5. The Hamilton-Jacobi-Bellman equation	50
3.6. Examples	52
3.6.1. Merton portfolio allocation problem.	52
Chapter 4. Defined Contribution and Defined Benefit Pension Plans	55
4.1. Introduction.	55
4.2. The defined benefit method	56
4.3. The defined contribution method	57
4.3.1. The model	57
4.3.2. The capitalization system	58
4.4. The notional defined contribution (NDC) method	58
4.4.1. Historical preliminaries	58
4.4.2. The Dini reform transformation coefficients	60
4.4.3. Theoretical preliminaries	63
4.4.4. The construction of a unitary pension present value	65
4.4.5. Numerical example and results comparison	78
4.5. Conclusions.	93

Chapter 5. Fair and Market Values and Interest Rate Stochastic Models	95
5.1. Fair value	95
5.2. Market value of financial flows	96
5.3. Yield curve	97
5.4. Yield to maturity for a financial investment and for a bond	99
5.5. Dynamic deterministic continuous time model for an instantaneous interest rate	100
5.5.1. Instantaneous interest rate	100
5.5.2. Particular cases	101
5.5.3. Yield curve associated with an instantaneous interest rate.	101
5.5.4. Examples of theoretical models	102
5.6. Stochastic continuous time dynamic model for an instantaneous interest rate.	104
5.6.1. The OUV stochastic model	105
5.6.2. The CIR model (1985).	111
5.7. Zero-coupon pricing under the assumption of no arbitrage.	114
5.7.1. Stochastic dynamics of zero-coupons	114
5.7.2. Application of the no arbitrage principle and risk premium	116
5.7.3. Partial differential equation for the structure of zero coupons.	117
5.7.4. Values of zero coupons without arbitrage opportunity for particular cases	118
5.8. Market evaluation of financial flows	130
5.9. Stochastic continuous time dynamic model for asset values	132
5.9.1. The Black-Scholes continuous time model.	132
5.9.2. The solution of the Black-Scholes-Samuelson model	132
5.9.3. Prediction	135
5.10. VaR of one asset	136
5.10.1. Motivation	136
5.10.2. Definition of VaR for one asset	137
5.10.3. Normal distribution case	138
5.10.4. Lognormal distribution case	140
5.10.5. Trajectory simulation	143
5.10.6. VaR extensions: TVaR and conditional VaR.	144
Chapter 6. Risk Modeling and Solvency for Pension Funds	149
6.1. Introduction.	149
6.2. Risks in defined contribution	149
6.3. Solvency modeling for a DC pension scheme	150
6.3.1. The model	150
6.3.2. Maturity risk	151
6.3.3. Liquidity risk	156

6.3.4. Lifecycle strategy in DC schemes	163
6.3.5. Introduction of the longevity risk	166
6.4. Risks in defined benefit.	170
6.5. Solvency modeling for a DB pension scheme	171
6.5.1. The model	171
6.5.2. Maturity risk.	173
6.5.3. Liquidity risk	177
6.5.4. Introduction of longevity risk.	180
Chapter 7. Optimal Control of a Defined Benefit Pension Scheme.	181
7.1. Introduction.	181
7.2. A first discrete time approach: stochastic amortization strategy.	181
7.2.1. The problem	181
7.2.2. Stochastic evolution of the fund	182
7.2.3. Asymptotic evolution of the fund and the contribution.	184
7.2.4. Optimal amortization period	191
7.3. Optimal control of a pension fund in continuous time.	194
7.3.1. The problem	194
7.3.2. The model	195
Chapter 8. Optimal Control of a Defined Contribution Pension Scheme	207
8.1. Introduction.	207
8.2. Stochastic optimal control of annuity contracts.	208
8.2.1. The problem	208
8.2.2. The general model	209
8.2.3. Case with single contribution and no annuitization	215
8.2.4. Case with regular contributions and no annuitization.	216
8.2.5. Case with single contribution and annuitization	216
8.2.6. Case with regular premiums and annuitization	218
8.2.7. Extension: model with several risky assets	219
8.3. Stochastic optimal control of DC schemes with guarantees and under stochastic interest rates	223
8.3.1. The problem	223
8.3.2. The financial market	223
8.3.3. The pension scheme	226
8.3.4. The optimal control formulation	226
8.3.5. The solution	228

