Contemporary Mathematicians

F. Alberto Grünbaum
Pierre van Moerbeke
Victor H. Moll
Editors

Henry P. McKean Jr.

Selecta

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Contemporary Mathematicians

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To my teachers Will Feller, Kiyoshi Ito, Mark Kac, Norman Levinson and Gretchen Warren

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1 Personal Recollections

Henry P. McKean Jr.[1]

1.1. Norman Levinson

Thinking of Norman Levinson, I remember how much I learned from him as a very young and inexperienced person, and how much I found to admire, both in his mathematical work and in himself, as a man.

Looking back, his choice of mathematical questions seems memorable enough: mostly close to applications, rich in their details, suggestive of general phenomena, as in the wonderful papers on the forced vander Pol equation, etc. foreshadowing the current vogue of attractors, chaos, and all that.

What I could better appreciate then was his mastery of the kind of hard analysis such questions require, the kind in which every equality costs you two opposing inequalities. When we got stuck, working together, he'd always take an example, and he'd estimate things with an understanding and a speed that impressed me equally, and soon we'd be back on solid ground. It was excitingly easy, Norman doing all the hard work, as I understood later. *Gap and Density Theorems* (Chap. 8, vol. 2) and the extraordinary papers on Riemann's zeta function (Chap. 11, vol. 2) is where you can see this expertise at its best.

Other lessons I was not so ready to digest though happy to benefit from. I mean his unobtrusive, remarkably effective administrative style, the way he seemed to run the department with the back of his hand, and, on the personal side, his patience and gentle encouragement.

He was shy. Fagi said: "Norman, he's terrible. He never wants to go out. He's afraid he'll meet somebody he doesn't know." I was shy, too, but slowly we got to know each other a little, coming from as different backgrounds as you could imagine, and I thought myself lucky when he told me bits about his early life. He said: "We were very poor, but we didn't think of ourselves as poor." I take the liberty to transpose that and to say he was a rich man in his particular way, spreading about his riches quietly, with an open hand.

Henry McKean, New York, May 1997.

1.2. Will Feller

Will Feller was born in Zagreb, Yugoslavia on July 7, 1906, the ninth of 12 children of a well to-do family. They named him Willy in the then popular German style. This changed to William upon his coming here (1939), but everybody called him plain Will. His early studies called him from Zagreb (1923–1925) to Gottingen (1925–1928) where he got his degree in 1926, aged 20! Then to Kiel as Privatdozent (1928–1933). Nazi times: Will taught a class on the new ideas in probability of Kolmogorov, etc., attended, by chance, by a person of some importance in the SS. One day, this person and some

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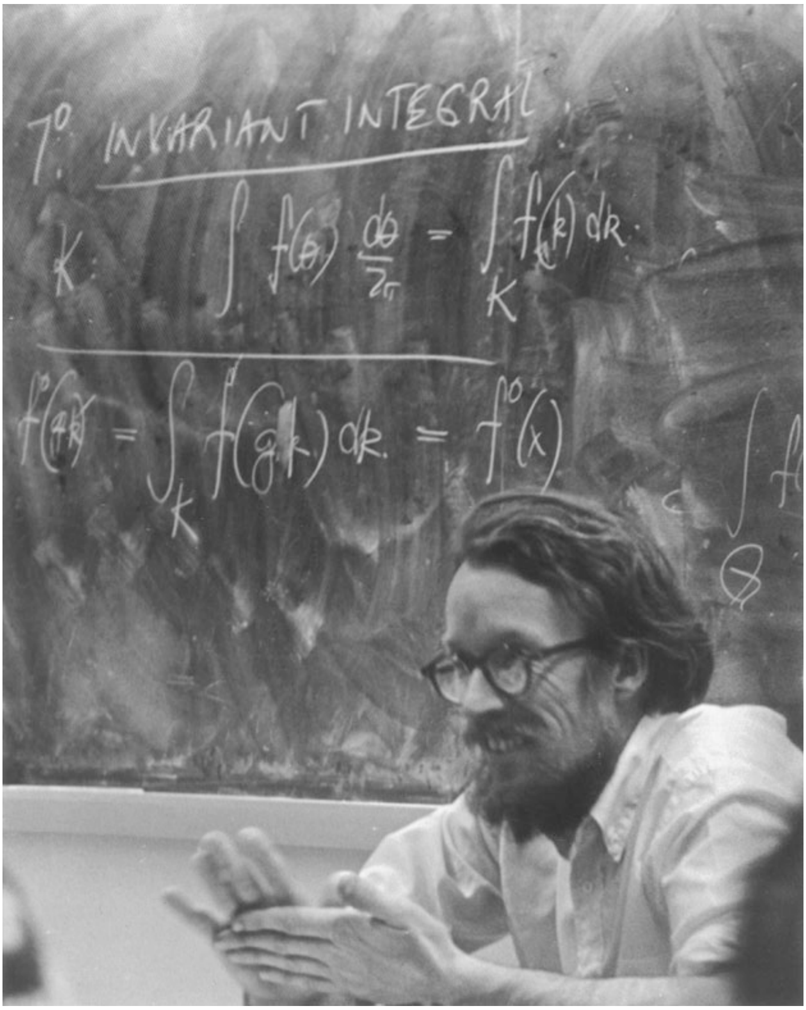

FIGURE 1.1. Henry P. McKean Jr.

two or three of his men present themselves at Will's apartment. Will lets them in in fear and trembling, whereupon the boss says how much he loves Will's lectures and if there is somebody Will would like them to beat up, just to say the word. A courtesy call I suppose. Will declined this civility and after a subsequent refusal to sign a Nazi oath, packed his bags for Copenhagen where he stayed a year (1933). Then to Stockholm (1934–1939), Providence (1939–1944), Ithaca (1945–1950), and Princeton (1950–) which is where I first knew him (1953). He died, with difficulty, in New York on January 14, 1970, aged 63.

I think it is fair to say that Kolmogorov, Paul Lévy, and Will made probability an honest woman. They are the people chiefly responsible for its rise from a not quite respectable rule of thumb to the ubiquitous, precise, intuitively appealing subject it is today. I think that, of the three, Will had the wider view. He understood Kolmogorov's mostly analytical way and also Paul Lévy's way with sample paths, and was a master of both, as can best be seen in his splendid book, *An Introduction to Probability Theory* (John Wiley & Sons, 1950) and its subsequent amplifications and revisions (1957/1968 and 1966/1971). Here you can see him endlessly

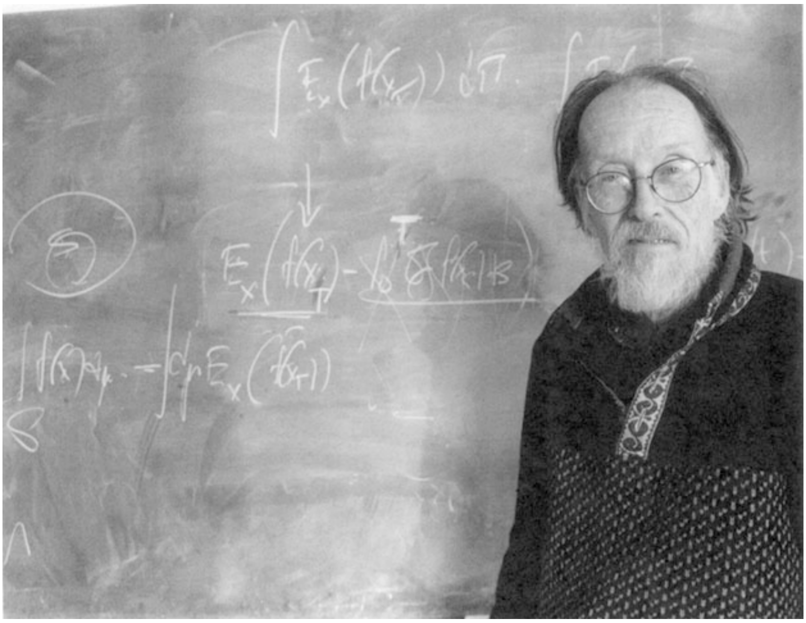

FIGURE 1.2. Henry P. McKean Jr.

perfecting proofs and pictures. I point, for example, to his simple way with supposedly hard Tauberian theorems and to his so appealing presentation of Sparre-Andersen's combinatorial way with random walks. This is *the* book for beginners. It is so full of interesting things, both mathematical and practical, looked at from an astonishing number of aspects, and full of Will's own self: his enthusiasm, his high standards, his indefatigable desire to make you understand "what's really going on".

That was also his watch-word when he lectured. He would get quite excited, his audience in his hand, and come (almost) to the point. Then the hour would be over, and he would promise to tell us "what's really going on" next time. Only next time the subject would be not quite the same, and so a whole train of things was left hanging, somewhat in the manner of Tristram Shandy. But it didn't matter. We loved it and couldn't wait for the next (aborted) revelation.

I was too young to appreciate my luck in coming so accidentally into his orbit, but soon realized that I had not only a teacher but a friend whose generosity and sense of the ridiculous in the gloomy moments of the young could be counted on for sure. I remember a note he sent me at such a moment in which he said: "You were made for success and the Lord God himself is kicking Himself that you do not understand his good intentions". (Will's capitals). That's how he was with me and many others, too: good company, a raiser of spirits, smart, kind.

I learned so much from him as a man, and also in mathematics, from him and others of my elders and betters: D. Ray, G. Hunt, and K. Itô.

