TURNPIKE PROPERTIES IN THE CALCULUS OF VARIATIONS AND OPTIMAL CONTROL

Nonconvex Optimization and Its Applications

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TURNPIKE PROPERTIES IN THE CALCULUS OF VARIATIONS AND OPTIMAL CONTROL

By

ALEXANDER J. ZASLAVSKI
The Technion—Israel Institute of Technology, Haifa, Israel

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Preface

This monograph is devoted to recent progress in the turnpike theory. Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson who showed that an efficient expanding economy would for most of the time be in the vicinity of a balanced equilibrium path (also called a von Neumann path) [78, 79]. These properties were studied by many authors for optimal trajectories of a Neumann–Gale model determined by a superlinear set-valued mapping. In the monograph we discuss a number of results concerning turnpike properties in the calculus of variations and optimal control which were obtained by the author in the last ten years. These results show that the turnpike properties are a general phenomenon which holds for various classes of variational problems and optimal control problems.

Turnpike properties are studied for optimal control problems on finite time intervals $[T_1, T_2]$ of the real line. Solutions of such problems (trajectories) always depend on the time interval $[T_1, T_2]$, an optimality criterion which is usually determined by a cost function, and on data which is some initial conditions. In the turnpike theory we are interested in the structure of solutions of optimal problems. We study the behavior of solutions when an optimality criterion is fixed while T_1, T_2 and the data vary. To have turnpike properties means, roughly speaking, that the solutions of a problem are determined mainly by the optimality criterion (a cost function), and are essentially independent of the choice of time interval and data, except in regions close to the endpoints of the time interval. If a point t does not belong to these regions, then the value of a solution at t is closed to a trajectory ("turnpike") which is defined on the infinite time interval and depends only on the optimality criterion. This phenomenon has the following interpretation. If one wishes to reach a point A from a point B by a car in an optimal way, then one should enter onto a turnpike, spend most of one's time on it and then leave the turnpike to reach the required point.

The turnpike phenomenon was discovered by Samuelson in a specific situation. In further numerous studies turnpike properties were established under strong assumptions on an optimality criterion (a cost function). The usual assumptions were that a cost function is time independent and is convex as a function of all its variables. Under these assumptions the "turnpike" is a stationary trajectory (a singleton). The simple form of the "turnpike" with a convex cost function allowed one to discover the turnpike property in this case. Since convexity plays an important role in mathematical economics, turnpike theory has many applications in this area of research. It should be mentioned that there are several interesting results concerning turnpike properties without convexity assumptions. In these results convexity was replaced by other assumptions. The verification of these assumptions was rather difficult and they hold for a narrow class of problems. Thus the turnpike phenomenon was considered by experts as an interesting property of some very particular problems arising in mathematical economics for which a "turnpike" was usually a singleton or a half-ray. This situation has changed in the last ten years. In this monograph we discuss results which were obtained during this period and allow us today to think about turnpike properties as a general phenomenon which holds for various classes of variational problems and optimal control problems. To establish these properties we do not need convexity of a cost function and its time independence.

It was my great pleasure to receive on October 2000 the following letter from Paul A. Samuelson, the discoverer of the turnpike phenomenon.

Dear Professor Zaslavski:

I note with interest your long paper "The Turnpike Property ...Functions" in *Nonlinear Analysis* 42 (2000), 1465-98.

It may be of interest to report that this property and name originated just over half a century ago when, as a Guggenheim Fellow on a 1948-49 sabbatical leave from MIT, I conjectured it in a memo written at the RAND Corporation in Santa Monica, California. In *The Collected Scientific Papers of Paul A. Samuelson*, MIT Press, 1966, 1972, 1977, 1986, it is reproduced. R. Dorfman, P.A. Samuelson, R.M. Solow, *Linear Programming and Economic Analysis*, McGraw-Hill, 1958 gives a pre-Roy Radner exposition. I believe that somewhere Lionel McKenzie has given a nice survey of the relevant mathematical-economics literature.

With admiration,
Paul A. Samuelson

Our studies are based on the following ideas. A "turnpike" is not necessarily a singleton or a half-ray. It can be an absolutely continuous time-dependent function (trajectory) or a compact subset of R^n. To establish a turnpike property we consider a space of cost functions equipped with a natural complete metric and show that a turnpike property holds for most elements of this space in the sense of Baire categories. We obtain a turnpike theorem in the following way. We consider an optimality criterion (a cost function f) and show that for a problem with this criterion there exists an optimal trajectory, say X_f, on an infinite time interval. Then we perturb our cost function by some nonnegative small perturbation which is zero only on X_f. We show that for our new cost function $\bar{f}$ the trajectory X_f is a turnpike, and that optimal solutions of the problem with a cost function g which is closed to $\bar{f}$, are also most of the time close to X_f.

ALEXANDER J. ZASLAVSKI

June 2005

Introduction

Let us consider the following problem of the calculus of variations:

$$\int_0^T f(v(t), v'(t))dt \to \min, \qquad (P_0)$$

$$v : [0, T] \to R^n \text{ is an absolutely continuous function}$$

$$\text{such that } v(0) = y, \ v(T) = z.$$

Here T is a positive number, y and z are elements of the n-dimensional Euclidean space R^n and an integrand $f : R^n \times R^n \to R^1$ is a continuous function.

We are interested in the structure of solutions of the problem (P_0) when y, z and T vary and T is sufficiently large.

Assume that the function f is strictly convex and differentiable and satisfies the following growth condition:

$$f(y, z)/(|y| + |z|) \to \infty \text{ as } |y| + |z| \to \infty.$$

Here we denote by $| \cdot |$ the Euclidean norm in R^n and by $< \cdot, \cdot >$ the scalar product in R^n. In order to analyse the structure of minimizers of the problem (P_0) we consider the auxiliary minimization problem:

$$f(y, 0) \to \min, \ y \in R^n. \qquad (P_1)$$

It follows from the growth condition and the strict convexity of f that the problem (P_1) has a unique solution which will be denoted by $\bar{y}$. Clearly,

$$\partial f/\partial y(\bar{y}, 0) = 0.$$

Define an integrand $L : R^n \times R^n \to R^1$ by

$$L(y, z) = f(y, z) - f(\bar{y}, 0) - < \nabla f(\bar{y}, 0), (y, z) - (\bar{y}, 0) >$$

$$= f(y, z) - f(\bar{y}, 0) - < (\partial f/\partial z)(\bar{y}, 0), z > .$$

Clearly L is also differentiable and srictly convex and satisfies the same growth condition as f:

$$L(y,z)/(|y|+|z|) \to \infty \text{ as } |y|+|z| \to \infty.$$

Since f and L are strictly convex we obtain that

$$L(y,z) \geq 0 \text{ for all } (y,z) \in R^n \times R^n$$

and

$$L(y,z) = 0 \text{ if and only if } y = \bar{y},\ z = 0.$$

Consider the following auxiliary problem of the calculus of variations:

$$\int_0^T L(v(t), v'(t))dt \to \min, \tag{P_2}$$

$$v : [0,T] \to R^n \text{ is an absolutely continuous function}$$

$$\text{such that } v(0) = y,\ v(T) = z,$$

where $T > 0$ and $y, z \in R^n$. It is easy to see that for any absolutely continuous function $x : [0,T] \to R^n$ with $T > 0$,

$$\int_0^T L(x(t), x'(t))dt$$

$$= \int_0^T [f(x(t), x'(t)) - f(\bar{y}, 0) - < (\partial f/\partial z)(\bar{y}, 0), x'(t) >]dt$$

$$= \int_0^T f(x(t), x'(t))dt + Tf(\bar{y}, 0) - < (\partial f/\partial z)(\bar{y}), x(T) - x(0) > .$$