Chapter 9. Simulation Models	231
9.1. Introduction.	231
9.2. The direct method	233
9.2.1. The model	233
9.2.2. A real life example	244
9.3. The Monte Carlo models	250
9.3.1. The MAGIS model (individual as operational variable)	250
9.3.2. Time as an operational variable	251
9.4. Salary lines construction	252
9.4.1. A direct generalization of the Bernoulli process.	253
9.4.2. The salary line construction by means of the generalized Bernoulli process	257
9.4.3. A real data application	264
9.4.4. The studied cases	266
 Chapter 10. Discrete Time Semi-Markov Processes (SMP) and Reward SMP	 277
10.1. Discrete time semi-Markov processes	277
10.1.1. Purpose	277
10.1.2. DTSMP Definition	278
10.2. DTSMP numerical solutions	280
10.3. Solution of DTHSMP and DTNHSMP in the transient case: a transportation example	284
10.3.1. Principle of the solution	284
10.3.2. Semi-Markov transportation example	286
10.4. Discrete time reward processes	294
10.4.1. Classification and notation.	294
10.4.2. Undiscounted SMRWP.	297
10.4.3. Discounted SMRWP	301
10.5. General algorithms for DTSMRWP.	304
 Chapter 11. Generalized Semi-Markov Non-homogeneous Models for Pension Funds and Manpower Management	 307
11.1. Application to pension funds evolution.	307
11.1.1. Introduction	308
11.1.2. The non-homogeneous semi-Markov pension fund model.	310
11.1.3. The reserve structure	317
11.1.4. The impact of inflation and interest variability	319
11.1.5. Solving evolution equations	322
11.1.6. The dynamic population evolution of the pension funds	327
11.1.7. Financial equilibrium of the pension funds	330

11.1.8. Scenario and data	333
11.1.9. The usefulness of the NHSMPFM	337
11.2. Generalized non-homogeneous semi-Markov model for manpower management	338
11.2.1. Introduction	338
11.2.2. GDTNHSMP salary lines construction	339
11.2.3. GDTNHSMRWP for a reserve structure	342
11.2.4. Reserve structure with stochastic interest rate	343
11.2.5. The dynamics of population evolution.	344
11.2.6. The computation of salary cost present value.	346
11.3. Algorithms	347
11.3.1. The algorithm for the GNHSMP with a 2 time random variable.	347
11.3.2. The algorithm for the pension model	352
APPENDICES	359
Appendix 1. Basic Probabilistic Tools for Stochastic Modeling	361
A1.1. Probability space and random variables	361
A1.2. Expectation and independence	364
A1.3. Main distribution probabilities	367
A1.3.1. Binomial distribution	367
A1.3.2. Negative exponential distribution	368
A1.3.3. Normal (or Laplace Gauss) distribution	369
A1.3.4. Poisson distribution	371
A1.3.5. Lognormal distribution	372
A1.3.6. Gamma distribution	372
A1.3.7. Pareto distribution	373
A1.3.8. Uniform distribution	374
A1.3.9. Gumbel distribution	375
A1.3.10. Weibull distribution	375
A1.3.11. Multidimensional normal distribution	375
A1.4. Conditioning	378
A1.5. Stochastic processes.	386
A1.6. Martingales	390
A1.7. Brownian motion	394
Appendix 2. Itô Calculus and Diffusion Processes	397
A2.1. Problem of stochastic integration	397
A2.2. Stochastic integration of simple predictable processes and semi-martingales.	399
A2.3. General definition of the stochastic integral.	403

A2.4. Itô's formula	410
A2.4.1. Quadratic variation of a semi-martingale	410
A2.4.2. Itô's formula	412
A2.5. Stochastic integral with a standard Brownian motion as the integrator process	413
A2.5.1. Case of predictable simple processes	414
A2.5.2. Extension to general integrator processes	416
A2.6. Stochastic differentiation	417
A2.6.1. Definition	417
A2.6.2. Examples	417
A2.7. Back to the Itô's formula	419
A2.7.1. Stochastic differential of a product	419
A2.7.2. Examples	419
A2.7.3. The Itô's formula with time dependence	420
A2.7.4. Interpretation of the Itô's formula	421
A2.7.5. Other extensions of the Itô's formula	422
A2.8. Stochastic differential equations	425
A2.8.1. Existence and unicity general theorem [GIK 68]	425
A2.8.2. Solution of stochastic differential equations	429
A2.9. Diffusion processes	429
A2.10. Multidimensional diffusion processes	432
Bibliography	437
Index	449

Preface

In recent years quantitative finance has become an extraordinary field of research and interest, from an academic point of view as well as for practical applications.

