Topics in Physical Mathematics

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Dedicated to the memory of my mother,
Indumati (1920 – 2005),
who passed on to me her love of learning.

Memories

Your voice is silent now.
But the sound of your soft
Music will always resonate
In my heart.
The wisp of morning incense
Floats in the air no more.
It now resides only in my
Childhood memories.

Contents

Preface

Physikalische Mathematik	**Physical Mathematics**
Die meisten	**Most**
Mathematiker	**Mathematicians**
glauben.	**believe.**
Aber alle	**But all**
Physiker	**Physicists**
wissen.	**know.**

The marriage between gauge theory and the geometry of fiber bundles from the sometime warring tribes of physics and mathematics is now over thirty years old. The marriage brokers were none other than Chern and Simons. The 1978 paper by Wu and Yang can be regarded as the announcement of this union. It has led to many wonderful offspring. The theories of Donaldson, Chern–Simons, Floer–Fukaya, Seiberg–Witten, and TQFT are just some of the more famous members of their extended family. Quantum groups, CFT, supersymmetry, string theory and gravity also have close ties with this family. In this book we will discuss some topics related to the areas mentioned above where the interaction of physical and mathematical theories has led to new points of view and new results in mathematics. The area where this is most evident is that of geometric topology of low-dimensional manifolds. I coined the term "physical mathematics" to describe this new and fast growing area of research and used it in the title of my paper [265]. A very nice discussion of this term is given in Zeidler's book on quantum field theory [417], which is the first volume of a six-volume work that he has undertaken (see also [418]).

Historically, mathematics and physics were part of what was generally called "natural philosophy." The intersection of ideas from different areas of natural philosophy was quite common. Perhaps the earliest example of this is to be found in the work of Kepler. Kepler's laws of planetary motion caused a major sensation when they were announced. Newton's theory of gravitation and his development of the calculus were the direct result of his successful attempt to provide a mathematical explanation of Kepler's laws. We may consider this the beginning of modern mathematical physics or, in the spirit of this book, **physical mathematics**.

Kepler was an extraordinary observer of nature. His observations of snowflakes, honeycombs, and the packing of seeds in various fruits led him to his lesser known study of the sphere packing problem. For dimensions 1, 2, and 3 he found the answers to be 2, 6, and 12 respectively. The lattice structures on these spaces played a crucial role in Kepler's "proof." The three-dimensional problem came to be known as Kepler's conjecture. The slow progress in the solution of this problem led John Milnor to remark that here was a problem that nobody could solve but its answer was known to every schoolboy. It was only solved in 1998 by Tom Hales and the problem in higher dimensions is still wide open. It was the study of the symmetries of a special lattice (the 24-dimensional Leech lattice) that led John Conway to the discovery of his sporadic simple groups. Conway's groups and other sporadic simple groups are closely related to the automorphisms of lattices and algebras. The study of representations of the largest of these sporadic groups (called the Friendly Giant or Fischer–Griess Monster) has led to the creation of a new field of mathematics called vertex algebras. They turn out to be closely related to the chiral algebras in conformal field theory.

It is well known that physical theories use the language of mathematics for their formulation. However, the original formulation of a physical law often does not reveal its appropriate mathematical context. Indeed, the relevant mathematical context may not even exist when the physical law is first formulated. The most well known example of this is Maxwell's equations, which were formulated as a system of partial differential equations for the electric and magnetic fields. Their formulation in terms of the electromagnetic field tensor came later, when Minkowski space and the theory of special relativity were introduced. The classical theory of gravitation as developed by Newton offers another example of a theory that found later mathematical expression as a first approximation in Einstein's work on gravitation. Classical Riemannian geometry played a fundamental role in Einstein's general theory of relativity, and the search for a unified theory of electromagnetism and gravitation led to continued interest in geometrical methods for some time.

However, communication between physicists and mathematicians has been rather sporadic. Indeed, one group has sometimes developed essentially the same ideas as the other without being aware of the other's work. A recent example of this missed opportunity (see [115] for other examples) for communication is the development of Yang–Mills theory in physics and the theory of connections in a fiber bundle in mathematics. Attempts to understand the precise relationship between these theories has led to a great deal of research by mathematicians and physicists. The problems posed and the methods of solution used by each have led to significant contributions towards better mutual understanding of the problems and the methods of the other. For example, the solution of the positive mass conjecture in gravitation was obtained as a result of the mathematical work by Schoen and Yau [339]. Yau's solution of the Calabi conjecture in differential geometry led to the definition of Calabi–Yau manifolds. Manifolds are useful as models in superstring compactification in string theory.

A complete solution for a class of Yang–Mills instantons (the Euclidean BPST instantons) was obtained by using methods from differential geometry by Atiyah, Drinfeld, Hitchin, and Manin (see [19]). This result is an example of a result in mathematical physics. Donaldson turned this result around and studied the topology of the moduli space of BPST instantons. He found a surprising application of this to the study of the topology of four-dimensional manifolds. The first announcement of his results [106] stunned the mathematical community. When combined with the work of Freedman [136, 137] one of its implications, the existence of exotic $\mathbf{R}^4$ spaces, was a surprising enough piece of mathematics to get into the New York Times. Since then Donaldson and other mathematicians have found many surprising applications of Freedman's work and have developed a whole area of mathematics, which may be called **gauge-theoretic mathematics**. In a series of papers, Witten has proposed new geometrical and topological interpretations of physical quantities arising in such diverse areas as supersymmetry, conformal and quantum field theories, and string theories. Several of these ideas have led to new insights into old mathematical structures and some have led to new structures. We can regard the work of Donaldson and Witten as belonging to physical mathematics.