These equations imply that the problems (P_0) and (P_2) are equivalent: a function $x : [0,T] \to R^n$ is a solution of the problem (P_0) if and only if it is a solution of the problem (P_2).

The integrand $L : R^n \times R^n \to R^1$ has the following property:

(C) If $\{(y_i, z_i)\}_{i=1}^{\infty} \subset R^n \times R^n$ satisfies $\lim_{i\to\infty} L(y_i, z_i) = 0$, then $\lim_{i\to\infty} y_i = \bar{y}$ and $\lim_{i\to\infty} z_i = 0$.

Indeed, assume that

$$\{(y_i, z_i)\}_{i=1}^{\infty} \subset R^n \times R^n \text{ and } \lim_{i\to\infty} L(y_i, z_i) = 0.$$

By the growth condition the sequence $\{(y_i, z_i)\}_{i=1}^{\infty}$ is bounded. Let (y, z) be a limit point of the sequence $\{(y_i, z_i)\}_{i=1}^{\infty}$. Then,

$$L(y,z) = \lim_{i\to\infty} L(y_i, z_i) = 0$$

$$\text{and } (y, z) = (\bar{y}, 0).$$

This implies that $(\bar{y}, 0) = \lim_{i\to\infty}(y_i, z_i)$.

Let $y, z \in R^n$, $T > 2$ and a function $\bar{x} : [0, T] \to R^n$ be an optimal solution of the problem (P_0). Then $\bar{x}$ is also an optimal solution of the problem (P_2). We will show that

$$\int_0^T L(\bar{x}(t), \bar{x}'(t))dt \le 2c_0(|y|, |z|)$$

where $c_0(|y|, |z|)$ is a constant which depends only on $|y|$ and $|z|$.

Define a function $x : [0, T] \to R^n$ by

$$x(t) = y + t(\bar{y} - y),\ t \in [0, 1],\ x(t) = \bar{y},\ t \in [1, T - 1],$$

$$x(t) = \bar{y} + (t - (T - 1))(z - \bar{y}),\ t \in [T - 1, T].$$

It follows from the definition of $\bar{x}$ and x that

$$\int_0^T L(\bar{x}(t), \bar{x}'(t))dt \le \int_0^T L(x(t), x'(t))dt$$

$$= \int_0^1 L(x(t), \bar{y} - y)dt + \int_1^{T-1} L(\bar{y}, 0)dt + \int_{T-1}^T L(x(t), z - \bar{y})dt$$

$$= \int_0^1 L(x(t), \bar{y} - y)dt + \int_{T-1}^T L(x(t), z - \bar{y})dt.$$

It is not difficult to see that the integrals

$$\int_0^1 L(x(t), \bar{y} - y)dt \text{ and } \int_{T-1}^T L(x(t), z - \bar{y})dt$$

do not exceed a constant $c_0(|y|, |z|)$ which depends only on $|y|$, $|z|$. Thus

$$\int_0^T L(\bar{x}(t), \bar{x}'(t))dt \le 2c_0(|y|, |z|).$$

It is very important that in this inequality the constant $c_0(|y|, |z|)$ does not depend on T.

We denote by $\mathrm{mes}(E)$ the Lebesgue measure of a Lebesgue mesurable set $E \subset R^1$.

Now let ϵ be a positive number. By the property (C) there is $\delta > 0$ such that if $(y, z) \in R^n \times R^n$ and $L(y, z) \le \delta$, then $|y - \bar{y}| + |z| \le \epsilon$. Then by the choice of δ and the inequality $\int_0^T L(\bar{x}(t), \bar{x}'(t))dt \le 2c_0(|y|, |z|)$,

$$\mathrm{mes}\{t \in [0, T] : |(\bar{x}(t), \bar{x}'(t)) - (\bar{y}, 0)| > \epsilon\}$$

$$\leq \text{mes}\{t \in [0,T] : L(\bar{x}(t), \bar{x}'(t)) > \delta\}$$
$$\leq \delta^{-1} \int_0^T L(\bar{x}(t), \bar{x}'(t))dt \leq \delta^{-1} 2c_0(|y|, |z|)$$

and

$$\text{mes}\{t \in [0,T] : |\bar{x}(t) - \bar{y}| > \epsilon\} \leq \delta^{-1} 2c_0(|y|, |z|).$$

Therefore the optimal solution $\bar{x}$ spends most of the time in an ϵ-neighbor- hood of the point $\bar{y}$. The Lebesgue measure of the set of all points t, for which $\bar{x}(t)$ does not belong to this ϵ-neighborhood, does not exceed the constant $2\delta^{-1}c_0(|y|, |z|)$ which depends only on $|y|, |z|$ and ϵ and does not depend on T. Following the tradition, the point $\bar{y}$ is called the turnpike. Moreover we can show that the set

$$\{t \in [0,T] : |\bar{x}(t) - \bar{y}| > \epsilon\}$$

is contained in the union of two intervals $[0, \tau_1] \cup [T - \tau_2, T]$, where $0 < \tau_1, \tau_2 \leq 2\delta^{-1}c_0(|y|, |z|)$.

Under the assumptions posed on f, the structure of optimal solutions of the problem (P_0) is rather simple and the turnpike $\bar{y}$ is calculated easily. On the other hand the proof is strongly based on the convexity of f and its time independence. The approach used in the proof cannot be employed to extend the turnpike result for essentially larger classes of variational problems. For such extensions we need other approaches and ideas. The question of what happens if the integrand f is nonconvex and nonautonomous seems very interesting. What kind of turnpike and what kind of convergence to the turnpike do we have for general nonconvex nonautonomous integrands? The following example helps to understand the problem.