At the same time, the pension issue is clearly a major economic and financial topic for the coming decades, in the context of the well-known longevity risk. The emergence and development of pension schemes in our modern societies can essentially be explained by two factors:

- The individual approach to life: our modern world tends to substitute for large multi-generational families, for an individual model where each person is assumed to be self-supporting, before and after retirement age. As a consequence, personal pension planning, including social security pensions and other incomes, becomes a necessity for everyone.

- The longevity syndrome: longevity has increased extraordinarily in recent decades, and nothing indicates a stopping or a reverse in this phenomenon could occur in the next few years. In the past it was usual to retire at 65 and then have a life expectancy of only a few more years. Nowadays, and in the near future, in many countries, it will be quite common for all of us to hope to survive until after 85!

The future of our pension systems is clearly one of the main economic challenges of the world for the coming decades. In the huge majority of countries, collective pension schemes have been created by states as well as by private companies or professional organizations. These systems can use very different tools, techniques, funding approaches or legal forms.

As described in Gronchi [GRO 99], pensions can be classified as a function of the calculation method of the benefits as:

- defined benefit (DB): the pension is a function of salaries and the number of years of service of the retiring person;

- defined contribution (DC): the pension is a function of the contribution that the retiring person paid during his life.

It can also be classified as a function of the financing technique as:

- pay as you go (PAYG): the pensions are paid by the current workers to the previous generation;

- funding: the pensions are generated by the savings accumulated by the contribution of the workers.

Surprisingly few books are devoted to the application of modern stochastic tools to pension analysis.

Therefore, the aim of this book is to fill this gap and to show how recent stochastic methods can be useful for the risk management of pension funds and the computation of market values. Optimal control methods will be especially developed and applied to fundamental problems such as the optimal asset allocation of the fund or the cost spreading of a pension scheme. In these various problems, financial as well as demographic and economic risks will be addressed and modeled.

The layout of this book is as follows: Chapters 1 and 2 present the fundamental issues of the next decades and classical theory of pension funding.

In Chapter 3, we introduce the minimal basic results concerning control theory, both for deterministic and stochastic formulation in order to prepare the models presented in Chapters 7 and 8.

Chapter 4 is devoted to concepts of defined contribution and defined benefit pension plans, while Chapter 5 presents some basic definitions on fair and market values for the evaluation of financial flows and stocks in the future – not only to improve the classical concept of present value, but also to fill up the constraints of new IFRS rules as well as Basel II and III rules for banks and Solvency II for insurance companies.

Chapter 6 first develops stochastic models for DC and then for DB, in order to measure the various risks faced by the pension fund and to propose eventual solvency buffers in the philosophy of Basel II or Solvency II. In particular, it illustrates the importance of time in the risk assessment of a pension fund.

Chapters 7 and 8 give the main results of the pension theory as various models for the optimal control of the investment strategy and the contribution process for

both defined contribution pension schemes and defined benefit pension schemes are presented.

Chapter 7 presents – also for DB pension schemes in a stochastic environment – important asymptotic results of evolution of the fund and the contributions.

Chapter 9 presents the construction of algorithms for the management of pension fund liabilities necessary for simulation models. Fundamentally we give two approaches to these models, one that is a semi-deterministic approach that we will call the *direct method* and the other based on the Monte Carlo method.

In short, with these last three chapters, it is possible to carefully study the evolution of a pension fund in the future; they constitute the core of this book.

The presentation, in Chapter 10, of discrete time homogeneous semi-Markov processes (DTHSMP), discrete time non-homogeneous (DTNHSMP) semi-Markov processes and the discrete time semi-Markov reward processes (SMRWP), is followed by Chapter 11, the last chapter of this book, which is devoted to another type of stochastic model of pension funds and manpower management study, called generalized non-homogeneous semi-Markov models.

This model is a general, rigorous and tractable stochastic evolution time model for pension funds, called the discrete time non-homogeneous semi-Markov pension fund model, taking into account economic, financial and demographic evolution factors so that it becomes a real-life model using important factors such as *seniority*, *general age dependence*, *rate of inflation* and *salary lines*. It can be particularly useful for the study of private pension fund evolution.

In conclusion, this book presents realistic stochastic models to carefully study the evolution of pension funds in the future, not only from a theoretical point of view but also from a practical point of view, as we present algorithms for the construction of simulation models.