Scientists often wonder about the "unreasonable effectiveness of mathematics in the natural sciences." In his famous article [402] Wigner writes:

> The first point is that the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no rational explanation for it. Second, it is just this uncanny usefulness of mathematical concepts that raises the question of the uniqueness of our physical theories.

It now seems that mathematicians have received an unreasonably effective (and even mysterious) gift of classical and quantum field theories from physics and that other gifts continue to arrive with exciting mathematical applications.

Associated to the Yang–Mills equations by coupling to the Higgs field are the Yang–Mills–Higgs equations. If the gauge group is non-abelian then the Yang–Mills–Higgs equations admit smooth, static solutions with finite action. These equations with the gauge group $G_{ew} = U(1) \times SU(2)$ play a fundamental role in the unified theory of electromagnetic and weak interactions (also called the **electroweak theory**), developed in major part by Glashow [155], Salam [333], and Weinberg [397]. The subgroup of G_{ew} corresponding to $U(1)$ gives rise to the electromagnetic field, while the force of weak interaction corresponds to the $SU(2)$ subgroup of G_{ew}. The electroweak theory predicted the existence of massive vector particles (the intermediate bosons W^+, W^-, and Z^0) corresponding to the various components of the gauge potential, which mediate the weak interactions at short distances. The experimental verification of these predictions was an important factor in the renewed interest in gauge theories as providing a suitable model for the unification of fundamental forces of nature. Soon thereafter a theory was proposed to unify the electromagnetic, weak and strong interactions by adjoining the group $SU(3)$ of quantum chromodynamics to the gauge group of the electroweak theory. The resulting theory is called the standard model. It has had great success in describing the known fundamental particles and their interactions. An essential feature of the standard model is symmetry breaking. It requires the introduction of the Higgs field. The corresponding Higgs particle is as yet unobserved. Unified theory including the standard model and the fourth fundamental force, gravity, is still a distant dream. It seems that further progress may depend on a better understanding of the mathematical foundations of these theories.

The gulf between mathematics and physics widened during the first half of the twentieth century. The languages used by the two groups also diverged to the extent that experts in one group had difficulty understanding the work of those in the other. Perhaps the classic example of this is the following excerpt from an interview of Dirac by an American reporter during Dirac's visit to Chicago.

> Reporter: I have been told that few people understand your work. Is there anyone that you do not understand?
>
> Dirac: Yes.
>
> Reporter: Could you please tell me the name of that person?
>
> Dirac: Weyl.

Dirac's opinion was shared by most physicists. The following remark by Yang made at the Stoney Brook Festschrift honoring him illustrates this: Most physicists had a copy of Hermann Weyl's "Gruppentheorie und Quantenmechanik" in their study, but few had read it.

On the mathematical side the great emphasis on generality and abstraction driven largely by the work of the Bourbaki group and its followers further widened the gulf between mathematics and science. Most of them viewed the separation of mathematics and science as a sign of maturity for mathematics: It was becoming an independent field of knowledge. In fact, Dieudonné (one of the founders of the Bourbaki group) expressed the following thoughts in [99]:

> The nay-sayers who predicted that mathematics will be doomed by its separation from science have been proven wrong. In the sixty years or so after early 1900s, mathematics has made great progress, most of which has little to do with physical applications. The one exception is the theory of distributions by Laurent Schwartz, which was motivated by Dirac's work in quantum theory.

These statements are often quoted to show that mathematicians had little interest in talking to scientists. However, in the same article Dieudonné writes:

> I do not intend to say that close contact with other fields, such as theoretical physics, is not beneficial to all parties concerned.

He did not live to see such close contact and dialogue between physicists and mathematicians and to observe that it has been far more beneficial to the mathematicians than to the physicists in the last quarter century.

Gauss called mathematics the queen of sciences. It is well known that mathematics is indispensable in the study of the sciences. Mathematicians often gloat over this. For example, Atiyah has said that he and other mathematicians were very happy to help physicists solve the pseudoparticle (now called the **Euclidean instanton**) problem. His student, Donaldson, was not happy. He wanted to study the geometry and topology of the moduli space of instantons on a 4-manifold M and to find out what information it might provide on the topology of M. Donaldson's work led to totally unexpected results about the topology of M and made gauge theory an important tool for studying low-dimensional topology. At about the same time, the famous physicist Ed Witten was using ideas and techniques from theoretical physics to provide new results and new ways of understanding old ones in mathematics. It is this work that ushered in the study of what we have called "physical mathematics."

Nature is the ultimate arbiter in science. Predictions of any theory have to be tested against experimental observations before it can be called a physical theory. A theory that makes wrong predictions or no predictions at all must be regarded as just a toy model or a proposal for a possible theory. An appealing (or beautiful) formulation is a desirable feature of the theory, but it cannot sustain the theory without experimental verification. The equations of Yang–Mills gauge theory provide a natural generalization of Maxwell's equations. They have a simple and elegant formulation. However, the theory predicted massless bosons, which have never been observed. Yang has said that this was the reason he did not work on the problem for over two decades. Such a constraint does not exist in "Physical Mathematics." So the nonphysical pure Yang–Mills theory has been heartily welcomed, forming the basis for Donaldson's theory of 4-manifolds and Floer's instanton homology of 3-manifolds. However, it was Witten who brought forth a broad spectrum of physical theories to obtain new results and new points of view on old results in mathematics. His work created a whole new area of research that led me to coin the term "physical mathematics" to describe it. Perhaps we can now reverse Dirac's famous statement and say instead "Mathematics is now. Physics can wait." The mathematicians can now say to physicists, "give us your rejects, toy models and nonpredictive theories and we will see if they can give us new mathematics and let us hope that some day they may be useful in physics."

