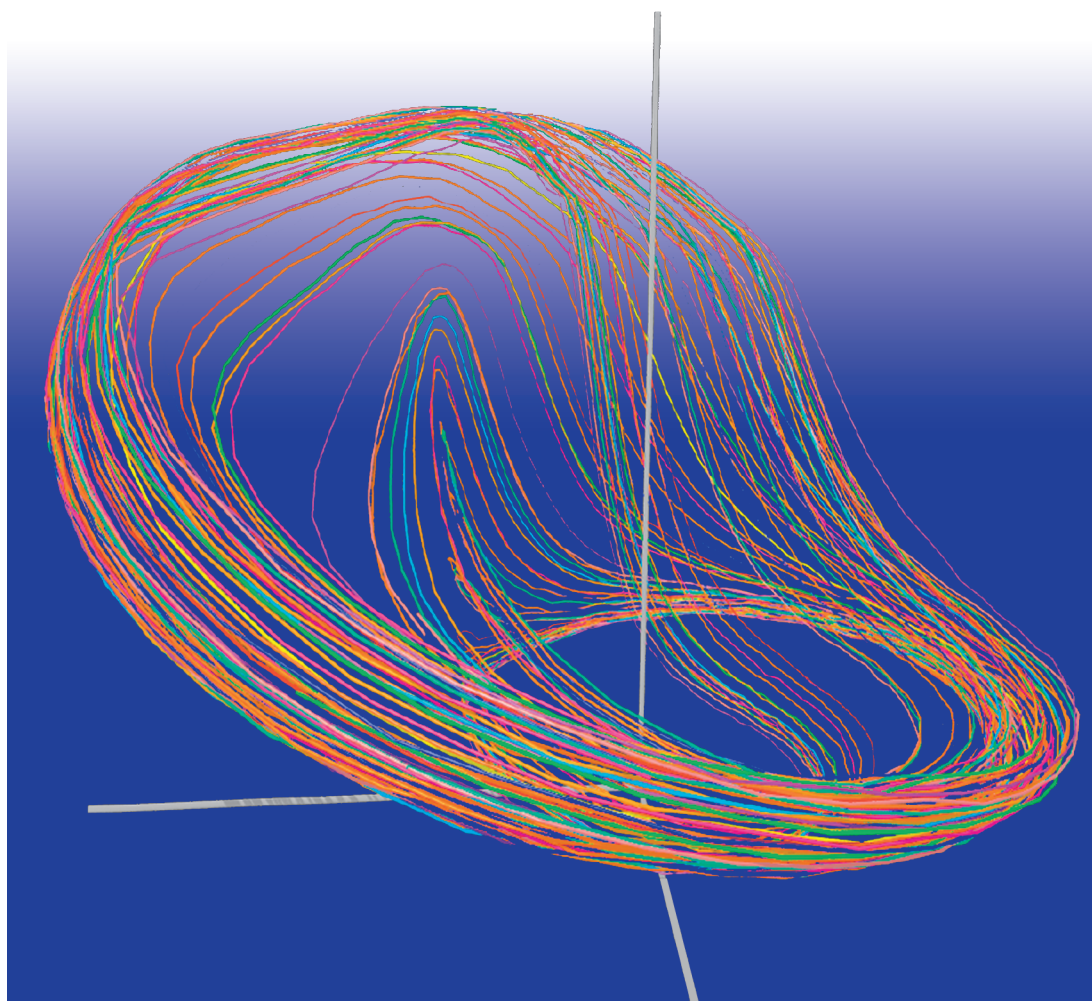


Robert Gilmore, Marc Lefranc

 WILEY-VCH

The Topology of Chaos

Alice in Stretch and Squeezeland
Second Revised and Enlarged Edition



Robert Gilmore and Marc Lefranc

The Topology of Chaos

Related Titles

Grigoriev, R. (ed.)

Microfluidics Fluid Mixing on the Microscale

2011

ISBN: 978-3-527-41011-8

Box, G. E. P., Jenkins, G. M., Reinsel, G. C.

Time Series Analysis Forecasting and Control

2008

ISBN: 978-0-470-27284-8

Schöll, E., Schuster, H. G. (eds.)

Handbook of Chaos Control

2008

ISBN: 978-3-527-40605-0

Schelter, B., Winterhalder, M., Timmer, J. (eds.)

Handbook of Time Series Analysis Recent Theoretical Developments and Applications

2006

ISBN: 978-3-527-40623-4

Banerjee, S.

Dynamics for Engineers

2005

ISBN: 978-0-470-86844-7

Tanaka, K., Wang, H. O.