We also present basic concepts in such a way that this book is relatively self-contained. Therefore, the book can be considered as the first textbook in the field of stochastic methods for pension funds and thus, it will be useful for graduate students in economics and actuarial science as well as for managers of pension funds and especially people involved in Solvency II for insurance companies and in Basel II and III for banks.

Chapter 1

Introduction: Pensions in Perspective

1.1. Pension issues

1.1.1. *The challenge*

The future of our pension systems is clearly one of the main economic challenges for the world in the coming decades. In the vast majority of countries, collective pension schemes have been created by states as well as by private companies or professional organizations. These systems can use very different tools, techniques, funding approaches or legal forms.

From a historical point of view, the concept of an organized pension institution is relatively new in human evolution and can be seen essentially as a creation of the 20th Century. Of course for centuries, societies have accepted the idea of solidarity between generations and the fact that at a given age people must stop working and then receive other forms of income to survive. But in the traditional way of life, this solidarity was guaranteed by familial or tribal assistance without need of complicated collective systems. For instance, it was common, in this context, to have families with 3 (or even 4) different generations living together in the same house and sharing a global income. At the same time, the mean longevity was not as important and the number of retirement years was limited in general.

2 Stochastic Methods for Pension Funds

The emergence and development of pension schemes in our modern societies can essentially be explained by two factors:

- the “individual approach” of life: our modern world tends to replace large “multi-generational” families by an individual model where each person is assumed to be self-supporting, before and after retirement age. As a consequence, personal pension planning, including social security pensions and other incomes, becomes a necessity for everyone;

- the “longevity syndrome”: longevity has increased extraordinarily in recent decades and nothing indicates that a stopping or reversing of this phenomenon could occur in the next few years. In the past it was usual to retire at 65 and then have a life expectancy of only a few years. Nowadays, and in the near future, in a lot of countries, it will be quite common for all of us to hope to survive until after 85!

This longevity evolution is a major driving force in motivating the development of various pension schemes under different forms. But at the same time it induces a huge challenge in terms of financial sustainability. Indeed, for the first time in history, we can simultaneously observe two major demographic evolutions: a continuous increase in longevity leading to an expected increase in the proportion of retirees and a decrease of fertility rates leading to a decrease of active workers.

As a consequence, we will live in increasingly ageing societies, with fewer contributors and more beneficiaries.

1.1.2. *Some figures*

In order to give a global view of the importance of the pension challenge and its worldwide coverage, we present some international demographical and financial figures [OEC 11].

1.1.2.1. Longevity at 65

The following table compares in various countries, for men and women, the life expectancy at 65 in 2005 and that projected for 2050. It shows that everywhere our longevity is expected to continue increasing in the coming years by more than 3 years. For the OECD34, a man aged 65 in 2045 is expected to survive until 84 (88 for a woman).

	Women			Men		
	2005-10	2045-50	Difference 2005-2050	2005-10	2045-50	Difference 2005-2050
Japan	23.33	27.30	4.0	18.10	21.28	3.2
France	22.17	25.53	3.4	17.61	21.15	3.5
Switzerland	21.64	25.19	3.5	18.37	21.97	3.6
Italy	21.47	25.08	3.6	17.53	20.51	3.0
Australia	21.42	24.89	3.5	18.15	21.37	3.2
Belgium	20.32	24.80	4.5	16.50	20.16	3.7
Spain	21.35	24.74	3.4	17.33	21.02	3.7
Iceland	20.71	24.42	3.7	18.36	21.44	3.1
Canada	20.93	24.39	3.5	17.73	21.04	3.3
Finland	20.44	24.26	3.8	16.24	19.43	3.2
Israel	20.25	24.13	3.9	17.82	21.13	3.3
Norway	20.45	24.08	3.6	17.18	20.48	3.3
Austria	20.16	24.02	3.9	16.88	20.69	3.8
Korea	20.19	24.02	3.8	15.90	18.97	3.1
Germany	20.21	23.97	3.8	16.55	19.90	3.4
New Zealand	20.44	23.86	3.4	17.58	20.82	3.2
Ireland	20.10	23.82	3.7	16.51	19.64	3.1
Sweden	20.59	23.79	3.2	17.35	20.70	3.3
Slovenia	19.77	23.68	3.9	14.90	18.50	3.6
Luxembourg	20.06	23.67	3.6	16.24	19.93	3.7
OECD34	19.88	23.52	3.6	16.36	19.47	3.1
United Kingdom	19.80	23.43	3.6	16.55	19.57	3.0
United States	20.49	23.29	2.8	17.32	19.45	2.1
Greece	18.54	23.29	4.7	16.62	19.49	2.9
Portugal	19.71	23.25	3.5	15.81	18.88	3.1
Netherlands	19.94	23.16	3.2	16.78	20.20	3.4
EU27	19.17	22.90	3.7	15.45	18.59	3.1