A bit about Will's one-dimensional "diffusions" which he was in the middle of when we were together (1953/1957). It exemplifies his simultaneous desire for generality and simplicity, taking what people thought to be quite complicated and making it obvious, as all good mathematics should be. In short, he reduced the general diffusion to the simplest one (Brownian motion) by: (1) a change of "scale" to make the motion appear unbiased, and (2) a change of "clock" or "speed" to make the (local, Gaussian) fluctuations the same at any place. This is easy enough in simple cases with stringent technical conditions imposed. But what Will realized is that, effectively, no technical conditions are needed at all; in the "natural" language, the technicalities evaporate and simple, perfectly general expressions for objects of interest are found. Here is an example. Let $\mathfrak{G} \equiv \frac{1}{2}e^2(x)\partial^2/\partial x^2 + f(x)\partial/\partial x$ be the infinitesimal operator of a one-dimensional diffusion $x(t) : t \geq 0$. Here, f specifies the "drift" or "bias" and e specifies the "fluctuations". Introduce the new "scale"

$$\int_0^x dy \exp\left[-\int_0^y (2f/e^2)\right]$$

and the "speed measure"

$$dm(x) \equiv 2\frac{dx}{e^2(x)} \exp\left[+\int_0^x (2f/e^2)\right],$$

in terms of which $\mathfrak{G} = (d/dm)(d/dx)$. Take $a < x < b$, start the diffusion at $x(0) = x$ and let T be the "exit time" $\min(t : x(t) = a \text{ or } b)$. Then in the new scale,

$$P\big[x(T) = a\big] = \frac{b-x}{b-a},\ P\big[x(T) = b\big] = \frac{x-a}{b-a},$$

$$\text{and } E(T) = \int_a^b G(x,y)dm(y)$$

with the (symmetric) Green's function $G(x,y) = (x-a)(b-y)/(b-a)$ for $x \leq y$. What could be simpler?

Back to Will himself. He was short; compact, with a mop of wooly gray hair; irrepressible. In conversation quick, always ready with an opinion (or two), addicted to exaggeration. If you knew the code, you applied the "Feller factor" (discount by 90 %). If you didn't, it could be awkward, as with the immigration official at Providence when they came to the question: "Do you advocate bigamy?" Will delivered a lengthy opinion on the distinction between practice and advocacy which, he said, must surely obtain in this great free land about to be his own. The official was not amused. So he could seem opinionated, even rude if you didn't know him. But the real Will took a wide view, meeting life with enthusiasm and good cheer. I think of him often, hearing his voice, remembering him so full of fun.

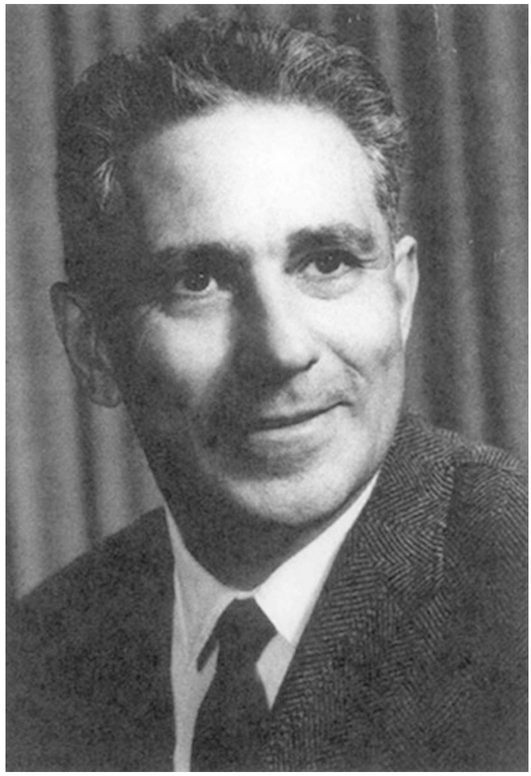

FIGURE 1.3. N. Levinson

Brooklyn,
April 15, 2005

1.3. Kiyoshi Itô: recollections of Kyôto 1957/1958

The following recollections formed a little talk I spoke in Kyôto on the occasion of K. Itô's 88th birthday. They bring the memory of the happy times we had together and congratulations on his Gauss Prize. The Gauss Committee will not find it easy to keep to the standard they have set.

Kyôto 1957/1958 It's a pleasure to think back for a little while to the happy days my family and I spent in Kyôto in 1957/1958.

I met Itô-*sensei* in Princeton 1954 where Kosaku Yosida also came for a shorter time. Dan Ray was there also Hunt & Trotter, and of course Feller who was the activator, the regisseur of it all. Feller had just formulated his ideas on diffusion and I was helping (feebly to be sure) to bring them together into a little

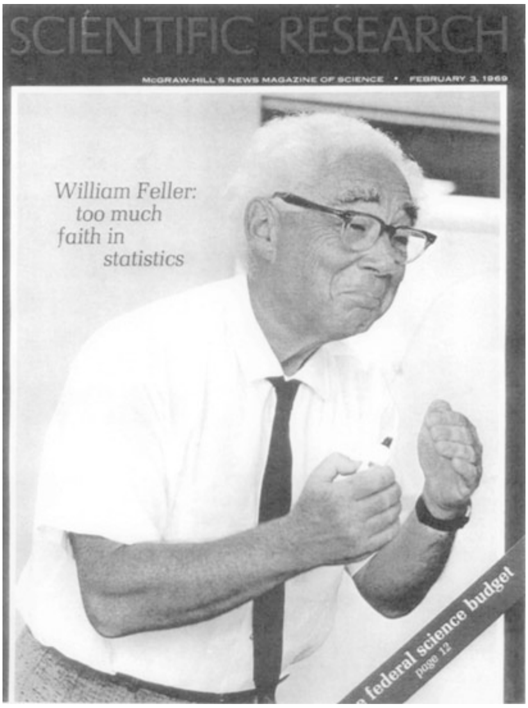

FIGURE 1.4. Will Feller

FIGURE 1.5. K. Ito
Picture by Konrad Jacobs. With friendly permission from the Archives of the Mathematisches Forschungsinstitut Oberwolfach

book. At the same time we were just hearing about Dynkin's ideas on stopping times so we had a seminar to digest these things. It was lucky for me, being so young and largely uneducated, to find myself at a moment when the understanding of diffusion was taking a big jump. I think I never worked so hard (without fatigue) or learned so much so fast (without tears) as I did then, and from two such patient kindly teachers as Feller and Itô.

Then Itô and I began to combine all that with ideas of Paul Lévy, especially his "measure du voisinage" or "local time" as we called it in English, and we understood quite quickly how this local time is a sort of 1/2-dimensional measure on the zeros of the Brownian path, how it could be used to implement the elastic Brownian motion, and so on. I say "we" understood. I should say Itô understood, and since he was patient and I was pretty quick though ignorant, pretty soon I understood, too. But invariably, at that happy moment when you say to yourself "I see", it was Itô who saw the whole. It was my first real taste of how mathematics is done and I cannot think of it now without excitement and gratitude. So far Princeton.

I stayed on another year or two and then to Japan (1957/1958) where we continued to work on our book. Never ever write a book before everything is proved and never ever let the junior partner write it! It would have been a better book had it been written half as many times at twice the length. But never mind: Regrets are not very interesting, what's more interesting is that year 1957/1958 in Kyôto.

It was a long trip by a little Japanese freighter from Los Angeles, up past the Aleutians where it was cold and rough with a sick child of 2 years, but after a couple of weeks we arrived at Yokohama to be met by Itô and

Itô-*san no okusan* and Junko, and after that all was well, though naturally new and more than a little foreign.

I might tell you that I already knew a little about Japan as my great-grandfather in Boston was an enthusiastic collector of Japanese art. That was in the days of Fenellosa who first made such things known in America and who was a friend of that grandfather, so the chance to see some of the treasures of that marvelous art with my own eyes and to do mathematics with Itô was a combination not to be resisted. Now back to Kyôto.

Itô had rented us a spacious house right on Takanogawa in the Shimogamo district, just a short walk from the university. I still see in my mind's eye the lovely printed cloths washing in the current of the river and the moon coming up from behind the Hieisan, and I hear in my mind's ear the call of the noodle-seller at night. Now the cloths and the noodle-seller are gone but the river is sparkling clean, which it was not then. I went to the university every day walking, and back at night. I had a vast office with a couch and many chairs and tables and a huge glass-topped desk, and a coal stove for the winter which smoked horribly when the wind was wrong and the kind old librarian (whose name I regret to have forgotten) would try to make it work. I had a sink, too, to wash my hands, and a big safe—was it to keep my theorems in? I never figured out. And after the morning, we had lunch all seated about a long table, or else if the weather was fine, we went to a little out door restaurant a few steps away and had some kind of *domburi* and a beer.

Once a week, we had a seminar with so many quick and eager people, then very young like me, now not so young like me: Nobuyuki Ikeda was there and Hiroshi Tanaka, and Shinzo Watanabe and Takesi Watanabe, Minoru Motoo, Tunekiti Sirao, Tadashi Ueno, Takeyuki Hida, Makiko Nisio. I must have forgotten some names. If so please forgive me. Itô said that we were "sowing the seeds of diffusion in the mathematical fields of Japan". It was all very welcoming and exciting and I loved it.

To come back for a moment to our joint work and the way Itô taught me. I remember we were flying to some place (Fukuoka I think) and trying to understand Feller's most general boundary conditions for Brownian motion on a half-line. Itô sat beside me drawing pictures of sample paths. These did not please him until he got to the right one (it didn't take him very long) and as soon as he showed it to me I understood perfectly, but lacking his experience and deep feeling could not have thought of it myself. Some things we missed entirely like the deep facts about the spatial dependence of stopped local time. We had prototype formulas in front of us, all perfectly explicit, and never imagined what they meant: That the stopped local time was itself a diffusion in its spatial parameter. Oh well.