Let

$$f(t, x, u) = (x - \cos(t))^2 + (u + \sin(t))^2, \ (t, x, u) \in R^1 \times R^1 \times R^1$$

and consider the family of the variational problems

$$\int_{T_1}^{T_2} [(v(t) - \cos(t))^2 + (v'(t) + \sin(t))^2]dt \to \min, \qquad (P_3)$$

$$v : [T_1, T_2] \to R^1 \text{ is an absolutely continuous function}$$
$$\text{such that } v(T_1) = y, \ v(T_2) = z,$$

where $y, z, T_1, T_2 \in R^1$ and $T_2 > T_1$. The integrand f depends on t, for each $t \in R^1$ the function $f(t, \cdot, \cdot) : R^2 \to R^1$ is convex, and for each $x, u \in R^1 \setminus \{0\}$ the functon $f(\cdot, x, u) : R^1 \to R^1$ is nonconvex. Thus the function $f : R^1 \times R^1 \times R^1 \to R^1$ is also nonconvex and depends on t.

Assume that $y, z, T_1, T_2 \in R^1$, $T_2 > T_1 + 2$ and $\widehat{v} : [T_1, T_2] \to R^1$ is an optimal solution of the problem (P_3). Note that the problem (P_3) has a solution since f is continuous and $f(t, x, \cdot) : R^1 \to R^1$ is convex and grows superlinearly at infinity for each $(t, x) \in [0, \infty) \times R^1$.

Define $v : [T_1, T_2] \to R^1$ by

$$v(t) = y + (\cos(1) - y)(t - T_1),\ t \in [T_1, T_1 + 1],$$

$$v(t) = \cos(t),\ t \in [T_1 + 1, T_2 - 1],$$

$$v(t) = \cos(T_2 - 1) + (t - T_2 + 1)(z - \cos(T_2)),\ t \in [T_2 - 1, T_2].$$

It is easy to see that

$$\int_{T_1+1}^{T_2-1} f(t, v(t), v'(t))dt = 0$$

and

$$\int_{T_1}^{T_2} f(t, \widehat{v}(t), \widehat{v}'(t))dt \le \int_{T_1}^{T_2} f(t, v(t), v'(t))dt$$

$$= \int_{T_1}^{T_1+1} f(t, v(t), v'(t))dt + \int_{T_2-1}^{T_2} f(t, v(t), v'(t))dt$$

$$\le 2 \sup\{|f(t, x, u)| :\ t, x, u \in R^1,\ |x|, |u| \le |y| + |z| + 1\}.$$

Thus

$$\int_{T_1}^{T_2} f(t, \widehat{v}(t), \widehat{v}'(t))dt \le c_1(|y|, |z|),$$

where

$$c_1(|y|, |z|) = 2 \sup\{|f(t, x, u)| :\ t, x, u \in R^1,\ |x|, |u| \le |y| + |z| + 1\}.$$

For any $\epsilon \in (0, 1)$ we have

$$\text{mes}\{t \in [T_1, T_2] :\ |\widehat{v}(t) - \cos(t)| > \epsilon\}$$

$$\le \epsilon^{-2} \int_{T_1}^{T_2} f(t, \widehat{v}(t), \widehat{v}'(t))dt \le \epsilon^{-2} c_1(|y|, |z|).$$

Since the constant $c_1(|y|, |z|)$ does not depend on T_2 and T_1 we conclude that if $T_2 - T_1$ is sufficiently large, then the optimal solution $\widehat{v}(t)$ is equal to $\cos(t)$ up to ϵ for most $t \in [T_1, T_2]$. Again, as in the case of convex time independent problems we can show that

$$\{t \in [T_1, T_2] :\ |x(t) - \cos(t)| > \epsilon\} \subset [T_1, T_1 + \tau] \cup [T_2 - \tau, T_2]$$

where $\tau > 0$ is a constant which depends only on ϵ, $|y|$ and $|z|$.

This example shows that there exist nonconvex time dependent integrands which have the turnpike property with the same type of convergence as in the case of convex autonomous variational problems. The difference is that the turnpike is not a singleton but an absolutely continuous time dependent function defined on the infinite interval $[0, \infty)$. This leads us to the following definition of the turnpike property for general integrands.

Let us consider the following variational problem:

$$\int_{T_1}^{T_2} f(t, v(t), v'(t))dt \to \min, \qquad (P)$$

$$v : [T_1, T_2] \to R^n \text{ is an absolutely continuous function}$$

$$\text{such that } v(T_1) = y,\ v(T_2) = z.$$

Here $T_1 < T_2$ are real numbers, y and z are elements of the n-dimensional Euclidean space R^n and an integrand $f : [0, \infty) \times R^n \times R^n \to R^1$ is a continuous function.

We say that the integrand f has the *turnpike property* if there exists a locally absolutely continuous function $X_f : [0, \infty) \to R^n$ (called the "turnpike") which depends only on f and satisfies the following condition:

For each bounded set $K \subset R^n$ and each $\epsilon > 0$ there exists a constant $T(K, \epsilon) > 0$ such that for each $T_1 \geq 0$, each $T_2 \geq T_1 + 2T(K, \epsilon)$, each $y, z \in K$ and each optimal solution $v : [T_1, T_2] \to R^n$ of variational problem (P), the inequality $|v(t) - X_f(t)| \leq \epsilon$ holds for all $t \in [T_1 + T(K, \epsilon), T_2 - T(K, \epsilon)]$.

The turnpike property is very important for applications. Suppose that the integrand f has the turnpike property, K and ϵ are given, and we know a finite number of "approximate" solutions of the problem (P). Then we know the turnpike X_f, or at least its approximation, and the constant $T(K, \epsilon)$ which is an estimate for the time period required to reach the turnpike. This information can be useful if we need to find an "approximate" solution of the problem (P) with a new time interval $[T_1, T_2]$ and the new values $y, z \in K$ at the end points T_1 and T_2. Namely instead of solving this new problem on the "large" interval $[T_1, T_2]$ we can find an "approximate" solution of problem (P) on the "small" interval $[T_1, T_1 + T(K, \epsilon)]$ with the values $y, X_f(T_1 + T(K, \epsilon))$ at the end points and an "approximate" solution of problem (P) on the "small" interval $[T_2 - T(K, \epsilon), T_2]$ with the values $X_f(T_2 - T(K, \epsilon)), z$ at the end points. Then the concatenation of the first solution, the function $X_f : [T_1 + T(K, \epsilon), T_2 - T(K, \epsilon)]$ and the second solution is an "approximate" solution of problem (P) on the interval $[T_1, T_2]$ with the values y, z at the end points.