Table 1.1. Life expectation at 65 in 2005 and 2045 (source: OECD 2011)

1.1.2.2. Fertility rates

The following table compares, in various countries, the evolution of the fertility rate between 1975 and 2050. It illustrates the persistent low level of this rate (less than 2.1, which is often seen as the necessary rate in order to guarantee the strict renewal of the population). For OECD34, this rate is expected to stay around 1.7.

Total fertility rates. 1975-2050

	1975-80	1985-90	1995-00	2005-10	2015-20	2025-30	2035-40	2045-50
OECD members								
Australia	1.99	1.86	1.78	1.83	1.85	1.85	1.85	1.85
Austria	1.65	1.44	1.37	1.38	1.46	1.56	1.66	1.76
Belgium	1.71	1.56	1.60	1.77	1.84	1.85	1.85	1.85
Canada	1.73	1.62	1.56	1.57	1.67	1.77	1.85	1.85
Chile	2.80	2.65	2.21	1.94	1.85	1.85	1.85	1.85
Czech Republic	2.31	1.92	1.17	1.41	1.57	1.67	1.77	1.85
Denmark	1.68	1.54	1.76	1.84	1.85	1.85	1.85	1.85
Estonia	2.06	2.20	1.33	1.64	1.84	1.85	1.85	1.85
Finland	1.66	1.66	1.74	1.83	1.85	1.85	1.85	1.85
France	1.86	1.81	1.76	1.89	1.85	1.85	1.85	1.85
Germany	1.52	1.43	1.34	1.32	1.39	1.49	1.59	1.69
Greece	2.32	1.53	1.30	1.38	1.46	1.56	1.66	1.76
Hungary	2.12	1.82	1.38	1.35	1.47	1.57	1.67	1.77
Iceland	2.29	2.12	2.06	2.10	1.98	1.86	1.85	1.85
Ireland	3.48	2.29	1.90	1.96	1.87	1.85	1.85	1.85
Israel	3.41	3.05	2.94	2.81	2.46	2.22	2.04	1.90
Italy	1.94	1.34	1.22	1.38	1.44	1.54	1.64	1.74
Japan	1.83	1.66	1.37	1.27	1.30	1.40	1.50	1.60
Korea	2.92	1.60	1.51	1.22	1.29	1.39	1.49	1.59
Luxembourg	1.49	1.47	1.72	1.66	1.74	1.84	1.85	1.85
Mexico	5.25	3.63	2.67	2.21	1.89	1.85	1.85	1.85
Netherlands	1.60	1.56	1.60	1.74	1.81	1.85	1.85	1.85
New Zealand	2.18	2.03	1.95	2.02	1.95	1.85	1.85	1.85
Norway	1.81	1.80	1.85	1.89	1.85	1.85	1.85	1.85
Poland	2.26	2.15	1.48	1.27	1.34	1.44	1.54	1.64
Portugal	2.41	1.62	1.46	1.38	1.44	1.54	1.64	1.74
Slovakia	2.47	2.15	1.40	1.28	1.40	1.50	1.60	1.70
Slovenia	2.20	1.66	1.25	1.36	1.52	1.62	1.72	1.82
Spain	2.57	1.46	1.18	1.43	1.65	1.75	1.84	1.85
Sweden	1.66	1.91	1.56	1.87	1.85	1.85	1.85	1.85
Switzerland	1.53	1.53	1.47	1.45	1.54	1.64	1.74	1.83
Turkey	4.72	3.28	2.57	2.13	1.97	1.85	1.85	1.85
United Kingdom	1.72	1.81	1.70	1.84	1.85	1.85	1.85	1.85
United States	1.79	1.92	1.99	2.09	1.95	1.85	1.85	1.85
OECD34	2.26	1.91	1.68	1.69	1.71	1.73	1.77	1.80

Table 1.2. Evolution of the fertility rates 1975–2050 (source: OECD 2011)

1.1.2.3. *Public pension expenditure*

The following table compares, in various countries, the evolution of the public pension expenditure between 1990 and 2007 (in % of the GDP). Absolute levels and evolutions can be quite different from one country to another, depending on the importance of social security and the eventual financial measures already taken by some states (for instance Sweden or Luxemburg). But from a global point of view, the increasing trend seems to be clear. For OECD34, nowadays public pension expenditures represent 7% of the GDP.