Leaving mathematics, I remember a continual attentive kindness from Itô and Itô-*san no okusan*. I remember happy suppers at their house when Itô, knowing next to nothing about cooking, would explain what his wife had placed before us by saying: "You take it and you put it and then it's very good." I remember also excursions near and far: To Shugakuin, Hieisan, Kokedera, Nara, and so on, with Itô always on the lookout that we should be comfortable and at ease. Once at the zoo, when the little Japanese children were staring at my odd-looking American children, Itô said to them sharply: "You are here to look at the animals!" and they did that.

So now you can see what a load of obligation (of *on* or *giri* if you like) I must carry, but that is not Itô's way. Itô's father was a very traditional, correct man who kept a record of every kindness done to himself and to his family over the years. When he died, Itô discharged each recorded debt meticulously, as his father would have wished him to do. After that, he went his own way, marrying his dear Shizue by inclination and for love, giving always freely: to myself, to many of you in this room, and I must suppose to many many others unknown to me.

Arigato gozaimashita, arigato gozaimasu.

FIGURE 1.6. M. Kac
Picture by Konrad Jacobs. With friendly permission from the Archives of the Mathematisches Forschungsinstitut Oberwolfach

1.4. Mark Kac, August 16, 1914–October 25, 1984

Poland. Mark Kac was born "to the sound of the guns of August on the 16th day of that month, 1914," in the town of Krzemieniec—then in Russia, later in Poland, now in the Soviet Ukraine (1985, 1, p. 6). In this connection Kac liked to quote Hugo Steinhaus, who, when asked if he had crossed the border replied, "No, but the border crossed me."

In the early days of the century Krzemieniec was a predominantly Jewish town surrounded by a Polish society generally hostile to Jews. Kac's mother's family had been merchants in the town for three centuries or more. His father was a highly educated person of Galician background, a teacher by profession, holding degrees in philosophy from Leipzig, and in history and philosophy from Moscow.

As a boy Kac was educated at home and at the Lycée of Krzemieniec, a well-known Polish school of the day. At home he studied geometry with his father and discovered a new derivation of Cardano's formula for the solution of the cubic—a first bite of the mathematical bug that cost Kac *père* five Polish zlotys in prize money. At school, he obtained a splendid general education in science, literature, and history. He was grateful to his early teachers to the end of his life.

In 1931 when he was 17, he entered the John Casimir University of Lwów, where he obtained the degrees M. Phil. in 1935 and Ph.D. in 1937.

This was a period of awakening in Polish science. Marian Smoluchowski had spurred a new interest in physics, and mathematics was developing rapidly: in Warsaw, under Waclaw Sierpinski, and in Lwów, under Hugo Steinhaus. In his autobiography (1985, 1, p. 29), Kac called this renaissance "wonderful." Most wonderful for him was the chance to study with Steinhaus, a mathematician of perfect taste, wide culture, and wit; his adored teacher who became his true friend and introduced him to the then undigested subject of probability. Kac would devote most of his scientific life to this field and to its cousin, statistical mechanics, beginning with a series of papers prepared jointly with Steinhaus on statistical independence (1936, l–4; and 1937, l–2).

Kac's student days saw Hitler's rise and consolidation of power, and he began to think of quitting Poland. In 1938 the opportunity presented itself in the form of a Polish fellowship to Johns Hopkins in Baltimore. Kac was 24. He left behind his whole family, most of whom perished in Krzemieniec in the mass executions of 1942–1943. Years later he returned, not to Krzemieniec but to nearby Kiev. I remember him rapt, sniffing about him and saying he had not smelled such autumn air since he was a boy. On this trip he met with a surviving female cousin who asked him, at parting, "Would you like to know how it was in Krzemieniec?" then added, "No. It is better if you don't know".

These cruel memories and their attendant regrets surely stood behind Kac's devotion to the plight of Soviet *refusniks* and others in like distress. His own life adds poignancy to his selection of the following quote from his father's hero, Solomon Maimon: "In search of truth I left my people, my country and my family. It is not therefore to be assumed that I shall forsake the truth for any lesser motives" ([1], p. 9).

America. Kac came to Baltimore in 1938 and wrote of his reaction to his new-found land:

> "I find it difficult... to convey the feeling of decompression, of freedom, of being caught in the sweep of unimagined and unimaginable grandeur. It was life on a different scale with more of everything—more air to breathe, more things to see, more people to know. The friendliness and warmth from all sides, the ease and naturalness of social contacts. The contrast to Poland... defied description."

After spending 1938–1939 in Baltimore, Kac moved to Ithaca, where he would remain until 1961. Cornell was at that time a fine place for probability: Kai-Lai Chung, Feller, Hunt, and occasionally the peripatetic Paul Erdös formed, with Kac, a talented and productive group. His mathematics bloomed there. He also courted and married Katherine Mayberry, shortly finding himself the father of a family. So began, as he said, the healing of the past.

From 1943 to 1947 Kac was associated off and on with the Radiation Lab at MIT, where he met and began to collaborate with George Uhlenbeck. This was an important event for him. It reawakened his interest in statistical mechanics and was a decisive factor in his moving to be with Uhlenbeck at The Rockefeller University in 1962. There Detlev Bronk, with his inimitable enthusiasm, was trying to build up a small, top-flight school. While this ideal was not fully realized either then or afterwards, it afforded Kac the opportunity to immerse himself in the statistical mechanics of phase transitions in the company of Ted Berlin and Uhlenbeck, among others. Retiring in 1981, Kac moved to the University of Southern California, where he stayed until his death on October 25, 1984, at the age of seventy.

I am sure I speak for all of Kac's friends when I remember him for his wit, his personal kindness, and his scientific style. One summer when I was quite young and at loose ends, I went to MIT to study mathematics, not really knowing what that was. I had the luck to have as my instructor one M. Kac and was enchanted not only by the content of the lectures but by the person of the lecturer. I had never seen mathematics like that nor anybody who could impart such (to me) difficult material with so much charm.

As I understood more fully later, his attitude toward the subject was in itself special. Kac was fond of Poincaré's distinction between God-given and man-made problems. He was particularly skillful at pruning away superfluous details from problems he considered to be of the first kind, leaving the question in its simplest interesting form. He mistrusted as insufficiently digested anything that required fancy technical machinery—to the extent that he would sometimes insist on clumsy but elementary methods. I used to kid him that he had made a career of noting with mock surprise that $e^x = 1 + x + x^2/2 +$ etc. when the whole thing could have been done without expanding anything. But he did wonders with these sometimes awkward tools. Indeed, he loved computation (*Desperationmatemtik* included) and was a prodigious, if secret, calculator all his life.

I cannot close this section without a Kac story to illustrate his wit and kindness. Such stories are innumerable, but I reproduce here a favorite Kac himself recorded in his autobiography:

> "The candidate [at an oral examination] was not terribly good—in mathematics at least. After he had failed a couple of questions, I asked him a really simple one... to describe

> the behavior of the function $1/z$ in the complex plane. ‘The function is analytic, sir, except at $z = 0$, where it has a singularity,’ he answered, and it was perfectly correct. ‘What is the singularity called?’ I continued. The student stopped in his tracks. ‘look at me,’ I said. ‘What am I?’ His face lit up. ‘A simple Pole, sir,’ which was the correct answer.”

[1] *Enigmas of Chance.* (Autobiography). New York: Harper and Row.

1.5. Gretchen Warren

I add to these recollections another, from my boyhood. The first time I met Gretchen Warren I must have been 10 or 12. It was her custom then to spend the summer at the house of my cousin Eleo (much older than me), looking out over the New England sea going all the way to Portugal, which I imagined I could perceive faintly, far away. Two more different women can hardly be imagined: Eleo the complete sports-woman, devoted to swimming, tennis, 100-mile walks, horses, gossip, handsome men and women. Grechen intellectual, learned in literature, philosophy and myth, especially the old things of East and West: the Icelandia Sagas, Chanson de Roland, Homer, Villon, the Bhagavad-Gita, Plotinus, the Bible, in no particular order; loving music and also Natural History; a friend of Santayana, Henry George, and A.K. Coomaraswami; a mixture you might say, of Emerson and Agassiz. How these two became (and stayed) friends I do not know, but they did.

Gretchen was of another generation, maybe 60 then or more, but she spoke to me as an equal in a way I have never forgotten. She took my childish love of Natural History with perfect seriousness, sharing her books and her marvelous collection of shells: the violet snail, thinner than paper, making a raft of foam to carry her eggs, far out in the uttermost parts of the ocean; Cuban land snails with their wonderful variegated colors; things I still keep, making me think of her. She introduced me to other things she loved like Homer and Plotinus, not then but later on. She encouraged me to think that I might do something of my own, in science perhaps, or writing. And nothing heavy here, only that serious attention to a little child which was her great charm and kind gift to me. She was a beautiful woman. You may have seen her at the Fine Arts in Boston in Sargent’s painting: Mrs. Warren and her Daughters. We met less often as time went by. I saw her last in Boston, Beacon Hill, in 1949 or so. Then she died, leaving these memories.

1.6. Acknowledgements

It is my pleasant duty to thank F. Alberto Grünbaum, Pierre van Moerbeke, and Victor H. Moll for putting all this together. It has been a long job, not without my grateful acknowledgement which I offer here. Thanks also to David Williams and Hermann Flaschka for kind words and true understanding of what I have tried imperfectly to do.

My Debt to Henry P. McKean Jr. 2

David Williams[1]

I was very privileged to have had as research supervisors, David Kendall and Harry Reuter. I learnt a great deal from them and from Eugene Dynkin, André Meyer, and of course, Paul Lévy. But it has been to Henry McKean that I have most often turned for inspiration.