Fuzzy Control Systems Design and Analysis A Linear Matrix Inequality Approach

2001

ISBN: 978-0-471-32324-2

Robert Gilmore and Marc Lefranc

The Topology of Chaos

Alice in Stretch and Squeezeland

Second Revised and Enlarged Edition



WILEY-VCH Verlag GmbH & Co. KGaA

The Authors

Prof. Robert Gilmore

Drexel University
Dept. of Physics
Philadelphia, USA
bob@bach.physics.drexel.edu

Prof. Marc Lefranc

Laboratoire de Physique des Lasers, Atomes,
Molécules
Université des Sciences et Technologies de Lille
Villeneuve d'Ascq
France
marc.lefranc@univ.lille1.fr

■ All books published by Wiley-VCH are carefully produced. Nevertheless, authors, editors, and publisher do not warrant the information contained in these books, including this book, to be free of errors. Readers are advised to keep in mind that statements, data, illustrations, procedural details or other items may inadvertently be inaccurate.

Library of Congress Card No.: applied for

British Library Cataloguing-in-Publication Data:

A catalogue record for this book is available from the British Library.

Bibliographic information published by the Deutsche Nationalbibliothek

The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available on the Internet at <http://dnb.d-nb.de>.

© 2011 WILEY-VCH Verlag GmbH & Co. KGaA, Boschstr. 12, 69469 Weinheim, Germany

All rights reserved (including those of translation into other languages). No part of this book may be reproduced in any form – by photoprinting, microfilm, or any other means – nor transmitted or translated into a machine language without written permission from the publishers. Registered names, trademarks, etc. used in this book, even when not specifically marked as such, are not to be considered unprotected by law.

Typesetting le-tex publishing services GmbH, Leipzig

Printing and Binding Fabulous Printers Pte Ltd, Singapore

Cover Design Adam-Design, Weinheim

Printed in Singapore

Printed on acid-free paper

ISBN Print 978-3-527-41067-5

ISBN eBook 978-3-527-63940-3

ISBN ePDF 978-3-527-63942-7

ISBN ePub 978-3-527-63941-0

ISBN Mobi 978-3-527-63943-4

Contents

Preface to Second Edition XVII

Preface to the First Edition XIX

1	Introduction	1
1.1	Brief Review of Useful Concepts	2
1.2	Laser with Modulated Losses	4
1.3	Objectives of a New Analysis Procedure	11
1.4	Preview of Results	12
1.5	Organization of This Work	14
2	Discrete Dynamical Systems: Maps	19
2.1	Introduction	19
2.2	Logistic Map	20
2.3	Bifurcation Diagrams	22
2.4	Elementary Bifurcations in the Logistic Map	25
2.4.1	Saddle–Node Bifurcation	25
2.4.2	Period-Doubling Bifurcation	29
2.5	Map Conjugacy	32
2.5.1	Changes of Coordinates	32
2.5.2	Invariants of Conjugacy	33
2.6	Fully Developed Chaos in the Logistic Map	34
2.6.1	Iterates of the Tent Map	35
2.6.2	Lyapunov Exponents	36
2.6.3	Sensitivity to Initial Conditions and Mixing	37
2.6.4	Chaos and Density of (Unstable) Periodic Orbits	38
2.6.4.1	Number of Periodic Orbits of the Tent Map	38
2.6.4.2	Expansiveness Implies Infinitely Many Periodic Orbits	39
2.6.5	Symbolic Coding of Trajectories: First Approach	40
2.7	One-Dimensional Symbolic Dynamics	42
2.7.1	Partitions	42
2.7.2	Symbolic Dynamics of Expansive Maps	44
2.7.3	Grammar of Chaos: First Approach	48
2.7.3.1	Interval Arithmetics and Invariant Interval	48