We begin our monograph with a discussion of the problem (P). In Chapter 1 we introduce a space $\mathcal{M}$ of continuous integrands $f : [0, \infty) \times R^n \times R^n \to R^1$. This space is equipped with a natural complete metric. We show that for any initial condition $x_0 \in R^n$ there exists a locally absolutely continuous function $x : [0, \infty) \to R^n$ with $x(0) = x_0$ such that for each $T_1 \geq 0$ and $T_2 > T_1$ the function $x : [T_1, T_2] \to R^n$ is a solution of problem (P) with $y = x(T_1)$ and $z = x(T_1)$. We also establish that for every bounded set $E \subset R^n$ the $C([T_1, T_2])$ norms of approximate solutions $x : [T_1, T_2] \to R^n$ for the problem (P) with $y, z \in E$ are bounded by some constant which does not depend on T_1 and T_2.

In Chapter 2 we establish the turnpike property stated above for a generic integrand $f \in \mathcal{M}$. We establish the existence of a set $\mathcal{F} \subset \mathcal{M}$ which is a countable intersection of open everywhere dense sets in $\mathcal{M}$ such that for each $f \in \mathcal{F}$ the turnpike property holds. Moreover we show that the turnpike property holds for approximate solutions of variational problems with a generic integrand f and that the turnpike phenomenon is stable under small pertubations of a generic integrand f.

In Chapters 3-5 we study turnpike properties for autonomous problems (P) with integrands $f : R^n \times R^n \to R^1$ which do not depend on t. Since the turnpike theorems of Chapter 2 are of generic nature and the subset of $\mathcal{M}$ which consists of all time independent integrands are nowhere dense, the results of Chapter 2 can not be applied for this subset. Moreover, we cannot expect to obtain the turnpike property stated above for the general autonomous case. Indeed, if an integrand f does not depend on t and has a turnpike, then this turnpike should also be time independent. It means that the turnpike is a stationary trajectory (a singleton). But it is not true when a time independent integrand f is not a convex function.

Consider the following example. Let

$$f(x_1, x_2, u_1, u_2) = (x_1^2 + x_2^2 - 1)^2 + (u_1 + x_2)^2 + (u_2 - x_1)^2,$$

$$(x_1, x_2, u_1, u_2) \in R^2 \times R^2$$

and consider the family of the variational problems

$$\int_0^T f(v_1(t), v_2(t), v_1'(t), v_2'(t))dt \to \min, \qquad (P_4)$$

$$(v_1, v_2) : [0, T] \to R^2 \text{ is an absolutely continuous function}$$

$$\text{such that } (v_1, v_2)(0) = y, \ (v_1, v_2)(T) = z,$$

where $y = (y_1, y_2),\ z = (z_1, z_2) \in R^2$ and $T > 0$. The integrand f does not depend on t. Since f is continuous and for each $x = (x_1, x_2) \in R^2$ the function $f(x, \cdot) : R^2 \to R^1$ is convex and grows superlinearly at infinity, the problem (P_4) has a solution for each $T > 0$ and each $y, z \in R^2$. Clearly, if $T > 0$, $y = (\cos(0), \sin(0))$ and $z = (\cos(T), \sin(T))$, then the function

$$\widehat{x}_1(t) = \cos(t),\ \widehat{x}_2(t) = \sin(t),\ t \in [0, T]$$

is a solution of the problem (P_4). Thus, if the integrand f has a turnpike property, then the turnpike is not a singleton.

Let $T > 2$, $y, z \in R^2$ and let $\bar{v} = (\bar{v}_1, \bar{v}_2) : [0, T] \to R^2$ be a solution of the problem (P_4). Define a function $v = (v_1, v_2) : [0, T] \to R^n$ by

$$v(t) = y + t((\cos(1), \sin(1)) - y),\ t \in [0, 1],$$

$$v(t) = (\cos(t), \sin(t)),\ t \in [1, T-1],$$

$$v(t) = (\cos(T-1), \sin(T-1)) + (t - T + 1)(z - (\cos(T-1), \sin(T-1)),$$
$$t \in [T-1, T].$$

Then

$$\int_1^{T-1} f(v(t), v'(t))dt = 0$$

and

$$\int_0^T (\bar{v}_1(t)^2 + \bar{v}_2(t)^2 - 1)^2 dt \le \int_0^T f(\bar{v}(t), \bar{v}'(t))dt$$
$$\le \int_0^T f(v(t), v'(t))dt$$
$$= \int_0^1 f(v(t), v'(t))dt + \int_{T-1}^T f(v(t), v'(t))dt$$
$$\le \sup\{f(x_1, x_2, u_1, u_2) :\ x_1, x_2, u_1, u_2 \in R^1$$
$$\text{and } |x_i|, |u_i| \le 2|y| + 2|z| + 2,\ i = 1, 2\}.$$

Thus

$$\int_0^T (\bar{v}_1(t)^2 + \bar{v}_2(t)^2 - 1)^2 dt \le c_2(|y|, |z|)$$

with

$$c_2(|y|, |z|) = \sup\{f(x_1, x_2, u_1, u_2) :\ x_1, x_2, u_1, u_2 \in R^1$$
$$\text{and } |x_i|, |u_i| \le 2|y| + 2|z| + 2\}.$$

Here $c_2(|y|, |z|)$ depends only on $|y|, |z|$ and does not depend on T. For any $\epsilon \in (0, 1)$ we have

$$\text{mes}\{t \in [0, T] : ||(\bar{v}_1(t), \bar{v}_2(t))| - 1| > \epsilon\}$$

$$\le \text{mes}\{t \in [0,T] : |\bar{v}_1(t)^2 + \bar{v}_2(t)^2 - 1| > \epsilon^2\}$$
$$\le \epsilon^{-4} \int_0^T (\bar{v}_1(t)^2 + \bar{v}_2^2 - 1)^2 dt$$
$$\le \epsilon^{-4} c_2(|y|, |z|).$$

It means that for most $t \in [0,T]$, $\bar{v}(t)$ belongs to the ϵ-neighborhood of the set $\{x \in R^2 : \ |x| = 1\}$. Thus we can say that the integrand f has a weakened version of the turnpike property and the set $\{|x| = 1\}$ can be considered as the turnpike for f.

For a general autonomous nonconvex problem (P) we also have a version of the turnpike property in which a turnpike is a compact subset of R^n. This subset depends only on the integrand f.

Consider the following autonomous variational problem:

$$\int_0^T f(z(t), z'(t))dt \to \min,\ z(0 = x,\ z(T) = y, \qquad (P_a)$$

$$z : [0,T] \to R^n \text{ is an absolutely continuous function}$$

where $T > 0$, $x, y \in R^n$ and $f : R^{2n} \to R^1$ is an integrand.

We say that a time independent integrand $f = f(x,u) \in C(R^{2n})$ has the *turnpike property* if there exists a compact set $H(f) \subset R^n$ such that for each bounded set $K \subset R^n$ and each $\epsilon > 0$ there exist numbers $L_1 > L_2 > 0$ such that for each $T \ge 2L_1$, each $x, y \in K$ and an optimal solution $v : [0,T] \to R^n$ for the variational problem (P_a), the relation

$$\text{dist}(H(f), \{v(t) :\ t \in [\tau, \tau + L_2]\}) \le \epsilon$$

holds for each $\tau \in [L_1, T - L_1]$. (Here $\text{dist}(\cdot,\cdot)$ is the Hausdorff metric).