There is a simple reason for this. If anyone else writes a book on Stochastic Integrals, Fourier Series and Integrals, or Elliptic Curves, they might produce a fine book on the topic. But Henry (either alone, or with Harry Dym, or with Victor H. Moll) writes *Mathematics*, not mere exposition of a topic. One is left awestruck by the rich interconnectedness of the subject as evidenced by a dazzling array of examples and (usually very challenging) exercises. What a great antidote to the too prevalent 'elegant abstraction' culture in which (for example) Number Theory is OK provided one keeps away from those common-or-garden numbers, and in which even the Generalized Riemann Hypothesis, astoundingly deep though it is, is perhaps rather closer to the ground than one should be flying.

A few years ago, I had to have a brain tumour removed in something of an emergency. It was explained to me that the operation might seriously impede my ability to understand Mathematics. (I had great surgeons, and I don't think it has!) How glad I was that I had McKean and Moll [14] with me for what might have been my last few hours of Mathematics! Yes, there are a few slips in the book, but these are fussed over only by those who could never write anything a tenth as inspirational.

I started my research career on Markov-chain theory, and soon became haunted by the then recent paper by William Feller and McKean [4]. This showed that there exists a chain with all states instantaneous, counter to what Lévy had thought, though it was he who then gave the beautiful probabilistic construction of the F-M chain. From the viewpoint of the time, the F-M chain was even more amazing because all its off-diagonal jump rates are zero. I became rather obsessed by the Q-matrix problem of characterizing what could be the off-diagonal jump rates of a totally instantaneous chain.

When I realized that I could then make no progress with this problem, I decided to switch fields and to read the great Itô-McKean book on diffusion processes. Again I had to work very hard to do the exercises. I felt that Itô and McKean had calculated everything there is to calculate about Brownian motion. (This was in the days before Marc Yor and coworkers had found, and solved, lots more explicit problems.)

When it came to the famous Section 2.8 of Itô-McKean on local time, I despaired of

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having a full understanding. I therefore tried hard to decompose the problem into simpler ones. Henry included a nice exposition of my efforts as part of his paper [13].

What I had never expected was that thinking hard about Brownian local time would lead me to solve that Q-matrix problem. (I also made heavy use of ideas of Kendall, Reuter, Jacques Neveu and Lèvy.) Things really are interconnected!

For many years, I had been intrigued by McKean's paper [9] on a winding problem driven by white noise. In this, he looks at windings around the origin of the two-dimensional process with Brownian motion as one component and its integral up to current time as the other. The winding process is not at all easy to analyze. However, in this case, the joint process is Gaussian, and this allows one to describe a key distribution by an integral equation. In a typical *tour de force* of transform theory, McKean obtained the explicit solution as an unfamiliar distribution, and derived striking probabilistic consequences. This was the first paper on what came to be known as Markovian Wiener-Hopf theory.

I became interested in more general Wiener-Hopf winding problems in which one component of the two-dimensional process is a Markov process, and the other a fluctuating additive functional of that process. One of the simplest problems required calculation of a jump distribution from 0 of an induced Brownian motion on a half-line of the type studied by Feller and, more fully, by Ito and McKean [5]. When Henry visited Swansea, I told him that I conjectured that the jump distribution in the W-H context must be totally monotone. A day later (I recall Michael Atiyah's saying to me that Henry thinks at a million miles per hour!), Henry told me that he had proved both this conjecture and, by using Krein theory as in Dym and McKean, that all totally monotone jump laws arise this way. See London, McKean, Rogers and Williams [6].

My most recent paper (`http://arxiv.org/abs/1011.6513`), which I hope to be an amusing *divertissement*, is on the simplest non-linear version of Markovian W-H problems. Its W-H aspects can all be traced back to McKean on windings.

The paper's non-linear aspects can be traced back to McKean's paper on travelling waves and the KPP equation [12] which prompted massive developments on branching diffusions, measure-valued processes, and the like. How often he has sparked off new fields of investigation! He was way ahead of the field even in financial mathematics [10].

McKean has also been interested in winding problems of non-W-H type: in particular, topological problems on windings of the Brownian path. See, in particular, the papers [7, 15] by McKean with Lyons and with Sullivan, a formidable trio indeed, and doing Mathematics, not mere Probability.

His expertise in Gaussian processes was also used to great effect in his paper [8] on Lèvy's Brownian motions in multi-dimensional (and even Hilbert-space) time. There is profound work here on splitting fields, etc., and there are really surprising results. See also papers [2, 3] with Dym.

In 1980, I organized a conference at Durham in which the then-brand-new Malliavin calculus played a large part. I decided to write an introduction to the proceedings, and found McKean's paper [11] on the geometry of differential space invaluable for this. The fundamental Malliavin process is Henry's Brownian motion on an infinite-dimensional sphere of radius the square-root of infinity. (Surreal?!)

Though perhaps primarily interested in seeing principles put to good use in the concrete, he is a master of the abstract too. His paper [1] with Blumenthal and Getoor, proves that two Markov processes with identical hitting distributions are time-changes of each other. No result in Markov-process theory is deeper than this.

The above is just a hint as to how Henry's work has enriched the life of one probabilist. I have concentrated on books and papers which, as it were, have become part of me:

ones on which I do not need to refresh my memory. Other, better, probabilists could say much more.

I know that Henry's work is regarded with the same admiration and gratitude by people working in differential equations and in other fields.

If there is an explicit solution to be found, then Henry is the man to find it. But his almost unique skill at calculation is always combined with deep new insights into the underlying principles.

My sincere thanks, and my very best wishes, Henry!

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Henry P. McKean Jr. and Integrable Systems

3

Hermann Flaschka[1]

Henry McKean's first contribution to integrable systems appeared in 1975 [15]. Over a period of more than 30 years since, in some 50 papers, he has explored integrable systems from uniquely original points of view. His selecta could have included the pioneering work, with Airault and Moser, on the time-dynamics of poles of meromorphic solutions of KdV [2]; or the series on invariant measures for wave equations, integrable or otherwise; or one of the seminal papers with Trubowitz [12, 13] that created a theory of infinite-genus hyperelliptic curves (= Riemann surfaces) and infinite-dimensional Jacobian varieties, which they applied to KdV under periodic boundary conditions, or subsequent extensions of that theory motivated by the desire to understand all the iconic integrable partial differential equation on the circle.

There is another project, of great scope, in which "Geometry of KdV (1)" [7], in this volume, is the first step. Read in isolation, without appreciation of the developments in papers [8] and [10] at least, this paper is incomplete. It sets out the vocabulary for the striking results to follow. I thought that Henry's invention of unimodular spectral classes and additive classes would serve as exemplar of his imagination and technical virtuosity (and fearlessness, one might add). Moreover, there are many questions to be thought about; some are natural, though probably hard, but the most difficult task is to think of workable examples that will reveal something new.

The papers [7, 8, 10] are themselves building blocks of a sweeping conjecture. Henry proposes to interpret the spectral theory of differential operators[2] $Q = -D^2 + q(x)$ as a reflection of infinite-dimensional algebraic geometry in the space of all such operators. The space is stratified into classes labeled by spectral data of some sort, and parametrized by an additive group; the classes are Jacobian varieties of objects resembling, somehow, quadratic algebraic curves with perhaps a continuum of branch cuts or singular points; and the coefficients $q(x)$ and properly normalized eigenfunctions $\mathfrak{e}(x, \lambda)$ on each class are represented by an object resembling, somehow, a Riemann theta function, and these representations, quoting from [10], "may be viewed as uniformizing, class by class, the eigenvalue problem $Q\mathfrak{e} = \lambda\mathfrak{e}$. It seems that such a geometrical attitude would be new to spectral theory".

Substantial evidence suggests that yes, "something must be going on". Supporting examples draw on quantum scattering, analytic functions, algebraic curves, and the geometry of infinite-dimensional manifolds, all intertwined and placed into a framework of the infinite-dimensional dynamical systems

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[2]D stands for the operator d/dx, and dot and prime will indicate time and space derivatives.

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sometimes called "integrable". This name is often used more as a suggestion than a definition, and equally often merely indicates the origin of a piece of mathematics that has little to say about dynamical systems. The historical context, however, is still relevant to the story.

The modern theory of integrable systems was born in 1965, when Gardner, Greene, Kruskal, and Miura [4] invented a nonlinear version of the Fourier transform, based on the inverse scattering method from quantum mechanics, in order to solve the KdV equation. This is the partial differential equation $\dot{q} = -q''' + 6qq'$; it governs a certain asymptotic regime of the nonlinear motion of waves on water, or in a plasma, or in an atomic lattice. A few years later, Zakharov and Faddeev [16] interpreted the inverse scattering solution of KdV as a transformation of an infinite-dimensional Hamiltonian system to action-angle coordinates, thereby translating a successful, but unmotivated and mysterious, technique into the more familiar language of classical mechanics. As the supply of integrable Hamiltonian ordinary and partial differential equations grew, so did the scope of their applications, in mathematics and physics, and the breadth of techniques brought to bear on the analysis of their dynamics. In the early years, it was not terribly misleading to refer to this corner of mathematical physics as "the inverse scattering method", or "soliton theory", or "KdV theory". However, much as Fourier series, invented for the purpose of solving the heat equation, evolved into the all-encompassing paradigm called harmonic analysis, so has the invention of the inverse scattering method for the solution of KdV transformed mathematics through the creation of a new paradigm; one that has no official name, but "integrable systems" is as good a label as any, since, like "harmonic", it reminds one of the origin of the ideas that have emerged from the first examples.

I would have liked to spend time on theta functions and their incarnation in scattering theory [3] and in the quantum harmonic oscillator [14], since algebraic geometry is an integral part of the broad picture, but my description of the background and implications of [7] is of substantial length already. In any case, I could not improve on Henry's exposition, in [9], of the evidence for a theory of Riemann surfaces of countably and uncountably infinite genus.