The starting point of the present monograph was *The Mathematical Foundations of Gauge Theories* [274], which the author cowrote with Prof. Martucci (Firenze). That book was based in part on a course in differential geometric methods in physics" that the author gave at CUNY and then repeated at the Dipartimento di Fisica, Università di Firenze in 1986. The course was attended by advanced graduate students in physics and research workers in theoretical physics and mathematics. This monograph is aimed at a similar general audience. The author has given a number of lectures updating the material of that book (which has been out of print for some time) and presenting new developments in physics and their interaction with results in mathematics, in particular in geometric topology. This material now forms the basis for the present work. The classical and quantum theory of fields remains a very active area of research in theoretical physics as well as mathematics. However, the differential geometric foundations of classical gauge theories are now firmly established.

The latest period of strong interaction between theoretical physics and mathematics began in the early 1980s with Donaldson's fundamental work on the topology of 4-manifolds. A look at Fields Medals since then shows several going for work closely linked to physics. The Fields Medal is the highest honor bestowed by the mathematics community on a young (under 40 years of age) mathematician. The Noble Prize is the highest honor in physics but is often given to scientists many years after a work was done and there is no age bar. Appendix B contains more information on the Fields Medals.

Our aim in this work is to give a self-contained treatment of a mathematical formulation of some physical theories and to show how they have led to new results and new viewpoints in mathematical theories. This includes a differential geometric formulation of gauge theories and, in particular, of the theory of Yang–Mills fields. We assume that the readers have had a first course in topology, analysis and abstract algebra and an acquaintance with elements of the theory of differential manifolds, including the structures associated with manifolds such as tensor bundles and differential forms. We give a review of this mathematical background material in the first three chapters and also include material that is generally not covered in a first course.

We discuss in detail principal and associated bundles and develop the theory of connections in Chapter 4.

In Chapter 5 we introduce the characteristic classes associated to principal bundles and discuss their role in the classification of principal and associated bundles. A brief account of K-theory and index theory is also included in this chapter. The first five chapters lay the groundwork for applications to gauge theories, but the material contained in them is also useful for understanding many other physical theories.

Chapter 6 begins with an introduction and a review of the physical background necessary for understanding the role of gauge theories in high-energy physics. We give a geometrical formulation of gauge potentials and fields on a principal bundle P over an arbitrary pseudo-Riemannian base manifold M with the gauge group G. Various formulations of the group of gauge transformations are also given here. Pure gauge theories cannot describe interactions that have massive carrier particles. A resolution of this problem requires the introduction of matter fields. These matter fields arise as sections of bundles associated to the principal bundle P. The base manifold M may also support other fields such as the gravitational field. We refer to all these fields as **associated fields**. A Lagrangian approach to associated fields and coupled equations is also discussed in this chapter. We also discuss the generalized gravitational field equations, which include Einstein's equations with or without the cosmological constant, as well as the gravitational instanton equations as special cases.

Quantum and topological field theories are introduced in Chapter 7. Quantization of classical fields is an area of fundamental importance in modern mathematical physics. Although there is no satisfactory mathematical theory of quantization of classical dynamical systems or fields, physicists have developed several methods of quantization that can be applied to specific problems. We discuss the Feynman path integral method and some regularization techniques briefly.

In Chapter 8 we begin with some historical observations and then discuss Maxwell's electromagnetic theory, which is the prototype of gauge theories. Here, a novel feature is the discussion of the geometrical implications of Maxwell's equations and the use of universal connections in obtaining their solutions. This last method also yields solutions of pure (or source-free) Yang–Mills fields. We then discuss the most extensively studied coupled system, namely, the system of Yang–Mills–Higgs fields. After a brief discussion of various couplings we introduce the idea of spontaneous symmetry breaking and discuss the standard model of electroweak theory. The idea of spontaneous symmetry breaking was introduced by Nambu (who received the Nobel prize in Physics in 2008) and has been extensively studied by many physicists. Its most spectacular application is the Higgs mechanism in the standard model. A brief indication of some of its extensions is also given there.

Chapter 9 is devoted to a discussion of invariants of 4-manifolds. The special solutions of Yang–Mills equations, namely the instantons, are discussed separately. We give an explicit construction of the moduli space $\mathcal{M}_1$ of the BPST-instantons of instanton number 1 and indicate the construction of the moduli space $\mathcal{M}_k$ of the complete $(8k - 3)$-parameter family of instanton solutions over S^4 with gauge group $SU(2)$ and instanton number k. The moduli spaces of instantons on an arbitrary Riemannian 4-manifold with a semisimple Lie group as gauge group are then introduced. A brief account of Donaldson's theorem on the topology of moduli spaces of instantons and its implications for smoothability of 4-manifolds and Donaldson's polynomial invariants is then given. We then discuss Seiberg–Witten monopole equations. The study of $N = 2$ supersymmetric Yang–Mills theory led Seiberg and Witten to the now well-known monopole, or SW equations. The Seiberg–Witten theory provides new tools for the study of 4-manifolds. It contains all the information provided by Donaldson's theory and is much simpler to use. We discuss some applications of the SW invariants and their relation to Donaldson's polynomial invariants.

Chern–Simons theory and its application to Floer type homologies of 3-manifolds and other 3-manifold invariants form the subject of Chapter 10. Witten has argued that invariants obtained via Chern–Simons theory be related to invariants of a string theory. We discuss one particular example of such a correspondence between Chern–Simons theory and string theory in the last section. (String theory is expected to provide unification of all four fundamental forces. This expectation is not yet a reality and the theory (or its different versions) cannot be regarded as a physical theory. However, it has led to many interesting developments in mathematics.)