	Level (% of GDP)					Change (%)
	1990	1995	2000	2005	2007	1990-2007
Australia	3.0	3.6	3.8	3.3	3.4	11.2
Austria	11.4	12.3	12.3	12.5	12.3	7.8
Belgium	9.1	9.4	8.9	9.0	8.9	-2.9
Canada	4.2	4.7	4.3	4.2	4.2	-1.2
Chile		6.9	7.5	5.9	5.2	
Czech Republic	6.1	6.3	7.5	7.3	7.4	21.8
Denmark	5.1	6.2	5.3	5.4	5.6	8.6
Estonia			6.0	5.3	5.2	
Finland	7.3	8.8	7.7	8.4	8.3	13.3
France	10.6	12.0	11.8	12.3	12.5	17.5
Germany	9.0	10.7	11.2	11.5	10.7	19.1
Greece	9.9	9.6	10.7	11.7	11.9	20.9
Hungary			7.4	8.6	9.1	
Iceland	2.2	2.4	2.2	2.0	1.9	-14.7
Ireland	3.9	3.5	3.1	3.4	3.6	-7.7
Israel		4.7	4.9	5.1	4.8	
Italy	10.1	11.3	13.6	14.0	14.1	38.9
Japan	4.9	6.1	7.4	8.7	8.8	80.5
Korea	0.7	1.2	1.4	1.5	1.7	130.5
Luxembourg	8.2	8.8	7.5	7.2	6.5	-19.8
Mexico	0.5	0.7	0.9	1.2	1.4	202.0
Netherlands	6.7	5.8	5.0	5.0	4.7	-29.8
New Zealand	7.4	5.7	5.0	4.3	4.3	-41.8
Norway	5.6	5.5	4.8	4.8	4.7	-16.6
Poland	5.1	9.4	10.5	11.4	10.6	107.0
Portugal	4.9	7.2	7.9	10.3	10.8	119.8
Slovak Republic		6.3	6.3	6.2	5.8	
Slovenia			10.6	9.9	9.6	

6 Stochastic Methods for Pension Funds

Spain	7.9	9.0	8.6	8.1	8.0	1.5
Sweden	7.7	8.2	7.2	7.6	7.2	-6.8
Switzerland	5.6	6.7	6.6	6.8	6.4	14.2
Turkey	2.4	2.7	4.9	5.9	6.1	159.2
United Kingdom	4.8	5.4	5.3	5.6	5.4	11.0
United States	6.1	6.3	5.9	5.9	6.0	-1.5
OECD34	6.1	6.7	6.9	7.1	7.0	14.5

Table 1.3. Evolution of the public pension expenditure in % of GDP
(source: OECD 2011)

1.1.2.4. Assets of pension funds

The following table gives the pension assets (in % of the GDP and in USD) for 2009, in various countries. These figures illustrate the importance of the pension fund assets invested in the financial markets. For OECD34, these assets represent two-thirds of the GDP.

	Pension funds	
	% of GDP	USD
OECD members		
Australia	82.3	808,224
Austria	4.9	18,987
Belgium	3.3	16,677
Canada	62.9	806,350
Chile	65.1	106,596
Czech Republic	4.6	11,332
Denmark	43.3	133,980
Estonia	6.9	1,371
Finland	76.8	182,286
France	0.8	21,930
Germany	5.2	173,810
Greece	0.0	63
Hungary	13.1	16,886
Iceland	118.3	14,351
Ireland	44.1	100,278
Israel	46.9	95,257
Italy	4.1	86,818

Japan	25.2	1,042,770
Korea	2.2	29,632
Luxembourg	2.2	1,171
Mexico	7.5	107,135
Netherlands	129.8	1,028,077
New Zealand	11.8	13,755
Norway	7.3	27,852
Poland	13.5	58,143
Portugal	13.4	30,441
Slovak Republic	4.7	4,640
Slovenia	2.6	1,266
Spain	8.1	118,056
Sweden	7.4	35,307
Switzerland	101.2	496,957
Turkey	2.3	14,017
United Kingdom	73.0	1,589,409
United States	67.6	9,583,968
OECD 34	67.6	16,777,792

Table 1.4. *Importance of the assets of the pension funds in % of GDP and in US dollars (source: OECD 2011)*

1.2. Pension scheme

After having seen the worldwide importance of the ageing trend and the unavoidable pension challenge for the coming years we present, in this section, general concepts used in pension theory.