We also consider a weak version of this turnpike property for a time independent integrand $f(x,u)$. In this weak version, for an optimal solution of the problem (P_a) with $x, y \in R^n$ and large enough T, the relation

$$\text{dist}(H(f), \{v(t) :\ t \in [\tau, \tau + L_2]\}) \le \epsilon$$

with L_2, which depends on ϵ and $|x|, |y|$ and a compact set $H(f) \subset R^n$ depending only on the integrand f, holds for each $\tau \in [0,T] \setminus E$ where $E \subset [0,T]$ is a measurable subset such that the Lebesgue measure of E does not exceed a constant which depends on ϵ and on $|x|, |y|$.

These two turnpike properties for autonomous problems (P_a) are considered in Chapters 3-5.

In Chapter 3 we consider the space $\mathcal{A}$ of all time independent integrands $f \in \mathcal{M}$. We establish the existence of a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open everywhere dense sets in $\mathcal{A}$ such that for each $f \in \mathcal{F}$ the weakened version of the turnpike property holds.

The turnpike property for time independent integrands is established in Chapter 5 for a generic element of a subset $\mathcal{N}$ of the space $\mathcal{A}$. The space $\mathcal{N}$ is a subset of all integrands $f \in \mathcal{A}$ which satisfy some differentiability assumptions.

In the other chapters of the monograph we establish a number of turnpike results (generic and individual) for various classes of optimal control problems. We study optimal control of linear periodic systems with convex integrands (Chapter 6) and optimal solutions of linear systems with convex nonperiodic integrands (Chapter 7). In Chapter 8 we establish turnpike theorems for discrete-time control systems in Banach spaces and in complete metric spaces. Infinite-dimensional continuous-time optimal control problems in a Hilbert space are studied in Chapter 9. A turnpike theorem for a class of differential inclusions arising in economic dynamics is proved in Chapter 10 and structure of optimal trajectories of convex processes is studied in Chapter 11. In Chapter 12 we establish a turnpike property for a dynamic discrete-time zero-sum game.

Chapter 1

INFINITE HORIZON VARIATIONAL PROBLEMS

In this chapter we study existence and uniform boundedness of extremals of variational problems with integrands which belong to a complete metric space of functions. We establish that for every bounded set $E \subset R^n$ the $C([0,T])$ norms of approximate solutions $x: [0,T] \to R^n$ for the minimization problem on an interval $[0,T]$ with $x(0), x(T) \in E$ are bounded by some constant which does not depend on T. Given an $x_0 \in R^n$ we study the infinite horizon problem of minimizing the expression $\int_0^T f(t, x(t), x'(t))dt$ as T grows to infinity, where $x : [0, \infty) \to R^n$ satisfies the initial condition $x(0) = x_0$. We analyse the existence and the properties of approximate solutions for every prescribed initial value x_0.

1.1. Preliminaries

Variational and optimal control problems defined on infinite intervals are of interest in many areas of mathematics and its applications [10, 11, 16, 32, 62, 63, 88, 89, 95]. These problems arise in engineering [1, 3], in models of economic growth [14, 26, 27, 28, 29, 45, 46, 49-52, 60, 61, 67, 68, 72, 74, 80, 86, 94], in dynamic games theory [15, 17], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [6, 85] and in the theory of thermodynamical equilibrium of materials [20, 44, 53-55, 90-92, 95].

We consider the infinite horizon problem of minimizing the expression

$$\int_0^T f(t, x(t), x'(t))dt$$

as T grows to infinity where a function $x: [0,\infty) \to K$ is locally absolutely continuous (a.c.) and satisfies the initial condition $x(0) = x_0$, $K \subset R^n$ is a closed convex set and f belongs to a complete metric space of functions to be described below.

We say that an a.c. function $x : [0,\infty) \to K$ is (f)-overtaking optimal if

$$\limsup_{T\to\infty} \int_0^T [f(t,x(t),x'(t)) - f(t,y(t),y'(t))]dt \le 0$$

for any a.c. function $y : [0,\infty) \to K$ satisfying $y(0) = x(0)$.

This notion, known as the overtaking optimality criterion, was introduced in the economics literature by Atsumi [4], Gale [33] and von Weizsacker [81] and has been used in control theory [3, 13, 14, 16, 39, 40]. In general, overtaking optimal solutions may fail to exist. Most studies that are concerned with their existence assume convex integrands [13, 40, 72].

Another type of optimality criterion for infinite horizon problems was introduced by Aubry and Le Daeron [6] in their study of the discrete Frenkel–Kontorova model related to dislocations in one-dimensional crystals. More recently this optimality criterion was used in [44, 65, 66, 85]. A similar notion was introduced in Halkin [34] for his proof of the maximum principle.

Let I be either $[0,\infty)$ or $(-\infty,\infty)$. We say that an a.c. function $x : I \to K$ is an (f)-minimal solution if

$$\int_{T_1}^{T_2} f(t,x(t),x'(t))dt \le \int_{T_1}^{T_2} f(t,y(t),y'(t))dt \le 0$$

for each $T_1 \in I$, $T_2 > T_1$ and each a.c. function $y : [T_1,T_2] \to K$ which satisfies $y(T_i) = x(T_i),\ i = 1,2$.

It is easy to see that every (f)-overtaking optimal function is an (f)-minimal solution.

In this chapter we consider a functional space of integrands $\mathcal{M}$ described in Section 1.1. We show that for each $f \in \mathcal{M}$ and each $z \in R^n$ there exists a bounded (f)-minimal solution $Z : [0,\infty) \to R^n$ satisfying $Z(0) = z$ such that any other a.c. function $Y : [0,\infty) \to R^n$ is not "better" than Z. We also establish that given $f \in \mathcal{M}$ and a bounded set $E \subset R^n$ the $C([0,T])$ norms of approximate solutions $x : [0,T] \to R^n$ for the minimization problem on an interval $[0,T]$ with $x(0), x(T) \in E$ are bounded by some constant which depends only on f and E.

1.2. Main results

Let $a > 0$ be a constant and $\psi : [0, \infty) \to [0, \infty)$ be an increasing function such that $\psi(t) \to \infty$ as $t \to \infty$.