Classical and quantum invariants of 3-manifolds and knots and links in 3-manifolds are considered in Chapter 11. The relation of some of these invariants with conformal field theory and TQFT are also indicated there. The chapter concludes with a section on Khovanov's categorification of the Jones' polynomial and its extensions to categorification of other link invariants. The treatment of some aspects of these theories is facilitated by the use of techniques from analytic (complex) and algebraic geometry. A full treatment of these would have greatly increased the size of this work. Moreover, excellent monographs covering these areas are available (see, for example, Atiyah [15], Manin [257], Wells [399]). Therefore, topics requiring extensive use of techniques from analytic and algebraic geometry are not considered in this monograph.

There are too many other topics omitted to be listed individually. The most important is string theory. There are several books that deal with this still very active topic. For a mathematical treatment see for example, [95, 96] and [9, 212].

The epilogue points out some highlights of the topics considered. We note that the last three chapters touch upon some areas of active current research where a final definitive mathematical formulation is not yet available. They are intended as an introduction to the ever growing list of topics that can be thought of as belonging to physical mathematics. There are four appendices. Appendix A is a dictionary of terminology and notation between that used in physics and in mathematics. Background notes including historical and biographical notes are contained in Appendix B. The notions of categories and chain complexes are fundamental in modern mathematics. They are the subject of Appendix C. The cobordism category originally introduced and used in Thom's work is now the basis of axioms for TQFT. The general theory of chain complexes is basic in the study of any homology theory. Appendix D contains a brief discussion of operator theory and a more detailed discussion of the Dirac type operators.

Remark on References and Notation

Even though the foundations of electromagnetic theory (the prototype of gauge theories) were firmly in place by the beginning of nineteenth century, the discovery of its relation to the theory of connections and subsequent mathematical developments occurred only during the last three decades. As of this writing field theories remain a very active area of research in mathematical physics. However, the mathematical foundations of classical field theories are now well understood, and these have already led to interesting new mathematics. But we also use theories such as QFT, supersymmetry, and string theory for which the precise mathematical structure or experimental verification is not yet available. We have tried to bring the references up to date as of June 2009. In addition to the standard texts and monographs we have also included some books that give an elementary introductory treatment of some topics. We have included an extensive list of original research papers and review articles that have contributed to our understanding of the mathematical aspects of physical theories. However, many of the important results in papers published before 1980 and in the early 1980s are now available in texts or monographs and hence, in general, are not cited individually. The references to e-prints and private communication are cited in the text itself and are not included in the references at the end of the book.

As we remarked earlier, gauge theories and the theory of connections were developed independently by physicists and mathematicians, and as such have no standard notation. This is also true of other theories. We have used notation and terminology that is primarily used in the mathematical literature, but we have also taken into account the terminology that is most frequently used in physics. To help the reader we have included in Appendix A a correlation of terminology between physics and mathematics prepared along the lines of Trautman [378] and [409].

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The Einstein chair seminars on Topology and Quantum Objects organized by friend and colleague Dennis Sullivan at the CUNY Graduate Center is a continuing source of new ideas and information. Dennis' incisive questions and comments at these all-day seminars make them more accessible to nonexperts. I have enjoyed listening to and occasionally giving a substitute lecture in these seminars. Over the last 12 years I have held the MPG (Max Planck Gesselschaft) research fellowship at the MPI-MIS (Max Planck Institute for Mathematics in the Sciences) in Leipzig. I would like to thank my friend Eberhard Zeidler (founding director of MPI-MIS) for his interest in this work. I have attended many seminars by Prof. Jürgen Jost (director of MPI-MIS) on a wide range of topics in geometry and physics. I thank him for numerous discussions and for giving me a copy of his forthcoming book on geometry and physics before publication. The institute staff made this place a real ivory tower for my work. This work has required much more time and effort than I had anticipated. This would not have been possible without the continued support of MPI-MIS. I am deeply indebted to my colleague and friend Attila Mate who took over my duties on several occasions and who helped me with correcting the copy edited manuscript. His expertise with the typesetting system was also very useful in the production of this document. I would like to thank Stefan Wagner and Julia Plehnert, doctoral students at TU Darmstadt for their careful reading of some chapters of the first draft of the manuscript and for catching many errors of commission and omission. I would like to thank Dr. J. Heinze for his continued interest in my work. I met him about ten years ago when he asked me to write up my Shlosmann lecture for inclusion in Springer's Mathematics Unlimited. After reading my IMPRS lecture notes he suggested that I should write a new book updating my out of print book (with Prof. Martucci) on gauge theories. The present book is the outcome of this suggestion. After the project shifted to Springer UK, I dealt at first with Ms. Karen Borthwick and then with Ms. Lauren Stoney. My special thanks go to Karen and Lauren for providing several reviews of the manuscript at various stages and for making a number of useful suggestions that helped me navigate through a rather long period from initial to the final draft of the book. Thanks are also due to the reviewers for catching errors and obscure statements.

As a child I always learned something new during visits with my grandparents. My grandfather Moreshwar Marathe was a highly respected lawyer with a wide range of interests. He always had some article or poem for me to read. I was also strongly influenced by the great interest in learning that my mother, Indumati, and my grandmother, Parvatibai Agashe, exhibited. Neither of them had university education but they exemplified for me what a truly educated person should be. My grandmother passed away a long time ago. My mother passed away on January 26, 2005. This book is dedicated to her memory.