I start with a summary of that part of KdV and the inverse scattering method that is needed for an appreciation of Henry's ideas, and only that part. Maybe it is of use to a non-expert. KdV (1)–(3) enter in Section 3, where I explain the content of [7]. Section (4) deals with a remarkable result, from [8], about the class of operators known as "finite-gap" operators. It says that if the orbit of a certain group of transformations acting on the space of all $-D^2 + q$ is finite dimensional, then q is an Abelian function.

3.1. KdV Manifolds in the Scattering Class

The inverse scattering method was created to solve KdV with localized initial conditions, and this is also the starting point here; the coefficient q is taken to be in the space $C^\infty_\downarrow$ of rapidly decreasing smooth functions. The spectrum of $-D^2 + q$ then consists of a continuous part, $0 \leq \lambda < \infty$, and possibly a finite number of negative eigenvalues. These are of great importance in applications of KdV, but complicate the analysis. Therefore it is assumed throughout that there is no discrete spectrum.[3] The collection $\mathcal{S}$ of such Q is called the *scattering class*. The KdV equation defines a flow on $\mathcal{S}$.

The first order of business is to introduce an equivalence relation that stratifies $\mathcal{S}$ into infinite-dimensional tori[4] invariant under the

[3] This assumption leads to incorrect conclusions in the Hamiltonian approach to KdV, but that difficulty is ignored.

[4] It would be (usually) possible to give precise characterizations, in terms of analyticity, growth, smoothness, etc., of the function classes introduced below. Statements about geometry, for instance, that something is "stratified" or a "manifold" or a "torus", are more problematical; the intuition is very important, but there is often no proof or even precise formulation. I do not try to separate fact from useful fiction.

KdV flow. These are called *KdV manifolds*. They are labeled by a kind of integral transform $Z(k)$ of $q(x)$ constructed from the generalized eigenfunctions of Q. The functions Z are known as *action variables* in Hamiltonian mechanics.

KdV is in fact only one of an infinite family of commuting vector fields on $\mathcal{S}$ that are tangent to the KdV manifolds and span the tangent spaces. The next step will be to understand their flows; they can be interpreted as action of a huge additive group of operators, the so-called group $\mathfrak{A}$ of *additions*. It generates the KdV manifold from a reference potential. The coordinates on the tori introduced by $\mathfrak{A}$ are (related to) the *angle variables* of Hamiltonian mechanics.

Finally, one needs an algorithm to pass from the torus coordinates Z and $A \in \mathfrak{A}$ back to q. This is implemented by an operator determinant indexed by the torus label Z. Schematically, $q(x) = \det(A(x); Z)$, where $A(x)$ is a 1-parameter subgroup of $\mathfrak{A}$.

This picture is a rewording of the inverse scattering solution of KdV, formulated with an eye towards generalizations to q that are not of scattering class. The geometric interpretation is enriched by incorporation of ideas from the theory of algebraic curves. That comes later.

3.1.1. Model: A Geometric Picture of Linear KdV. The KdV manifolds are infinite-dimensional, except for the operator $Q_0 := -D^2$ which is a fixed point–the only one in $\mathcal{S}$–of KdV. One can linearize KdV, and in fact the whole (yet to be explained) construction at Q_0; the tangent space $T_{Q_0}\mathcal{S}$ is naturally identified with $C^\infty_\downarrow$, and the linearized KdV manifolds are infinite-dimensional tori indexed by the modulus of the Fourier transform. This familiar setting affords a convenient starting point. It is meant to introduce the idea of stratification by tori, their labeling, and coordinates on them.

Consider the KdV equation linearized at $q = 0$,

(LKdV)
$$\dot q = M_0 q := -D^3 q, \quad q(x,0) \text{ given}, \ x \in \mathbb{R}.$$

It is diagonalized by the Fourier transform, which we define by

$$\hat q(k) = \int q(x) \mathrm{e}^{-2ikx}\, dx. \tag{3.1.1}$$

The Fourier transformed LKdV equation,

$$\dot{\hat q}(k,t) = 8ik^3 \hat q(k,t), \tag{3.1.2}$$

is solved by inversion. Evidently, $|\hat q(k,t)|^2$ is independent of time t, signifying that the energy in the k-th Fourier mode is conserved by LKdV. More generally, every equation of the form

$$\dot q = D\Omega(-D^2)q, \text{ with transform}$$
$$\dot{\hat q} = 2ik\Omega(4k^2)\hat q, \tag{3.1.3}$$

preserves modal energy; here Ω denotes a real function of slow growth. The PDEs

$$\dot q = D(-D^2)^m q, \ m = 1, 2, 3, \dots,$$

belong to this class. They are the linearizations, about $q = 0$, of the equations in the so-called *KdV hierarchy*, which will be introduced shortly.

Equations (3.1.3) define commuting vector fields $\mathbb{Y}_\Omega$ on $C^\infty_\downarrow$. They are tangent to the subsets of $C^\infty_\downarrow$ characterized by common modal energies. Following [3], the sets of q with Fourier transforms $\hat q = |\hat q_0| \mathrm{e}^{i\psi}$ whose modulus $|\hat q_0|$ is fixed and phase $\psi(k)$ is a slowly increasing odd function are called *LKdV manifolds* and denoted by $\mathbb{J}(|\hat q_0|)$. An LKdV manifold is shaped like an infinite-dimensional torus. It is a product of continuum many circles, one for each k; the radii are labeled by $|\hat q_0(k)|$, and the circumferences are coordinatized by $\exp(i\psi(k))$.

Even though LKdV appears initially to be the central object, the goal, were we to continue in the linear approximation, would really be to understand the totality of vector fields $\mathbb{Y}_\Omega$, their interaction with the LKdV manifolds, and the stratification of $C^\infty_\downarrow$ into tori. We now begin this program in the nonlinear setting.

The review of scattering theory contains nothing that is not known to experts. It is in the nature of an appendix, but placed where it should logically appear.

3.1.2. Scattering Theory: A Nonlinear Fourier Transform. The KdV equation is a nonlinear modification of LKdV:

(KdV)
$$\dot{q} = -q''' + 6qq' = (M_0 + 2(qD + Dq))\, q := Mq.$$

It, too, can be diagonalized, in a basis formed by eigenfunctions Φ, not quite of M but of the pseudodifferential operator $\mathcal{L} := D^{-1}M$ and its adjoint. There is a Φ-transform of q, denoted by $R(k)$, whose time dependence is governed by the same decoupled system, (3.1.2), as the linearized equation. However, because M depends on q, the expansion basis $\{\Phi\}$ will also depend on q, and to invert this Φ-transform one must reconstruct both q and $\{\Phi\}$ from R.

The KdV miracles happen because M is a very special object. In differential Galois theory, it is known as the *2nd symmetric power* of the Schrödinger operator $Q := -D^2 + q(x)$. The name signifies that products $\Phi = f^2, fg, g^2$ of two solutions f, g of $Qy = k^2 y$ satisfy the generalized eigenvalue equation

$$M\Phi = 4k^2\Phi', \tag{3.1.4}$$

Therefore, to understand the Fourier-like expansions in the squared eigenfunctions Φ, one must first study the solutions of $Qy = k^2 y$.

This reasoning "explains" why the Schrödinger operator Q is so fundamental in KdV theory.[5]

3.1.2.1. *Scattering Matrix.* Since $-D^2+q$ is approximately $-D^2$ for large $|x|$, there are two solutions $f_\pm$ of $Qy = k^2 y$ normalized as shown in the table.

	$x \to \infty$	$x \to -\infty$
$f_+(x,k)$	$T_+(k)\exp(ikx)$	$\exp(ikx)$ $+R_-(k)\exp(-ikx)$
$f_-(x,k)$	$\exp(-ikx)$ $+R_+(k)\exp(ikx)$	$T_-(k)\exp(-ikx)$

Incoming plane waves $\exp(\pm ikx)$ are scattered by a "potential" $q(x)$. A portion carrying energy $|T_\pm(k)|^2$ is transmitted, and $|R_\pm(k)|^2$ worth is reflected. The *scattering matrix*

$$S(k) = \begin{bmatrix} T_+(k) & R_-(k) \\ R_+(k) & T_-(k) \end{bmatrix} \tag{3.1.5}$$

is unitary. The condition $|R_\pm|^2 + |T_\pm|^2 = 1$ signifies conservation of energy at each wavenumber k. One finds that $T_+ = T_-$; the common value is the transmission coefficient T. We will rarely need R_-, and write R for R_+. It is the (right) reflection coefficient.

The reflection coefficient $R(k)$ determines $q(x)$. This crucial fact will be taken for granted. The inversion formula is stated below in the section on the quantum harmonic oscillator.

3.1.2.2. *Tools from Complex Function Theory.* Analyticity properties of $f_\pm$ and T in the upper half plane $\operatorname{Im} k > 0$ play an essential role. Key pieces of the evidence for Henry's conjecture, presented in [8], are essentially theorems about analytic functions. To give a flavor of the tools required, I list some essential facts that are used in many proofs.

Analyticity of $f_\pm(x,k)$ for $\operatorname{Im} k > 0$, for each fixed x, is a simple side product of the Neumann series argument that proves existence of these functions. The Wronskian of $f_\pm$ is $-2ik/T(k)$, so T is also analytic. The reflection coefficient R is rapidly decreasing as $|k| \to \infty$ on the real axis.