Let $K \subset R^n$ be a closed convex set. Denote by $|\cdot|$ the Euclidean norm in R^n and denote by $\mathcal{M}$ the set of continuous functions $f : [0, \infty) \times K \times R^n \to R^1$ which satisfy the following assumptions:

A(i) for each $(t, x) \in [0, \infty) \times K$ the function $f(t, x, \cdot) : R^n \to R^1$ is convex;

A(ii) the function f is bounded on $[0, \infty) \times E$ for any bounded set $E \subset K \times R^n$;

A(iii) for each $(t, x, u) \in [0, \infty) \times K \times R^n$,

$$f(t, x, u) \geq \max\{\psi(|x|), \psi(|u|)|u|\} - a;$$

A(iv) for each $M, \epsilon > 0$ there exist $\Gamma, \delta > 0$ such that

$$|f(t, x_1, u_1) - f(t, x_2, u_2)| \leq \epsilon \max\{f(t, x_1, u_1), f(t, x_2, u_2)\}$$

for each $t \in [0, \infty)$, each $u_1, u_2 \in R^n$ and each $x_1, x_2 \in K$ which satisfy

$$|x_i| \leq M,\ |u_i| \geq \Gamma,\ i = 1, 2, \quad \max\{|x_1 - x_2|, |u_1 - u_2|\} \leq \delta;$$

A(v) for each $M, \epsilon > 0$ there exist $\delta > 0$ such that

$$|f(t, x_1, u_1) - f(t, x_2, u_2)| \leq \epsilon$$

for each $t \in [0, \infty)$, each $u_1, u_2 \in R^n$ and each $x_1, x_2 \in K$ which satisfy

$$|x_i|, |u_i| \leq M,\ i = 1, 2, \quad \max\{|x_1 - x_2|, |u_1 - u_2|\} \leq \delta.$$

When $K = R^n$ it is an elementary exercise to show that an integrand $f = f(t, x, u) \in C^1([0, \infty) \times R^n \times R^n)$ belongs to $\mathcal{M}$ if f satisfies Assumptions A(i), A(iii),

$$\sup\{|f(t, 0, 0)| : t \in [0, \infty)\} < \infty$$

and there exists an increasing function $\psi_0 : [0, \infty) \to [0, \infty)$ such that

$$\sup\{|\partial f/\partial x(t, x, u)|,\ |\partial f/\partial u(t, x, u)|\} \leq \psi_0(|x|)(1 + \psi(|u|)|u|)$$

for each $t \in [0, \infty)$, $x, u \in R^n$.

Therefore the space $\mathcal{M}$ contains many functions.

Example 1. It is not difficult to see that if $\psi(t) = t$ for all $t \geq 0$, $n = 1$, $K = R^1$, if functions $h_1, h_2, h_3 \in C^1(R^1)$ satisfy

$$h_1(t) \geq 0,\ t \in [0, \infty),\ \sup\{h_1(t) : t \in [0, \infty)\} < \infty,$$

$$h_2(x) \geq |x| + 1, \ x \in R^1$$

and if the function $h_3 : R^1 \to R^1$ is convex and

$$u^2 + 1 \leq h_3(u) \leq c_0(u^2 + 1), \ |h_3'(u)| \leq c_0(u^2 + 1)$$

for all $u \in R^1$, where c_0 is a positive constant, then the function

$$f(t, x, u) = h_1(t) + h_2(x)h_3(u), \ (t, x, u) \in [0, \infty) \times R^1 \times R^1$$

belongs to $\mathcal{M}$.

In Chapters 1-5 we consider variational problems with integrands belonging to the space $\mathcal{M}$ or to its subspaces. The Assumption A(i) and the inequality $f(t, x, u) \geq \psi(|u|)|u| - a$ in the Assumption A(iii) guarantee the existence of minimizers of the variational problems. These assumptions are common in the literature. We need the inequality $f(t, x, u) \geq \psi(|x|) - a$ in A(iii) in order to show that for every bounded set $E \subset R^n$ the $C([0, T])$ norms of approximate solutions $x : [0, T] \to R^n$ for the variational problems on intervals $[0, T]$ with $x(0), x(T) \in E$ are bounded by some constant which does not depend on T. We need the Assumptions A(ii) and A(v) in order to obtain certain properties of approximate solutions for variational problems on intervals $[T_1, T_2]$ which depend on $T_2 - T_1$ and do not depend of T_1 and T_2. Note that if a function f is Frechet differentiable, then the Assumption A(v) means that the growth of the partial derivatives of f does not exceed the growth of f. We use it in order to establish the continuity of the function U^f which is defined below.

We equip the set $\mathcal{M}$ with the uniformity which is determined by the following base:

$$E(N, \epsilon, \lambda) = \{(f, g) \in \mathcal{M} \times \mathcal{M} : \ |f(t, x, u) - g(t, x, u)| \leq \epsilon \tag{2.1}$$

$$\text{for each } t \in [0, \infty), \text{ each } u \in R^n \text{ each } x \in K \text{ satisfying } |x|, |u| \leq N\}$$

$$\cap\{(f, g) \in \mathcal{M} \times \mathcal{M} : \ (|f(t, x, u)| + 1)(|g(t, x, u)| + 1)^{-1} \in [\lambda^{-1}, \lambda]$$

$$\text{for each } t \in [0, \infty), \text{ each } u \in R^n \text{ and each } x \in K \text{ satisfying } |x| \leq N\}$$

where $N > 0$, $\epsilon > 0$, $\lambda > 1$ [37].

Clearly, the uniform space $\mathcal{M}$ is Hausdorff and has a countable base. Therefore $\mathcal{M}$ is metrizable. We will prove in Secton 1.3 that the uniform space $\mathcal{M}$ is complete.

Put

$$I^f(T_1, T_2, x) = \int_{T_1}^{T_2} f(t, x(t), x'(t))dt \tag{2.2}$$

where $f \in \mathcal{M}$, $0 \le T_1 < T_2 < \infty$ and $x : [T_1, T_2] \to K$ is an a.c. function.

For $f \in \mathcal{M}$, $a, b \in K$ and numbers T_1, T_2 satisfying $0 \le T_1 < T_2$, put

$$U^f(T_1, T_2, a, b) = \inf\{I^f(T_1, T_2, x) : \ x : [T_1, T_2] \to K \tag{2.3}$$

$$\text{is an a.c. function satisfying } x(T_1) = a, \ x(T_2) = b\},$$

$$\sigma^f(T_1, T_2, a) = \inf\{U^f(T_1, T_2, a, b) : \ b \in K\}. \tag{2.4}$$

It is easy to see that $-\infty < U^f(T_1, T_2, a, b) < \infty$ for each $f \in \mathcal{M}$, each $a, b \in K$ and each pair of numbers T_1, T_2 satisfying $0 \le T_1 < T_2$.

Let $f \in \mathcal{M}$. We say that an a.c. function $x : [0, \infty) \to K$ is an (f)-good function if for any a.c. function $y : [0, \infty) \to K$,

$$\inf\{I^f(0, T, y) - I^f(0, T, x) : \ T \in (0, \infty)\} > -\infty. \tag{2.5}$$

In this chapter we study the set of (f)-good functions and prove the following results.