Kishore B. Marathe
Brooklyn, New York, May 2010

Chapter 1
Algebra

1.1 Introduction

We suppose that the reader is familiar with basic structures of algebra such as groups, rings, fields, and vector spaces and their morphisms, as well as the elements of representation theory of groups. Theory of groups was discovered by Cauchy. He called it "theory of substitutions." He found it so exotic that he is said to have remarked: "It is a beautiful toy, but it will not have any use in the mathematical sciences." In fact, quite the opposite was revealed to be true. The concept of group has proved to be fundamental in all mathematical sciences. In particular, the theory of Lie groups enjoys wide applicability in theoretical physics. We will discuss Lie groups in Chapter 3. Springer has started to reissue the volumes originally published under the general title "Éléments de mathématique" by Nicholas Bourbaki (see the note in Appendix B). The volumes dealing with Lie groups and Lie algebras are [57, 56]. They can be consulted as standard reference works, even though they were written more than 20 years ago.

In the rest of this chapter we discuss some algebraic structure that may not be included in a first year course in algebra. These are some of the structures that appear in many physical theories. Section 1.2 considers the general structure of algebras, including graded algebras. Kac–Moody algebras are discussed in Section 1.3. Clifford algebras are introduced in Section 1.4. The gamma matrices in Dirac's equation for the electron wave function generate one such special Clifford algebra. Section 1.5 is devoted to the classification of finite simple groups and in particular to some strange coincidences dubbed "monstrous moonshine" related to the largest sporadic group called the monster. The quantum dimension of representations of the monster are encoded in various classical Hauptmoduls. Surprising relations between the the monster and vertex algebras, conformal field theory, and string theory have emerged, and these remain a very active area of research.

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1.2 Algebras

Let K denote a field of characteristic zero. All the structures considered in this section are over K, and hence we will often omit explicit reference to K. In most applications K will be either the field $\mathbf{R}$ of real numbers or the field $\mathbf{C}$ of complex numbers. Recall that an **algebra** A over K (or simply an algebra) is a vector space with a bilinear function from $A \times A$ to A (multiplication) and denoted by juxtaposition of elements of A. Note that in general the multiplication in A need not be associative. A is called an **associative algebra** (resp., a **commutative algebra**) if A has a two-sided multiplicative identity (usually denoted by 1) and the multiplication is associative (resp., commutative). A vector subspace B of an algebra A is called a **subalgebra** if it is an algebra under the product induced on it by the product on A. A subalgebra I of A is called a **left ideal** if $xI \subset I$, $\forall x \in A$. **Right ideal** and **two-sided ideal** are defined similarly.

If A, B are algebras, then a map $f : A \to B$ which preserves the algebra structure is called an (algebra) **homomorphism**; i.e., f is a linear map of the underlying vector spaces and $f(xy) = f(x)f(y)$, $\forall x, y \in A$. For associative algebras we also require $f(1) = 1$. If f has an inverse then the inverse is also a homomorphism and f is called an **isomorphism**. A homomorphism (resp., an isomorphism) $f : A \to A$ is called an **endomorphism**. (resp. an **automorphism**). A **derivation** $d : A \to A$ is a linear map that satisfies the Leibniz product rule, i.e.,

$$d(xy) = d(x)y + xd(y), \qquad \forall x, y \in A.$$

The set of all derivations of A has a natural vector space structure. However, the product of two derivations is not a derivation.

Example 1.1 *The set of all endomorphisms of a vector space V, denoted by* End(V) *has the natural structure of an associative algebra with multiplication defined by composition of endomorphisms. A choice of a basis for V allows one to identify the algebra* End(V) *with the* **algebra of matrices** *(with the usual matrix multiplication). Recall that the set $M_n(K)$ of $(n \times n)$ matrices with coefficients from the field K form an associative algebra with the usual operations of addition and multiplication of matrices. Any subalgebra of this algebra is called a* **matrix algebra** *over the field K.*

The set of all automorphisms of a vector space V, denoted by Aut(V) *or $GL(V)$, has the natural structure of a group with multiplication defined by composition of automorphisms. If $K = \mathbf{R}$ (resp., $K = \mathbf{C}$) and* $\dim(V) = n$ *then $GL(V)$ can be identified (by choosing a basis for V) with the group of invertible real (resp., complex) matrices of order n. These groups contain all the* **classical groups** *(i.e., orthogonal, symplectic, and unitary groups) as subgroups. The group operations are continuous in the topology on the groups induced by the standard topology on V (identified with $\mathbf{R}^n$ or $\mathbf{C}^n$). This makes them topological groups. In fact, the classical groups are Lie groups (i.e., they* *are differentiable manifolds and the group operations are differentiable maps). These groups play a fundamental role in the study of global and local symmetry properties of physical systems in classical as well as quantum theories. The tangent space to a Lie group G can be given a natural structure of a Lie algebra LG. This Lie algebra LG carries most of the local information about G. All groups with the same Lie algebra are locally isomorphic and can be obtained as quotients of a unique simply connected group modulo discrete central subgroups. Thinking of LG as a linearization of G allows one to study analytic and global properties of G by algebraic properties of LG. (This is discussed in greater detail in Chapter 3.)*

In physical applications, the most extensively used algebra structure is that of a **Lie algebra**. It is customary to denote the product of two elements x, y by the **bracket** $[x, y]$. Recall that an algebra $\mathfrak{g}$ is called a Lie algebra if its product is **skew-symmetric** and satisfies the well known **Jacobi identity**, i.e.

$$[x, y] = -[y, x], \qquad \forall x, y \in \mathfrak{g}, \tag{1.1}$$

and

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \qquad \forall x, y, z \in \mathfrak{g}, \tag{1.2}$$