1.2.1. Definition

A pension scheme can be defined as a systematic and organized mechanism, prescribed by law or by a convention, in order to provide after retirement regular incomes to a well defined category of people. This broad definition is based on the following keywords:

- *Systematic*: the benefits offered by a pension scheme are supposed to be defined by an objective rule and are not given “à la carte”. Two people in exactly the same situation will receive the same pension amount.

– *Organized*: a pension scheme must be initiated by a sponsor (a state, a private company, etc.) who is the guarantor of the continuity of the system.

– *By law*: when the sponsor is a state or an assimilated national or international institution (social security systems).

– *By convention*: when the sponsor is a private company or employer (occupational pension schemes).

– *Regular incomes*: the typical form of the benefits offered by a pension scheme is a lifetime annuity but sometimes other forms of benefits may exist (for instance, a lump sum paid at retirement age or temporary annuity).

– *Category*: the affiliates to the scheme are generally members of an objective category.

A pension scheme is therefore by definition a collective vehicle with precise rules, even if it may concern only a few people. Apart from this concept, individuals can also buy, during their active life, individual financial products or life insurances in order to increase their standard of living after retirement. This idea to accumulate different forms of pension arrangements, from the social security system to individual saving accounts, is a basic concept of the modern pension theory, sometimes known as the multi-pillar philosophy.

1.2.2. The four dimensions of a pension scheme

When considering setting up a pension scheme, various fundamental choices arise. In order to choose a specific pension scheme, we must first understand the alternatives that arise and measure the benefits and disadvantages of different systems.

We can roughly summarize the strategy of choosing a pension scheme in four basic questions to be answered:

- Dimension 1: for whom? Affiliates: who is affected by the plan?
- Dimension 2: by whom? Sponsor: who organizes and manages the system?
- Dimension 3: how much? Benefits: how much to provide?
- Dimension 4: how? Funding: how can we finance these pension benefits?

1.2.2.1. The dimension of the affiliates

The first question to ask is who is the population to be covered by the pension plan.

The table below shows some community types encountered in practice, showing the large variety in terms of magnitude:

	POPULATION	EXAMPLE
1	The entire population of a country	Minimum pension income for all the citizens of a country
2	One category of workers of a country	Scheme for the civil servants
3	Members of a profession	Pension fund for accountants
4	One category of workers from one economic sector	Blue collar workers of the chemical industry
5	One category of workers of one private company	Workers of a company aged 25 or more
6	A key person of one company	CEO of the company

1.2.2.2. *The dimension of the sponsor: public or private*

The sponsor is the legal entity at the origin of the scheme. The table below shows some classical sponsors encountered in practice. Clearly the dimensions of the affiliates and of the sponsor are correlated.

	SPONSOR	EXAMPLE
1	A State	General social security scheme
2	A region	Scheme for the public servants of one region
3	A public company	Scheme for a national electricity company
4	A group of private companies	Common scheme for the chemical industry
5	A company	Company XY
6	A professional body	Association of Notaries

1.2.2.3. *The dimension of the benefits: DB or DC*

There are two main logics in designing the benefits of a pension scheme:

– defined benefits pension scheme (DB schemes): “we say what we want to receive as benefits at retirement age”;

– defined contributions pension scheme (DC schemes): “we say what we want to pay as contributions during the active life”.

In a *DB pension scheme*, the pension incomes to receive after retirement are explicitly defined by the rules of the plan and are generally related to the salaries and

the years of service of the affiliate. For instance, a pension plan could provide, after retirement age, fixed at 65, a pension income corresponding to 65% of the average of the last 5 salaries for 40 years of service in the company.

In a *DC pension scheme*, the contributions to be paid by the employer and/or the employee are explicitly defined by the rules of the plan and constitute the only liability of the sponsor. The benefits will be generated just by the accumulation of these contributions. For instance, a pension plan could provide a contribution of 5% of the salaries paid each year by the employer and 1% paid by the employee.

Clearly the sharing of risks between the affiliates and the sponsor are quite different in the two philosophies. This point will be illustrated in section 1.3.

1.2.2.4. *The dimension of the funding: pay as you go or funding*

In a DB pension scheme where benefits have been defined, we must finance these liabilities. In particular, we must compute a level of contributions in order to have an actuarial equilibrium between incomes and outcomes; this equilibrium is based on a long-term approach and not on a pure accounting point of view.