More careful analysis shows that the functions

$$k \mapsto e_\pm(x,k) := \frac{1}{T(k)} f_\pm(x,k) \mathrm{e}^{\mp ikx} \tag{3.1.6}$$

are outer functions in the Hardy space $(1 + H^{2+}) \cap H^{\infty+}$, and that $T(k)$ is determined for $\operatorname{Im} k > 0$ by its values along the real axis by the Poisson formula

(3.1.7)
$$\ln T(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln |T(k')|^2}{k' - k}\, dk', \ \operatorname{Im} k > 0.$$

The reflected functions $e^*_\pm(x,k) = e_\pm(x,-k)$ belong to $(1 + H^{2-}) \cap H^{\infty-}$. The

[5] The squared eigenfunction approach to integrable PDEs and the interpretation of inverse scattering as nonlinear Fourier transform were introduced in the early days of soliton theory in the seminal paper by Ablowitz et al. [1].

pairs $e^*_\pm$ and $e_\pm$ are patched across the real axis according to

$$e^*_\pm + R_\pm \mathrm{e}^{\mp 2ikx} e_\pm = T e_\mp. \tag{3.1.8}$$

The Fourier transform of this relation amounts to an integral equation from which q can be determined. This is inverse scattering.

3.1.2.3. *Squared Eigenfunction Expansion.* The so-called *squared eigenfunctions* $\Phi_\pm := f^2_\pm$ and $\Phi_0 := f_+ f_-$, are solutions of the pseudodifferential eigenvalue problem obtained by integrating (3.1.4),

$$D^{-1} M\Phi := \mathfrak{L}\Phi = 4k^2\Phi, \text{ where}$$
$$\mathfrak{L} = -D^2 + 4q - 2D^{-1}q'. \tag{3.1.9}$$

Their derivatives satisfy the adjoint equation, $\mathcal{L}^*\Phi'_\pm = 4k^2\Phi'_\pm$.

The product Φ_0 is Green's function G_{xy} for $x = y$, up to factor. The other two squared eigenfunctions $\Phi_\pm$ are of decisive importance for (at least) two reasons.

They are derivatives, with respect to $q(x)$, of the reflection coefficient $R(k)$. When one leaves the scattering class, and normalization at $|x| = \infty$ is no longer possible, the gradients of other spectral quantities are often natural substitutes (cf. the quantum harmonic oscillator, below). They always satisfy $M\Psi = 4\lambda\Psi'$.

The sets $\Phi_\pm$ and $\Phi'_\mp$ form the biorthogonal bases used to diagonalize KdV. The representation of q itself is simple and elegant:

$$\tilde{q}(k) = \int q\,\Phi'_-\,dx = 4k^2 R(k) \tag{3.1.10a}$$

$$q(x) = \frac{2i}{\pi}\int \frac{kR}{T^2}\,\Phi_+\,dk. \tag{3.1.10b}$$

This takes a familiar form when q is "small". To first order, $\Phi_\pm(x,k) \sim \mathrm{e}^{\pm 2ikx}$. The small scatterer q reflects but a small portion of the incoming wave, whence $T \sim 1$. The orthogonality relations between Φ and Φ' become the standard ones for exponentials, and the relations (3.1.10) reduce to

$$R(k) \sim -i\hat{q}(k)/2k.$$

The reflection coefficient is hereby revealed to be a nonlinearization of the Fourier transform. A number of later formulas are profitably understood as perturbations of familiar Fourier facts.

The orthogonality of Φ, Φ' can also be seen as consequence of a geometric property: the Φ' are tangent, and the Φ are normal, to the nonlinear analogs of the LKdV manifolds; these will now be introduced.

3.1.3. The KdV Manifold. In Sect. 3.1.1 we defined an LKdV manifold as a set of q with prescribed modal energies $|\hat{q}(k)|^2$. A KdV manifold[6] consists of the set of q with the *reflected* and *transmitted* energy in each mode, $|R(k)|^2$ and $|T(k)|^2$, prescribed (they add to 1).

Let $\mathbb{J}(r)$ denote such a set. It consists of all q whose reflection coefficient has the form $R(k) = r(k)\mathrm{e}^{i\psi(k)}$ with fixed r and odd function ψ of slow growth. Since $|R|$ determines T, one may also write $\mathbb{J}(T)$.

In complete analogy with the linear case (3.1.3), there is a distinguished family of vector fields tangent to $\mathbb{J}(r)$. Take the evolution equation (a priori nonlocal)

$$\dot{q} = \mathbb{Y}_\Omega q := D\Omega(\mathcal{L})q. \tag{3.1.11}$$

Substitution of (3.1.10b) for q and application of (3.1.9) lead to

$$\dot{R}(k,t) = 2ik\Omega(4k^2)R(k,t). \tag{3.1.12}$$

Thus, $|R(k)|^2$ is independent of time and the flows (3.1.11) preserve the KdV manifolds. Furthermore, they are simultaneously diagonalized, and so commute.

The vector fields

$$\begin{aligned}\dot{q} = \mathbb{X}_m q := D\mathcal{L}^m q &= (-1)^m D^{2m+1}q \\ &\quad + \text{nonlinear terms}, m = 0,1,2,3,\ldots,\end{aligned} \tag{3.1.13}$$

are known as the *KdV hierarchy* ($m = 0$ is translation of the potential, $\dot{q} = q'$, and $m = 1$ is KdV). The iterated antiderivatives in $D\mathcal{L}^m$ magically cancel, and the $\mathbb{X}_m$ are polynomial in q and its derivatives.

The nonlocal equation defined by the resolvent of $\mathcal{L}$, namely $\Omega(\mathcal{L}) = -2(\mathcal{L}+4\kappa_0^2)^{-1}$, is an *infinitesimal addition.* It will soon assume a place of prominence.

[6]The "KdV" indicates no more than invariance under the flows of a family of commuting vector fields that includes KdV; often, KdV per se has nothing to do with the matters of interest.

3.1.4. KdV as Integrable Hamiltonian System. At this point, we know[7] that the scattering class $\mathcal{S}$ is stratified by infinite-dimensional tori that are orbits of a huge group denoted by $\mathfrak{A}$ above. This is the group of translations of angles in the phase functions, generated by the vector fields $\mathbb{Y}_\Omega$. The torus-angle picture is familiar in classical Hamiltonian mechanics, and it will be very convenient to express KdV geometry with a Hamiltonian vocabulary. The list of definitions and facts is relegated to an appendix.

3.1.4.1. *Complete Integrability of KdV.* Hamiltonian systems are a skew analog of gradient systems. In Euclidean space, they have the form

$$\dot{\mathbf{x}} = P \,\mathrm{grad}\, H(\mathbf{x}).$$

For KdV, the so-called Poisson operator P is the derivative $D = \frac{d}{dx}$. Its non-invertibility causes complications, but these are ignored.

The Poisson bracket and Hamiltonian vector field have the form

$$\{F, G\}(q) = \int (\mathrm{grad}\, F)\,(\mathrm{grad}\, G)',$$

$$\dot{q} = \mathbb{X}_H q = (\mathrm{grad}\, H(q))'. \tag{3.1.14}$$

The Hamiltonian for KdV is $\frac{1}{2}\int (q')^2 + 2q^3$; the quadratic term by itself is the Hamiltonian for LKdV.

The KdV manifolds are interpreted as follows. They are common level sets $\mathbb{J}(r) = \cap_{k\in\mathbb{R}}\{q \mid |R(k)| = r(k)\}$ of the family of functions $q \mapsto |R(k)|$, indexed by k. These functions Poisson commute; for every pair k, ℓ one has

$$\{|R(k)|, |R(\ell)|\} = 0. \tag{3.1.15}$$

Remarkably, this geometric property is merely a rewording of bi-orthogonality of the squared eigenfunctions. The gradients of the right and left reflection coefficients are

$$\mathrm{grad}\, R_\pm = \frac{1}{2ik}\,\Phi_\mp,$$

and, up to factors,

$$\{R_-(k), R_+(\ell)\} = \int \Phi_+(x,k)\Phi_-'(x,\ell)\,dx = \delta(k-\ell). \tag{3.1.16}$$

[7]Once more: this is the guiding picture; what we *really* know may be a lot or a little.

Equation (3.1.15) for the moduli is then derived by appeal to unitarity of the scattering matrix (3.1.5).

The KdV manifolds $\mathbb{J}(|R|)$ are continuum tori labeled by $|R|$ and parametrized by $\arg R$; a little rewriting turns these coordinates into canonical variables of action-angle type,[8]

$$Z(k) = -\frac{k}{\pi}\ln|T(k)|^2 \quad \text{and} \quad \theta(k) = \arg\frac{R(k)}{T(k)}. \tag{3.1.17}$$

3.1.4.2. *Tangent Spaces to* $\mathbb{J}(r)$. The functions $T(i\kappa), \kappa > 0$, are in involution since the Poisson representation (3.1.7) determines T in the upper half plane from R on the real axis. The gradient of $T(i\kappa)$ is Green's function G_{xy} on the diagonal and generates the Hamiltonian vector field

$$\dot{q} = \mathbb{X}_\kappa q := (\mathrm{grad}\, T(i\kappa))' = -G_{xx}'(-\kappa^2|Q). \tag{3.1.18}$$

The $\mathbb{X}_\kappa$ span the tangent spaces to the KdV manifold.

The transform of $\mathbb{X}_\kappa$ to $R(k,t)$, which is needed later, can be computed in action-angle variables from the Poisson representation (3.1.7) of $T(i\kappa)$:

$$\dot{R}(k,t) = -\frac{ik}{k^2+\kappa^2}R(k,t). \tag{3.1.19}$$

According to (3.1.12), the vector field (3.1.18) can also be represented in terms of the resolvent of $\mathcal{L}$, namely, $\dot{q} = -2D(\mathcal{L}+4\kappa^2)^{-1}q$.

3.1.4.3. *KdV Hierarchy.* The role of the KdV hierarchy of local equations, (3.1.13), deserves special emphasis. Green's function satisfies $MG_{xx} = 4\lambda G_{xx}'$; the coefficients of the asymptotic expansion of $G_{xx}'(k^2|Q)$ in inverse powers of k can be determined recursively, and are nothing but the KdV vector fields $\mathbb{X}_m$:

$$G_{xx}'(k^2|Q) \sim \frac{1}{2}q'\frac{1}{k^3} + \frac{1}{8}(6qq' - q''')\frac{1}{k^5} + \mathbb{X}_3\frac{1}{k^7} + \cdots \tag{3.1.20}$$

Thus, Eq. (3.1.18) contain the KdV hierarchy, albeit in a very implicit manner.