The skew-symmetry property is equivalent to the following **alternating property** of multiplication.

$$[x, x] = 0, \qquad \forall x \in \mathfrak{g}, \tag{1.3}$$

This property is an immediate consequence of our assumption that the field K has characteristic zero. We say that the Lie algebra $\mathfrak{g}$ is m-dimensional if the underlying vector space is m-dimensional. If E_i , $1 \leq i \leq m$, is a basis for the Lie algebra $\mathfrak{g}$ then we have

$$[E_j, E_k] = c^i_{jk} E_i,$$

where we have used the Einstein summation convention of summing over repeated indices. The constants c^i_{jk} are called the **structure constants** of $\mathfrak{g}$ with respect to the basis $\{E_i\}$. They characterize the Lie algebra $\mathfrak{g}$ and satisfy the following relations:

1. $c^i_{jk} = -c^i_{kj}$,
2. $c^i_{jk} c^l_{im} + c^i_{km} c^l_{ij} + c^i_{mj} c^l_{ik} = 0$ (Jacobi identity).

The basis $\{E_i\}$ is called an **integral basis** if all the structure constants are integers. A vector subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called a **subalgebra** if it is a Lie algebra under the product (i.e., bracket) induced on it by the product on $\mathfrak{g}$. A subalgebra $\mathfrak{i}$ of $\mathfrak{g}$ is called an **ideal** if $x \in \mathfrak{g}, y \in \mathfrak{i}$ implies that $[x, y] \in \mathfrak{i}$. A Lie algebra ideal is always two-sided.

Given an associative algebra A, we can define a new product on A that gives it a Lie algebra structure. The new product, denoted by [. , .], is defined by

$$[x, y] := xy - yx\ , \qquad \forall x, y \in A. \tag{1.4}$$

The expression on the right hand side of (1.4) is called the **commutator** of x and y in A. It is easy to verify that the new product defined by (1.4) is skew-symmetric and satisfies the Jacobi identity. We denote this Lie algebra by $\mathrm{Lie}(A)$.

Example 1.2 *Let V be a vector space. The Lie algebra obtained by the above construction from* $\mathrm{End}(V)$ *is denoted by* $\mathfrak{gl}(V)$ *with multiplication defined by the commutator of endomorphisms. If $K = \mathbf{R}$ (resp., $K = \mathbf{C}$) and* $\dim(V) = n$ *then $\mathfrak{gl}(V)$ can be identified (by choosing a basis for V) with the Lie algebra of all real (resp., complex) matrices of order n. These Lie algebras contain all the classical (orthogonal, symplectic, and unitary) Lie algebras as Lie subalgebras.*

Given a Lie algebra $\mathfrak{g}$, there exists a unique (up to isomorphism) associative algebra $U(\mathfrak{g})$ called the **universal enveloping algebra** of $\mathfrak{g}$ such that $\mathrm{Lie}(U(\mathfrak{g})) = \mathfrak{g}$.

Example 1.3 *Let A be an associative algebra. A* **derivation** *of A is a linear map $d : A \to A$ satisfying the* **Leibniz rule**

$$d(xy) = xdy + (dx)y, \qquad \forall x, y \in A.$$

Let $\mathfrak{d}(A)$ denote the vector space of all derivations of A. It can be given a Lie algebra structure by defining the product of two derivations to be their commutator; i.e.,

$$[d_1, d_2] := d_1 d_2 - d_2 d_1, \qquad \forall d_1, d_2 \in \mathfrak{d}(A).$$

The commutator $[\mathfrak{g}\ , \mathfrak{g}]$ of $\mathfrak{g}$ with itself is called the **derived algebra** of $\mathfrak{g}$. The commutator $[\mathfrak{g}\ , \mathfrak{g}]$ is an ideal of $\mathfrak{g}$ which is zero if and only if $\mathfrak{g}$ is abelian. By induction one defines the **derived series** $\mathfrak{g}^{(k)}, k \in \mathbf{N}$, by

$$\mathfrak{g}^{(1)} := [\mathfrak{g}\ , \mathfrak{g}] \quad \text{and} \quad \mathfrak{g}^{(k)} := [\mathfrak{g}^{(k-1)}\ , \mathfrak{g}^{(k-1)}], \quad k > 1.$$

A Lie algebra $\mathfrak{g}$ is called **solvable** if $\mathfrak{g}^{(k)} = 0$, for some $k \in \mathbf{N}$. The **lower central series** $\mathfrak{g}^k, k \in \mathbf{N}$, is defined by

$$\mathfrak{g}^1 := [\mathfrak{g}\ , \mathfrak{g}] \quad \text{and} \quad \mathfrak{g}^k := [\mathfrak{g}^{(k-1)}\ , \mathfrak{g}], \quad k > 1.$$

A Lie algebra $\mathfrak{g}$ is called **nilpotent** if $\mathfrak{g}^k = 0$ for some $k \in \mathbf{N}$. Definitions given earlier for morphisms of algebras have their natural counterparts for Lie algebras. A **representation** of a Lie algebra $\mathfrak{g}$ on a vector space V is a homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. The vector space V becomes a left $\mathfrak{g}$-module under the action of $\mathfrak{g}$ on V induced by ρ. Conversely, given a **Lie algebra module** V we can obtain the representation ρ of the Lie algebra $\mathfrak{g}$ on V. In view of this observation we can use the language of representations and modules interchangeably. The dimension of V is called the **degree** of the representation. We now recall some basic facts about representations. A representation is called **faithful** if ρ is injective (i.e., a monomorphism). A **submodule** W of V is a subspace of V that is left invariant under the action of $\mathfrak{g}$ on V. It is called an **invariant subspace** of V. Clearly the zero subspace and V are invariant subspaces. A representation is called **irreducible** if zero and V are its only invariant subspaces. Otherwise, it is called **reducible**. A representation is called **fully reducible** if V is a direct sum of irreducible $\mathfrak{g}$-modules.