More precisely, the actuary will check on a certain time horizon and on a given population, the actuarial equivalence between the present value of the contributions and the present value of the benefits.

There are a lot of different actuarial techniques in order to finance a DB scheme depending on the way these discounted values are computed (definition of the time horizon and the population concerned). A major distinction is generally made between the pay as you go mechanism and funding techniques.

In a *pay as you go mechanism* (PAYG), the contributions paid one year by the active affiliates are directly used in order to pay the benefits to the retired people. Time horizon is typically one year. As a consequence, everybody is paying for somebody else! In particular, there are no (or nearly no) reserves in such a scheme.

The first condition in order to apply this technique is of course to be sure of the continuity of the sponsor. If one day, for one reason or another, the system must stop, clearly there is a big problem for the active affiliates! So in practice this technique can only be used for social security purposes. The ageing of the population, which was previously mentioned, is clearly a major concern for PAYG schemes.

On the other hand, PAYG systems are relatively protected against financial and inflation risks.

In a *funding approach*, the contributions paid by one generation are invested on the financial market and will be used by the same generation later upon retirement.

The time horizon can be very long and there are significant reserves. This technique can be used for any kind of sponsor. In case of stopping of the system, reserves exist in order to protect the rights of the affiliates. This method seems to be better adjusted against demographic risk (even if the longevity risk is also present as it will be explained in section 1.3). But inflation and market risks will also greatly affect the scheme. In particular, the search for a good investment strategy for the accumulated reserves is crucial.

This distinction between PAYG and funding is particularly meaningful for a DB pension scheme. But it can be also applied to a DC plan. When looking at a DC plan, it seems natural to associate a classical saving mechanism of individual accounts and therefore a funding approach. But DC schemes can also be based on pure PAYG techniques like notional accounts or point systems for social security purposes.

1.3. Pension and risks

Pension liabilities are exposed to a large variety of risks and the influence of these risks on the level of the benefits and on the funding mechanism may vary greatly from one scheme to another. For a given pension scheme, we must, for each kind of risk, answer two basic questions:

- Is the risk affecting the pension scheme and how much?
- If so, who is affected (the sponsor or/and the affiliate)?

We propose breaking the various risks down into two main categories: demographic risks and financial risks.

1.3.1. Demographic risks

By definition, a pension scheme is a collective vehicle covering a population.

Not surprisingly, the evolution of the affiliates and the beneficiaries is quite important. Among the demographic risks affecting the population of a pension fund, we have:

- the longevity risk: risk caused by the (unexpected) increase in the duration of human life; in a pension scheme this risk will typically increase the number of years of service of the benefits;

– the renewal risk: risk caused by a decrease in the number of entrances in the population; in a pension scheme this risk will typically decrease the amount of contributions;

– the lapse risk: risk caused by the number of people going out of the pension fund before retirement age (disability, dismissal, etc.).

1.3.2. *Financial risks*

In a pension scheme, we can also observe different financial risks:

– the market risk: risk caused by the variation of the market value of the financial assets underlying the pension benefits (stocks, investment funds, real estate, etc.);

– the interest rate risk: risk caused by the effect of a variation of the interest rates on the valuation of the assets and the liabilities of the pension scheme;

– the credit risk: risk caused by the change of rating of a counterparty or by a partial or total default of a debtor;

– the inflation risk: risk caused by the effect of inflation on the level of benefits to be paid by the fund.

1.3.3. *Impact of the risks on various kinds of pension schemes*

We will shortly analyze the effect of all these risks on 4 important kinds of pension schemes illustrated by the following table:

	<i>DB</i>	<i>DC</i>
<i>FUNDING</i>	<i>1</i>	<i>3</i>
<i>PAYG</i>	<i>2</i>	<i>4</i>

1.3.3.1. *DB funding scheme*

If we consider a DB scheme using a funding technique where contributions are (mainly) paid by the sponsor (for instance an occupational scheme organized by a private company for its employees), clearly a lot of risks are supported by this sponsor (the company) and the affiliates (the employees) have nearly no risk. The main risks for the sponsor are the different financial risks and the longevity risk. Market and inflation risks are particularly crucial in this combination. The contributions computed actuarially and to be paid by the sponsor can largely fluctuate depending on the level of indexation of the benefits (inflation risk) and on the financial return achieved on assets (market and interest rate risks). For a pension