[8]For small q, they reduce to $Z_0(k) = \frac{1}{4\pi k}|\hat{q}(k)|^2$ and $\theta_0(k) = \arg\hat{q}(k),\ k > 0$.

3.2. Geometry of KdV: Additive and Unimodular Classes

We now leave the scattering class $\mathcal{S}$ and only ask that q be a smooth function and that the spectrum of Q be positive. The class of such operators is denoted by $\mathfrak{Q}$.

The steps to be accomplished are:

(a) to find an equivalence relation on $\mathfrak{Q}$ whose equivalence classes in the scattering case are the tori $\mathbb{J}(r)$;
(b) to generate an equivalence class from one of its members by a procedure that generalizes translation of R by all possible phase functions;
(c) to find a family of commuting vector fields that span the tangent spaces of the equivalence classes and, in the scattering case, contains the KdV hierarchy.

Paper (1) in the *Geometry of KdV* series proposes solutions of these three problems. The equivalence classes are the *unimodular spectral classes* of the title. The analogs of the phase functions are generated by *additions*. The tangent vectors are *infinitesimal additions*, which generalize the vector fields (3.1.18) with Hamiltonians $T(i\kappa_0)$.

Paper (2) verifies that the scattering class, the class of operators with periodic potential (the Hill operators), and a certain finite-dimensional subclass thereof, all fit the framework. It will become clear that this is a highly nontrivial result.

Paper (3) integrates a large subset of the vector fields from (c), by the "simple" expedient of exhibiting an explicit solution as infinite determinant.

Some of these results, particularly (c), were motivated by earlier work [14] of McKean and Trubowitz[9] on the quantum mechanical harmonic oscillator, $Q_0 := -D^2 + q_0$ with $q_0 = x^2 - 1$. This operator is the complete opposite of the scattering type (and Hill's operator); it has pure discrete spectrum $\lambda_n = 2n, n \geq 0$, and the KdV vector field $-q''' + 6qq'$ cannot be integrated because the x^3 growth of the nonlinearity is not balanced by the linear terms. The analog of the KdV manifold is parametrized by the exponential map of Hamiltonian vector fields, $\mathbb{X}_n = (\operatorname{grad}\lambda_n)'$, that are no longer local in q,

$$(t_0, t_1, t_2, \ldots) : Q_0 \mapsto \exp(\sum_0^\infty t_n \mathbb{X}_n) \cdot Q_0,$$

In the generality of class $\mathfrak{Q}$, a continuum version of this map will be required; loosely speaking, it is a superposition of vector fields $\mathbb{X}_\lambda$ of infinitesimal additions, smeared by a measure μ,

(3.2.1)
$$t(\lambda) : Q_{\text{reference}} \mapsto \exp\left(\int_0^\infty t(\lambda)\mathbb{X}_\lambda \, d\mu(\lambda)\right) \cdot Q_{\text{reference}}.$$

For the oscillator, μ is concentrated on the nonnegative integers.

3.2.1. The Harmonic Oscillator and Paired Additions.

The goal is to describe the spectral manifold[10] $\mathbf{Q}$ of Q_0, meaning the set of operators of the form $Q = -D^2 + q$ whose eigenvalues are also $\lambda_n = 2n$, for q in the space $x^2 - 1 + C_\downarrow^\infty$. It is the analog of the KdV manifold in the scattering class (but recall that KdV has no meaning in $\mathbf{Q}$).

The spectral manifold $\mathbf{Q}$ will be imagined as submanifold of an ambient space of "all" operators Q that have eigenvalues $\lambda_n = \lambda_n(Q)$ "near" those of the harmonic oscillator. We picture $\mathbf{Q}$ as one of a stack of level sets $\mathbf{Q_c}$ of operators with all $\lambda_n(Q) = c_n$ prescribed. The geometry of $\Phi_\pm$ and $\Phi'_\pm$ is replayed simply and cleanly. The normal and tangent directions of $\mathbf{Q}$ are spanned, respectively, by[11] $\operatorname{grad}\lambda_n = e_n^2$ and $(e_n^2)'$. Bi-orthogonality translates into $\{\lambda_m, \lambda_n\} = 0$. The Hamiltonian vector fields $\mathbb{X}_n = (\operatorname{grad}\lambda_n)'$ commute, and it is possible to solve the equations

(3.2.2)
$$\frac{\partial q}{\partial t_m} = (e_m^2)', \; 0 \leq m \leq n,$$

[9] As far as I know, this is still the only paper to analyze an operator from outside the traditional KdV world.

[10] This time it really is a manifold.

[11] e_n is the normalized eigenfunction corresponding to λ_n.

simultaneously. Miraculously, there is an explicit formula:

(3.2.3)
$$q(x;t_0,t_1,\dots,t_n) = q_0 - 2\frac{d^2}{dx^2}\left(\ln\det\left[\delta_{k\ell} + (e^{t_k}-1)\int_x^\infty e_k^0 e_\ell^0,\ 0\le k,\ell\le n\right]\right)$$

(superscript zero denotes initial values). One can even take $n\to\infty$ as long as $\mathbf{t} = \{t_m\}_0^\infty$ decreases rapidly. The sequences $\mathbf{t}$ form a coordinate grid on the spectral manifold $\mathbf{Q}$, or equivalently, on the group $\mathfrak{A}$ of *additions* mentioned at the beginning of Section 2.

3.2.1.1. *Additions in General.* Additions are substitutes for the local flows, like KdV, that may no longer exist in the generality of the class $\mathfrak{Q}$. They are special instances of a classical transformation of 2nd-order ODEs, the Darboux transformation,[12] which takes a zero-free solution y_1 of $Q_1 y = -y'' + q_1 y = \mu y$ as input and creates a new operator according to

(3.2.4)
$$q_2 = q_1 - 2(\ln y_1)'', \quad \text{and} \quad Q_2 = -D^2 + q_2.$$

The map[13] $P : y \mapsto y_1^{-1} W(y, y_1)$ sends solutions of $Q_2 y = \lambda y$ to solutions of $Q_1 y = \lambda y$, for $\lambda \ne \mu$; for $\lambda = \mu$, the new solutions are

$$y_1^{-1}(a_1 + a_2 \int^x y_1^2). \tag{3.2.5}$$

Thus, everything about the new operator is known. Moreover, Darboux transformations commute and preserve the Poisson bracket.

The determinant (3.2.3) is constructed from repeated additions that start at the initial condition q_0 and use the data λ_m and e_m^0. However, because e_m^0 has zeros when $m > 0$, so that the new potential (3.2.4) has poles, the additions must be done in pairs. If y_1 is not zero-free, do the first Darboux transformation as above, simply ignore the singular nature of the resulting q_2, and perform a second transformation with the solution (3.2.5). The result of this paired addition will be the 1×1 version of (3.2.3),

(3.2.6)
$$\text{new } q := q_1 - 2\frac{d^2}{dx^2}\left(\ln[1 + a\int_x^\infty y_1^2]\right).$$

It is rather surprising that the new potential is smooth (when $a \ge 0$). The determinant in (3.2.3) is built by iterating paired additions. This procedure is not peculiar to the harmonic oscillator; if an operator Q has simple eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$, then the solution of (3.2.2) is given by the same formula.

3.2.1.2. *The Group of Additions Acts Freely and Transitively.* It is shown in [14] that the map $\mathbf{t} \mapsto \exp(\sum t_m \mathbb{X}_m) q_0$ is 1:1 onto $\mathbf{Q}$, by a method reminiscent of inverse scattering. For the normalized eigenfunctions of $-D^2 + q(x;\mathbf{t})$, define

$$c_\pm^n(\mathbf{t}) := \frac{e_n(x=\pm\infty,\mathbf{t})}{e_n^0(x=\pm\infty,\mathbf{t}=\mathbf{0})}.$$

These asymptotic data satisfy

$$c_+^n c_-^n = 1, \qquad \frac{c_+^n}{c_-^n} = e^{t_n}. \tag{3.2.7}$$

When a $\tilde q \in \mathbf{Q}$ is given, and is to be written as $\exp(\sum t_m \mathbb{X}_m) q_0$, look at the ratios (3.2.7) built from its eigenfunctions to determine what $\mathbf{t}$ must be, and then verify (difficult) that the formula (3.2.3) in fact reproduces the given $\tilde q$.

3.2.1.3. *The Dyson Determinant in Inverse Scattering.* The beautiful representation (3.2.3) of the solution of the system (3.2.2) is universal.

In [10], Henry shows that the commuting Hamiltonian flows $\dot q = (\operatorname{grad} T(i\kappa))'$ from the scattering class retain meaning in $\mathfrak{Q}$, and that their solutions have the form (3.2.3), but with a regularized continuum limit of the $\mathbb{Z}_{\ge 0}\times\mathbb{Z}_{\ge 0}$ determinant.

In the Hill class, there is a similar expression,

$$q(x) = -2\frac{d^2}{dx^2}\ln\Theta(A(x);Z). \tag{3.2.8}$$

[12] [6].

[13] W is the Wronskian.

The function Θ has two arguments. The second one labels the KdV manifolds; Z denotes a Riemann surface of (generically) infinite genus. The first one is a one-parameter subgroup of $\mathfrak{A}$, which is the group of translations on the Jacobian variety of Z. A summary and references may be found in [11].