Given an element x in a Lie algebra L we define the map

$$\operatorname{ad} x : L \to L \text{ by } (\operatorname{ad} x)(y) := [x, y], \qquad \forall y \in L.$$

It is easy to check that the map ad x is a linear transformation of the vector space L. The bilinear form on L defined by

$$\langle x, y\rangle := \operatorname{Tr}(\operatorname{ad} x \ \operatorname{ad} \ y) \tag{1.5}$$

is called the **Killing form** of L. We define the **adjoint map**

$$\operatorname{ad} : L \to \mathfrak{gl}(L) \text{ by } \operatorname{ad}(x) := \operatorname{ad} x, \qquad \forall x \in L.$$

A simple calculation shows that the adjoint map is a homomorphism of Lie algebras. It is called the **adjoint representation** of L. The kernel of the adjoint representation is the center $Z(L)$ (i.e., $\operatorname{Ker} \operatorname{ad} = Z(L)$). The **center** $Z(L) := \{x \in L \mid [x, y] = 0, \forall y \in L\}$ is an ideal of L.

A non-Abelian Lie algebra $\mathfrak{g}$ is called **simple** if its only ideals are zero and itself. A Lie algebra $\mathfrak{g}$ is called **semi-simple** if it can be written as a direct sum of simple Lie algebras. **Elie Cartan** (1869-1951) obtained a characterization of semi-simple Lie algebras in terms of their Killing form called the **Cartan criterion**. The Cartan criterion states:

> A Lie algebra $\mathfrak{g}$ is semi-simple if and only if its Killing form is non-degenerate. This is equivalent to saying that the Killing form is an inner product on $\mathfrak{g}$.

The simple summands of a semi-simple Lie algebra $\mathfrak{g}$ are orthogonal with respect to the inner product defined by the Killing form. A Lie group G is called semi-simple (resp., simple) if LG is semi-simple (resp., simple).

The classification of semi-simple Lie groups was initiated by **Wilhelm Killing** (1847–1923) at the end of the nineteenth century. It was completed by E. Cartan at the beginning of the twentieth century. The main tool in this classification is the classification of finite-dimensional complex, simple Lie algebras. We give a brief discussion of the basic structures used in obtaining this classification. They are also useful in the general theory of representations. Let $\mathfrak{g}$ be a finite dimensional complex, simple Lie algebra. A nilpotent subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that is **self-centralizing** is called a **Cartan subalgebra**. It can be shown that a non-zero Cartan subalgebra exists and is abelian. Any two Cartan subalgebras are isomorphic. The dimension of a Cartan subalgebra is an invariant of $\mathfrak{g}$. It is called the **rank** of $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra and $x \in \mathfrak{h}$. Then $\mathrm{ad}(x)$ is a **diagonalizable** linear transformation of $\mathfrak{g}$. Moreover, all these linear transformation are simultaneously diagonalizable. Let $\mathfrak{h}^*$ be the dual vector space of $\mathfrak{h}$. For $\lambda \in \mathfrak{h}^*$ define the space $\mathfrak{g}_\lambda$ by

$$\mathfrak{g}_\lambda := \{x \in \mathfrak{g} \mid [a, x] = \lambda(a)x, \qquad \forall a \in \mathfrak{h}. \tag{1.6}$$

We say that λ is a **root** of $\mathfrak{g}$ relative to the Cartan subalgebra $\mathfrak{h}$ if the space $\mathfrak{g}_\lambda$ is non-zero. There exist a set of non-zero roots $A := \{\alpha_i, 1 \leq i \leq s\}$ such that

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_1^s \mathfrak{g}_{\alpha_i} \right), \tag{1.7}$$

where each space $\mathfrak{g}_{\alpha_i}$ is one-dimensional. The decomposition of $\mathfrak{g}$ given in (1.7) is called a **root space decomposition**. Let r denote the rank of $\mathfrak{g}$ (dimension of $\mathfrak{h}$). Then we can find a set $B := \{\beta_j, 1 \leq j \leq r\} \subset A$ satisfying the following properties:

1. B is a **basis** for the space $\mathfrak{h}^*$.
2. Every root in A can be written as an integral linear combination of the elements of B, i.e.,
$$\alpha_i = k_i^j \beta_j, \qquad 1 \leq i \leq s.$$
3. For a given i all the coefficients k_i^j are either in $\mathbf{Z}+$ (non-negative) or are in $\mathbf{Z}-$ (non-positive). In the first case we say that α_i is a **positive root** (resp. **negative root**).

If B satisfies the above properties then we say that B is a set of **simple roots** of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$. The positive and negative roots are in one-to-one correspondence. Let $\mathfrak{g}_+$ denote the direct sum of positive root spaces. The algebra $\mathfrak{g}_+ \oplus \mathfrak{h}$ is called the **Borel subalgebra** relative to the basis of simple roots B. The classification is carried out by studying root systems that correspond to distinct (non-isomorphic) simple Lie algebras.

The finite dimensional complex, simple Lie algebras were classified by Killing and Cartan into four families of **classical algebras** and five **exceptional algebras**. The classical algebras are isomorphic to subalgebras of the matrix algebras $\mathfrak{gl}(n, \mathbf{C})$. Each exceptional Lie algebra is the Lie algebra of a unique simple Lie group. These Lie groups are called the **exceptional groups**. We list the classical Lie algebras, their dimensions, and a matrix representative for each in Table 1.1.