THEOREM 1.2.1 *For each $h \in \mathcal{M}$ and each $z \in K$ there exists an (h)-good function $Z^h : [0, \infty) \to K$ satisfying $Z^h(0) = z$ such that:*

1. For each $f \in \mathcal{M}$, each $z \in K$ and each a.c. function $y : [0, \infty) \to K$ one of the following properties holds:

(i) $I^f(0, T, y) - I^f(0, T, Z^f) \to \infty$ *as* $T \to \infty$*;*

(ii) $\sup\{|I^f(0, T, y) - I^f(0, T, Z^f)| : \ T \in (0, \infty)\} < \infty$,

$$\sup\{|y(t)| : \ t \in [0, \infty)\} < \infty.$$

2. For each $f \in \mathcal{M}$ and each number $M > \inf\{|u| : \ u \in K\}$ there exist a neighborhood U of f in $\mathcal{M}$ and a number $Q > 0$ such that

$$\sup\{|Z^g(t)| : \ t \in [0, \infty)\} \le Q$$

for each $g \in U$ and each $z \in K$ satisfying $|z| \le M$.

3. For each $f \in \mathcal{M}$ and each number $M > \inf\{|u| : \ u \in K\}$ there exist a neighborhood U of f in $\mathcal{M}$ and a number $Q > 0$ such that for each $g \in U$, each $z \in K$ satisfying $|z| \le M$, each $T_1 \ge 0$, $T_2 > T_1$ and each a.c. function $y : [T_1, T_2] \to K$ satisfying $|y(T_1)| \le M$ the following relation holds:

$$I^g(T_1, T_2, Z^g) \le I^g(T_1, T_2, y) + Q.$$

4. If $K = R^n$, then for each $f \in \mathcal{M}$ and each $z \in R^n$ the function $Z^f : [0, \infty) \to R^n$ is an (f)-minimal solution.

COROLLARY 1.2.1 *Let $f \in \mathcal{M}$, $z \in K$ and let $y : [0, \infty) \to K$ be an a.c. function. Then y is an (f)-good function if and only if condition (ii) of Assertion 1 of Theorem 1.2.1 holds.*

THEOREM 1.2.2 *For each $f \in \mathcal{M}$ there exist a neighborhood U of f in $\mathcal{M}$ and a number $M > 0$ such that for each $g \in U$ and each (g)-good function $x : [0, \infty) \to K$,*

$$\limsup_{t \to \infty} |x(t)| < M.$$

Our next result shows that for every bounded set $E \subset K$ the $C([0,T])$ norms of approximate solutions $x : \; [0,T] \to K$ for the minimization problem on an interval $[0,T]$ with $x(0), x(T) \in E$ are bounded by some constant which does not depend on T.

THEOREM 1.2.3 *Let $f \in \mathcal{M}$ and M_1, M_2, c be positive numbers. Then there exist a neighborhood U of f in $\mathcal{M}$ and a number $S > 0$ such that for each $g \in U$, each $T_1 \in [0, \infty)$ and each $T_2 \in [T_1 + c, \infty)$ the following properties hold:*

(i) if $x, y \in K$ satisfy $|x|, |y| \leq M_1$ and if an a.c. function $v : [T_1, T_2] \to K$ satisfies

$$v(T_1) = x, \; v(T_2) = y, \; I^g(T_1, T_2, v) \leq U^g(T_1, T_2, x, y) + M_2,$$

then

$$|v(t)| \leq S, \; t \in [T_1, T_2]; \tag{2.6}$$

(ii) if $x \in K$ satisfies $|x| \leq M_1$ and if an a.c. function $v : [T_1, T_2] \to K$ satisfies

$$v(T_1) = x, \; I^g(T_1, T_2, v) \leq \sigma^g(T_1, T_2, x) + M_2,$$

then the inequality (2.6) is valid.

Theorems 1.2.1-1.2.3 have been proved in [98].

In the sequel we use the following notation:

$$B(x, r) = \{y \in R^n : \; |y - x| \leq r\}, \; x \in R^n, \; r > 0, \tag{2.7}$$

$$B(r) = B(0, r), \; r > 0.$$

Chapter 1 is organized as follows. In Section 1.3 we study the space $\mathcal{M}$ and the dependence of the functionals U^f and I^f of f. In Section 1.4 we associate with any $f \in \mathcal{M}$ a related discrete-time control system and study its approximate solutions. Theorems 1.2.1-1.2.3 are proved in Section 1.5.

1.3. Auxiliary results

In this section we study the space $\mathcal{M}$ and continuity properties of the functionals I^f and U^f. The next proposition follows from Assumption A(iv).

PROPOSITION 1.3.1 *Let $f \in \mathcal{M}$. Then for each pair of positive numbers M and ϵ there exist $\Gamma, \delta > 0$ such that the following property holds:*

If $t \in [0, \infty)$ and if $u_1, u_2 \in R^n$ and $x_1, x_2 \in K$ satisfy

$$|x_i| \leq M, \ |u_i| \geq \Gamma, \ i = 1, 2, \ |u_1 - u_2|, \ |x_1 - x_2| \leq \delta, \tag{3.1}$$

then

$$|f(t, x_1, u_1) - f(t, x_2, u_2)| \leq \epsilon \min\{f(t, x_1, u_1), f(t, x_2, u_2)\}.$$

Proof. Let $M, \epsilon > 0$. Choose

$$\epsilon_0 \in (0, 8^{-1} \inf\{1, \epsilon\}). \tag{3.2}$$

It follows from Assumption A(iv) that there exist $\Gamma, \delta > 0$ such that the following property holds:

If $t \in [0, \infty)$ and if $u_1, u_2 \in R^n$ and $x_1, x_2 \in K$ satisfy (3.1), then

$$|f(t, x_1, u_1) - f(t, x_2, u_2)| \leq \epsilon_0 \sup\{f(t, x_1, u_1), f(t, x_2, u_2)\}. \tag{3.3}$$

Assume that $t \in [0, \infty)$, $u_1, u_2 \in R^n$ and $x_1, x_2 \in K$ satisfy (3.1). By the definition of Γ, δ, (3.2) and (3.3),

$$\min\{f(t, x_1, u_1), \ f(t, x_2, u_2)\} \geq (1 - \epsilon_0) \max\{f(t, x_1, u_1), f(t, x_2, u_2)\}$$

$$\geq (1 - \epsilon_0)\epsilon_0^{-1}|f(t, x_1, u_1) - f(t, x_2, u_2)| \geq \epsilon^{-1}|f(t, x_1, u_1) - f(t, x_2, u_2)|.$$

Proposition 1.3.1 is proved.