In the scattering class, the function Θ is the Fredholm determinant[14] of an integral operator,

(3.2.9)
$$\Theta(\theta; Z) = \det \left[I + \frac{1}{2\pi} \int e^{ik(x+y)} R(k)\, dk,\ 0 \le x, y \right].$$

The notation indicates that the dependence of Θ on R should be thought of as split into an angle part (the additions) and an action part (the unimodular invariant), as in (3.1.17). The inversion formula that recovers the potential from the scattering data again has the form (3.2.8), with $A(x) = \arg(\exp(ikx)R(k))$.

3.2.2. The Unimodular Class.

3.2.2.1. *Unimodular: Definition.* We now turn to the definition of KdV manifolds for operators in the class $\mathfrak{Q}$. By the spectral theorem, all self-adjoint Q give rise to an expansion in eigenfunctions, generalized or square integrable. The measure that appears in the inversion formula will be the replacement for the scattering matrix.

Fix an operator[15] $Q_0 = -D^2 + q_0$, and let $E_0(x, \lambda)$ be the fundamental system satisfying $[E_0, E_0'] = 2 \times 2$ identity at $x = 0$. In the resolution of the identity[16]

$$\delta(x-y) = \frac{1}{2\pi} \int_0^\infty E_0(x,\lambda)^\ddagger d\mathsf{F}_0(\lambda) E_0(y,\lambda),$$

the matrix $d\mathsf{F}_0(\lambda)$ is the *spectral weight*. If Q_0 and Q_1 are unitarily equivalent, their spectral weights are related by

$$d\mathsf{F}_1(\lambda) = \mathsf{G}(\lambda)\, d\mathsf{F}_0(\lambda) \mathsf{G}(\lambda)^\ddagger. \tag{3.2.10}$$

Conversely, this relation between spectral weights implies unitary equivalence.

Now define the *unimodular class* $\mathbb{U}(Q_0)$ to be the set of all $Q = -D^2 + q$ for which $\mathsf{G}(\lambda) \in SL(2)$, or equivalently, for which $\det d\mathsf{F} = \det d\mathsf{F}_0$.

The unimodular classes are Henry's substitute for the KdV manifolds. Some motivation is suggested in the next article.

For this proposal to be sensible, a unimodular class must coincide with a KdV manifold in the scattering case; i.e. if Q_0 has transmission coefficient T_0, then

$$\mathbb{J}(T_0) = \mathbb{U}(Q_0)$$

should be true. Now, for Q of scattering class the spectral weight can be computed in terms of the scattering matrix; in particular,

$$\det d\mathsf{F}/d\lambda = |T|^2. \tag{3.2.11}$$

For two operators Q_0, Q_1, relation (3.2.10) implies $|T_1|^2 = (\det \mathsf{G})^2 |T_0|^2$; if $Q_1 \in \mathbb{J}(Q_0)$, then $T_1 = T_0$ and so $\det \mathsf{G} = 1$. Hence a KdV manifold is contained in a unimodular class. The converse is much harder and is addressed in Sect. 3.2.4.

3.2.2.2. *Unimodular: Interpretation.* The condition $\det \mathsf{G} = 1$ is not as different from the parametrization of KdV manifolds by transmission coefficients as it appears on first glance. All operators of scattering class are unitarily equivalent, with identical continuous spectrum $\{\lambda \ge 0\}$. For every pair there is a scattering matrix $S(\lambda; Q_0, Q_1)$ that encodes the relation between their plane wave solutions ($\lambda = k^2$). When $Q_0 = -D^2$, this is our standard scattering matrix (3.1.5).

The scattering matrix relating operators in the same KdV manifold is unimodular: $\det S(\lambda; Q_0, Q_1) = 1$. This property has a very pretty and suggestive translation into an eigenvalue perturbation picture.

Let $d\mathfrak{P}^1_\lambda$ and $d\mathfrak{P}^0_\lambda$ be the spectral projections of $Q_1 = -D^2 + q_1$ and $Q_0 = -D^2$. The increment (Tr is the operator trace)

$$d\xi(\lambda) := \mathrm{Tr}(d\mathfrak{P}^1_\lambda - d\mathfrak{P}^0_\lambda)$$

should be the difference between the number of eigenvalues of Q_0, Q_1 in an infinitesimal interval about ω. This trace does not exist, but one can convert the formula into something meaningful. The *spectral shift function* $\xi(\lambda; Q_0, Q_1)$, introduced by Kreĭn, quantifies the displacement of "virtual eigenvalues" of

[14] Introduced into inverse scattering by Dyson.

[15] Note that Q_0 is not $-D^2$.

[16] The dagger ‡ denotes transpose.

Q_0 to "virtual eigenvalues" of Q_1. The scattering matrix $S(\lambda; Q_0, Q_1)$ instead looks at the "rotation" of the eigenfunctions at *fixed* eigenvalue λ. A general theorem of Birman and Kreĭn relates the two:

$$\det S(\lambda; Q_0, Q_1) = \mathrm{e}^{-2\pi i \xi(\lambda; Q_0, Q_1)}. \quad (3.2.12)$$

The determinant of the standard scattering matrix (3.1.5) is $T(k)/\overline{T}(k)$, and is the same throughout a KdV manifold $\mathbb{J}$. Then according to (3.2.12), the spectral shift from $-D^2$ remains the same as well. Put differently, the operators in $\mathbb{J}$ have identical "virtual" eigenvalues, and the pairwise scattering matrices are unimodular.

The condition $\det \mathsf{G} = 1$ similarly restricts the deformation of a basis at fixed λ, but I do not know a more transparent interpretation. It would be very pleasant if it were equivalent, for an interesting class of operators, to the rigidity of virtual eigenvalues, whatever that might mean.

3.2.3. Additive Class. The KdV manifolds in the scattering case were interpreted as tori of an integrable Hamiltonian system, labeled by $\det S(k)$. Instead of tori there are now unimodular strata inside a unitary equivalence class, labeled by $\det \mathsf{G}$. The KdV manifolds were swept out by the group of translations acting on the angular coordinate, $\arg R(k)$. We need a replacement for the angles. It was suggested earlier that the unimodular manifolds are generated by the action of a big additive group $\mathfrak{A}$. These are the additions, which now need to be described in a little more detail, first for the scattering class, followed by the generalization.

3.2.3.1. *Additions in Scattering.* Let $\lambda_0 = -\kappa_0^2 < 0$. Let $\mathfrak{p}$ denote the pair $(\lambda_0, +)$ or $(\lambda_0, -)$, and let $e(x, \mathfrak{p})$ be f_+ or f_- in accordance with that choice. Define *addition of* $\mathfrak{p}$ as before,

$$A^{\mathfrak{p}} : Q \mapsto Q^{\mathfrak{p}} := Q - 2(\ln e(x, \mathfrak{p}))''. \quad (3.2.13)$$

Repeated additions change Q by a Wronskian determinant,

$$A^{\mathfrak{p}_1} \cdots A^{\mathfrak{p}_n} Q = Q - 2\frac{d^2}{dx^2} \ln W(e(x, \mathfrak{p}_1), \ldots, e(x, \mathfrak{p}_n)).$$

From this it is clear that additions commute and that $A^{\mathfrak{p}} A^{-\mathfrak{p}} =$ identity.

Additions change R by a factor of modulus 1,

(3.2.14)
$$A^{\pm\mathfrak{p}} : R(k) \mapsto \frac{\kappa \mp ik}{\kappa \pm ik} R(k), \ \lambda = -\kappa^2;$$

they preserve $\mathbb{J}(|R|)$ and translate the coordinate $\arg R$ by an odd function.

An *infinitesimal addition* is the derivative of $A^{\mathfrak{p}}$ with respect to $\mathfrak{p}$. Set $\mathfrak{p} = (\lambda, +)$ and $\mathfrak{p}' = (\lambda + \Delta\lambda, +)$ and work out $A^{\mathfrak{p}'} A^{-\mathfrak{p}} Q$. The $\mathfrak{p}'$ factor has an f_+, the $-\mathfrak{p}$ has an f_-, and together they give $f_+ f_-$, which is Green's function:

(3.2.15)
$$A^{\mathfrak{p}'} A^{-\mathfrak{p}} Q = Q - 2G'_{xx}(\lambda|Q)\Delta\lambda + \cdots .$$

It is now clear why infinitesimal additions are so important: they are precisely the Hamiltonian vector fields $(\operatorname{grad} T(i\kappa))'$, which were denoted by $\mathbb{X}_\lambda$ in (3.1.18) (up to irrelevant factor). They span the tangent spaces to the KdV tori (redundantly), and generate the big additive group $\mathfrak{A}$ that acts on the tori. Integration of an infinitesimal addition vector field does not produce a global addition (3.2.14). To get one of those, one must piece together segments of integral curves of many $\mathbb{X}_\lambda$, along the lines of

$$\mathrm{e}^{\int t(\lambda)\mathbb{X}_\lambda \, d\mu(\lambda)} \, q_0.$$

3.2.3.2. *Additions in General.* No major adjustments are needed to define additions for general Q because there are natural substitutes for $f_\pm$. Take $\lambda_0 < 0$, to the left of the spectrum of Q. There exist solutions $h_\pm$ of $Qy = -\lambda^2 y$ characterized by[17]

$$h_\pm \in L^2(\mathbb{R}_\pm) \text{ and } h_\pm \notin L^2(\mathbb{R}_\mp)$$

and a normalization at $x = 0$. In the scattering class, $h_\pm$ are proportional to $f_\pm$. The vector fields of infinitesimal addition, $\mathbb{X}_\lambda := -2G'_{xx}(\lambda|Q)$, are everywhere defined, since every Q has a Green function; they include KdV when it makes sense, because the KdV vector field appears in the asymptotic expansion of $-2G'_{xx}(\lambda|Q)$; they preserve the unimodular class (this is not as obvious as in the scattering case, cf. (3.2.14)); and finally, remarkably,

[17] $\mathbb{R}_+ = [0, \infty)$ etc.