Table 1.1 Classical Lie algebras

Type	Dimension	Matrix algebra
$A_n, n \geq 1$	$n(n+2)$	$\mathfrak{sl}(n+1, \mathbf{C})$
$B_n, n \geq 2$	$n(2n+1)$	$\mathfrak{so}(2n+1, \mathbf{C})$
$C_n, n \geq 3$	$n(2n+1)$	$\mathfrak{sp}(2n, \mathbf{C})$
$D_n, n \geq 4$	$n(2n-1)$	$\mathfrak{so}(2n, \mathbf{C})$

The exceptional Lie groups are listed in Table 1.2 in increasing order of dimension.

Table 1.2 Exceptional Lie groups

Type	G_2	F_4	E_6	E_7	E_8
Dimension	14	52	78	133	248

We conclude this section with a discussion of weights for a finite dimensional $\mathfrak{g}$-module V with corresponding representation ρ. Our starting point is an important theorem due to Hermann Weyl.

Theorem 1.1 *If $\mathfrak{g}$ is a complex semi-simple Lie algebra, then every finite dimensional representation of $\mathfrak{g}$ is fully reducible.*

It follows from Weyl's theorem that $\rho(x), x \in \mathfrak{h}$ ($\mathfrak{h}$ a Cartan subalgebra) is a diagonalizable linear transformation of V. Moreover, all these linear transformations are simultaneously diagonalizable. We say that $\lambda \in \mathfrak{h}^*$ is a **weight** of the $\mathfrak{g}$-module V if the space

$$V_\lambda := \{v \in V \mid \rho(a)v = \lambda(a)v\}, \qquad \forall a \in \mathfrak{h} \tag{1.8}$$

is non-zero. It can be shown that the space V is the direct sum of all the **weight spaces** V_λ and that $\mathfrak{g}_\alpha V_\lambda \subset V_{\alpha+\lambda}$ whenever α is a root. A non-zero vector $v_0 \in V_\lambda$ is called a **highest weight vector** or a **vacuum vector** if $\mathfrak{g}_\alpha v_0 = 0$ for all positive roots α of $\mathfrak{g}$. The weight λ is then called a **highest weight**. The highest weight λ is maximal with respect to the partial order on $\mathfrak{h}^*$ defined by $\mu > \nu$ if $\mu - \nu$ is a sum of positive roots. It is easy to check that the highest weight vector is a simultaneous eigenvector of the Borel algebra of $\mathfrak{g}$. Given a vacuum vector v_0 we can generate an irreducible submodule V_0 of V as follows. Let $\{\alpha_1, \ldots, \alpha_k\}$ be a finite collection of negative roots (not necessarily distinct). Let V_o be the vector space generated by the vectors $(\mathfrak{g}_{\alpha_1} \ldots \mathfrak{g}_{\alpha_k})v_0$ obtained by the successive application of the negative root spaces to v_0. It can be shown that V_0 is an irreducible submodule of V. In particular, if V itself is irreducible, then we have $V_0 = V$. The module V_0 is called the **highest weight module** and the corresponding representation is called the **highest weight representation**. A highest weight vector always exists for a finite-dimensional representation of a semi-simple Lie algebra. It plays a fundamental role in the theory of such representations. In the following example we describe all the irreducible representations of the simple complex Lie algebra $\mathfrak{sl}(2, \mathbf{C})$ as highest weight representations.

Example 1.4 *The Lie algebra* $\mathfrak{sl}(2, \mathbf{C})$ *consists of 2-by-2 complex matrices with trace zero. A standard basis for it is given by the elements* h, e, f *defined by*

$$h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The commutators of the basis elements are given by

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

We note that the basis and commutators are valid for $sl(2, K)$ *for any field* K*. The Cartan subalgebra is one dimensional and is generated by* h*. The following theorem gives complete information about the finite dimensional irreducible representations of the Lie algebra* $\mathfrak{sl}(2, \mathbf{C})$*.*

Theorem 1.2 *For each* $n \in \mathbf{N}$ *there exists a unique (up to isomorphism) irreducible representation* ρ_n *of* $\mathfrak{sl}(2, \mathbf{C})$ *on a complex vector space* V_n *of dimension* n*. There exists a basis* $\{v_0, \ldots, v_{n-1}\}$ *of* V_n *consisting of eigenvectors of* $\rho_n(h)$ *with the vacuum vector (highest weight vector)* v_0 *satisfying the following properties:*

1. $h.v_i = (n - 1 - 2i)v_i$,
2. $e.v_i = (n - i)v_{i-1}$,
3. $f.v_i = v_{i+1}$,

where we have put $\rho_n(x)v = x.v, x \in \mathfrak{sl}(2, \mathbf{C}), v \in V$, $v_{-1} = 0 = v_n$ *and where* $0 \leq i \leq n - 1$.

1.2.1 Graded Algebras

Graded algebraic structures appear naturally in many mathematical and physical theories. We shall restrict our considerations only to $\mathbf{Z}$- and $\mathbf{Z}_2$-gradings. The most basic such structure is that of a graded vector space which we now describe. Let V be a vector space. We say that V is **Z-graded** (resp., **$\mathbf{Z}_2$-graded**) if V is the direct sum of vector subspaces V_i, indexed by the integers (resp., integers mod 2), i.e.

$$V = \bigoplus_{i \in \mathbf{Z}} V_i \quad (\text{resp., } V = V_0 \oplus V_1).$$