PROPOSITION 1.3.2 *The uniform space $\mathcal{M}$ is complete.*

Proof. Assume that $\{f_i\}_{i=1}^{\infty} \subset \mathcal{M}$ is a Cauchy sequence. Clearly, for each $(t, x, u) \in [0, \infty) \times K \times R^n$ the sequence $\{f_i(t, x, u)\}_{i=1}^{\infty}$ is a Cauchy sequence. Then there exists a function $f : [0, \infty) \times K \times R^n \to R^1$ such that

$$f(t, x, u) = \lim_{i \to \infty} f_i(t, x, u) \tag{3.4}$$

for each $(t, x, u) \in [0, \infty) \times K \times R^n$.

In order to prove the proposition it is sufficient to show that f satisfies Assumption A(iv).

Let M, ϵ be positive numbers. Choose a number $\lambda > 1$ for which

$$\lambda^2 - 1 < 8^{-1}\epsilon. \tag{3.5}$$

Since $\{f_i\}_{i=1}^{\infty}$ is a Cauchy sequence there exists an integer $j \geq 1$ such that

$$(f_i, f_j) \in E(M, \epsilon, \lambda) \text{ for any integer } i \geq j. \tag{3.6}$$

By (3.5) and the properties of ψ there exists a number Γ_0 such that

$$\Gamma_0 > 1,\ \psi(\Gamma_0) \geq 2a,\ \lambda^2(1 + 2\psi(\Gamma_0)^{-1})^2 - 1 < 8^{-1}\epsilon. \tag{3.7}$$

Choose $\epsilon_1 > 0$ such that

$$8\epsilon_1[\lambda(1 + 2\psi(\Gamma_0)^{-1})]^2 < \epsilon. \tag{3.8}$$

By Proposition 1.3.1 there exist numbers $\Gamma, \delta > 0$ such that

$$\Gamma > \Gamma_0$$

and that for each $t \in [0, \infty)$, each $u_1, u_2 \in R^n$ and each $x_1, x_2 \in K$ which satisfy (3.1) the inequality

$$|f_j(t, x_1, u_1) - f_j(t, x_2, u_2)| \leq \epsilon_1 \min\{f_j(t, x_1, u_1), f_j(t, x_2, u_2)\} \tag{3.9}$$

is true.

Assume that $t \in [0, \infty)$, $u_1, u_2 \in R^n$, $x_1, x_2 \in K$ satisfy (3.1). Then the inequality (3.9) follows from the definition of Γ, δ. (2.1), (3.4), (3.6) and (3.1) imply that

$$(|f(t, x_i, u_i)| + 1)(|f_j(t, x_i, u_i)| + 1)^{-1} \in [\lambda^{-1}, \lambda],\ i = 1, 2. \tag{3.10}$$

It follows from Assumption A(iii), (3.1), (3.7) and (3.9) that

$$\min\{f(t, x_i, u_i), f_j(t, x_i, u_i)\} \geq 2^{-1}\psi(\Gamma_0),\ i = 1, 2. \tag{3.11}$$

By (3.11) and (3.10),

$$f(t, x_i, u_i) f_j(t, x_i, u_i)^{-1}$$
$$\in [(\lambda(1 + 2\psi(\Gamma_0)^{-1}))^{-1}, \lambda(1 + 2\psi(\Gamma_0)^{-1})],\ i = 1, 2. \tag{3.12}$$

We may assume without loss of generality that

$$f(t, x_1, u_1) \geq f(t, x_2, u_2). \tag{3.13}$$

It follows from (3.12), (3.9), (3.8) and (3.7) that

$$f(t, x_1, u_1) - f(t, x_2, u_2) \leq \lambda(1 + 2\psi(\Gamma_0)^{-1}) f_j(t, x_1, u_1)$$

$$-(\lambda(1+2\psi(\Gamma_0)^{-1}))^{-1}f_j(t,x_2,u_2)$$
$$=\lambda(1+2\psi(\Gamma_0)^{-1})[f_j(t,x_1,u_1)-f_j(t,x_2,u_2)]$$
$$+f_j(t,x_2,u_2)[\lambda(1+2\psi(\Gamma_0)^{-1})-(\lambda(1+2\psi(\Gamma_0)^{-1}))^{-1}]$$
$$\leq\lambda(1+2\psi(\Gamma_0)^{-1})\epsilon_1 f_j(t,x_2,u_2)+f_j(t,x_2,u_2)[\lambda(1+2\psi(\Gamma_0)^{-1})$$
$$-(\lambda(1+2\psi(\Gamma_0)^{-1}))^{-1}]\leq\epsilon_1[\lambda(1+2\psi(\Gamma_0)^{-1})]^2 f(t,x_2,u_2)$$
$$+f(t,x_2,u_2)[\lambda^2(1+2\psi(\Gamma_0)^{-1})^2-1]\leq\epsilon f(t,x_2,u_2).$$

Therefore the function f satisfies Assumption A(iv). This completes the proof of the proposition.

The next auxiliary result will be used in order to establish the continuous dependence of the functional $U^f(T_1,T_2,y,z)$ of T_1,T_2,y,z and the continuous dependence of the functional $I^f(T_1,T_2,x)$ of f.

PROPOSITION 1.3.3 *Let $M_1>0$ and let $0<\tau_0<\tau_1$. Then there exists a number $M_2>0$ such that the following property holds:*

If $f\in\mathcal{M}$, numbers T_1,T_2 satisfy

$$0\leq T_1,\ T_2\in[T_1+\tau_0,T_1+\tau_1] \tag{3.14}$$

and if an a.c. function $x:[T_1,T_2]\to K$ satisfies

$$I^f(T_1,T_2,x)\leq M_1, \tag{3.15}$$

then

$$|x(t)|\leq M_2,\ t\in[T_1,T_2]. \tag{3.16}$$

Proof. By Assumption A(iii) and the properties of the function ψ there exists a number $c_0>0$ such that

$$f(t,x,u)\geq|u| \tag{3.17}$$

for each $f\in\mathcal{M}$ and each $(t,x,u)\in[0,\infty)\times K\times R^n$ satisfying $|u|\geq c_0$, and

$$f(t,x,u)\geq 2M_1(\min\{1,\tau_0\})^{-1} \tag{3.18}$$

for each $f\in\mathcal{M}$ and each $(t,x,u)\in[0,\infty)\times K\times R^n$ satisfying $|x|\geq c_0$. Fix a number

$$M_2>1+M_1+a\tau_1+c_0(1+\tau_1) \tag{3.19}$$

(recall a in Assumption A(iii)).

Let $f\in\mathcal{M}$, T_1,T_2 be numbers satisfying (3.14) and let $x:[T_1,T_2]\to K$ be an a.c. function satisfying (3.15). We will show that (3.16) holds.