2.7.3.2	Existence of Forbidden Sequences	49
2.7.4	Kneading Theory	51
2.7.4.1	Ordering of Itineraries	52
2.7.4.2	Admissible Sequences	54
2.7.5	Bifurcation Diagram of the Logistic Map Revisited	55
2.7.5.1	Saddle–Node Bifurcations	55
2.7.5.2	Period-Doubling Bifurcations	56
2.7.5.3	Universal Sequence	57
2.7.5.4	Self-Similar Structure of the Bifurcation Diagram	58
2.8	Shift Dynamical Systems, Markov Partitions, and Entropy	59
2.8.1	Shifts of Finite Type and Topological Markov Chains	59
2.8.2	Periodic Orbits and Topological Entropy of a Markov Chain	61
2.8.3	Markov Partitions	63
2.8.4	Approximation by Markov Chains	65
2.8.5	Zeta Function	65
2.8.6	Dealing with Grammars	66
2.8.6.1	Simple Grammars	67
2.8.6.2	Complicated Grammars	69
2.9	Fingerprints of Periodic Orbits and Orbit Forcing	70
2.9.1	Permutation of Periodic Points as a Topological Invariant	70
2.9.2	Topological Entropy of a Periodic Orbit	72
2.9.3	Period 3 Implies Chaos and Sarkovskii’s Theorem	74
2.9.4	Period 3 Does Not Always Imply Chaos: Role of Phase-Space Topology	75
2.9.5	Permutations and Orbit Forcing	75
2.10	Two-Dimensional Dynamics: Smale’s Horseshoe	77
2.10.1	Horseshoe Map	77
2.10.2	Symbolic Dynamics of the Invariant Set	78
2.10.3	Dynamical Properties	81
2.10.4	Variations on the Horseshoe Map: Baker Maps	82
2.11	Hénon Map	85
2.11.1	A Once-Folding Map	85
2.11.2	Symbolic Dynamics of the Hénon Map: Coding	87
2.11.3	Symbolic Dynamics of the Hénon Map: Grammar	93
2.12	Circle Maps	96
2.12.1	A New Global Topology	96
2.12.2	Frequency Locking and Arnold Tongues	96
2.12.3	Chaotic Circle Maps as Limits of Annulus Maps	100
2.13	Annulus Maps	100
2.14	Summary	104
3	Continuous Dynamical Systems: Flows	105
3.1	Definition of Dynamical Systems	105
3.2	Existence and Uniqueness Theorem	106
3.3	Examples of Dynamical Systems	107

3.3.1	Duffing Equation	107
3.3.2	Van der Pol Equation	109
3.3.3	Lorenz Equations	111
3.3.4	Rössler Equations	113
3.3.5	Examples of Nondynamical Systems	114
3.3.5.1	Equation with Non-Lipschitz Forcing Terms	115
3.3.5.2	Delay Differential Equations	115
3.3.5.3	Stochastic Differential Equations	116
3.3.6	Additional Observations	117
3.4	Change of Variables	120
3.4.1	Diffeomorphisms	120
3.4.2	Examples	121
3.4.3	Structure Theory	124
3.5	Fixed Points	125
3.5.1	Dependence on Topology of Phase Space	125
3.5.2	How to Find Fixed Points in R^n	126
3.5.3	Bifurcations of Fixed Points	127
3.5.4	Stability of Fixed Points	130
3.6	Periodic Orbits	131
3.6.1	Locating Periodic Orbits in $R^{n-1} \times S^1$	131
3.6.2	Bifurcations of Fixed Points	132
3.6.3	Stability of Fixed Points	133
3.7	Flows Near Nonsingular Points	134
3.8	Volume Expansion and Contraction	136
3.9	Stretching and Squeezing	137
3.10	The Fundamental Idea	138
3.11	Summary	139
4	Topological Invariants	141
4.1	Stretching and Squeezing Mechanisms	141
4.2	Linking Numbers	145
4.2.1	Definitions	146
4.2.2	Reidemeister Moves	147
4.2.3	Braids	148
4.2.4	Examples	151
4.2.5	Linking Numbers for a Horseshoe	153
4.2.6	Linking Numbers for the Lorenz Attractor	154
4.2.7	Linking Numbers for the Period-Doubling Cascade	154
4.2.8	Local Torsion	155
4.2.9	Writhe and Twist	156
4.2.10	Additional Properties	158
4.3	Relative Rotation Rates	159
4.3.1	Definition	160
4.3.2	Computing Relative Rotation Rates	160
4.3.3	Horseshoe Mechanism	163

4.3.4	Additional Properties	168
4.4	Relation between Linking Numbers and Relative Rotation Rates	169
4.5	Additional Uses of Topological Invariants	170
4.5.1	Bifurcation Organization	170
4.5.2	Torus Orbits	171
4.5.3	Additional Remarks	171
4.6	Summary	174
5	Branched Manifolds	175
5.1	Closed Loops	175
5.2	What Does This Have to Do with Dynamical Systems?	178
5.3	General Properties of Branched Manifolds	178
5.4	Birman–Williams Theorem	181
5.4.1	Birman–Williams Projection	182
5.4.2	Statement of the Theorem	183
5.5	Relaxation of Restrictions	184
5.5.1	Strongly Contracting Restriction	184
5.5.2	Hyperbolic Restriction	185
5.6	Examples of Branched Manifolds	186
5.6.1	Smale–Rössler System	186
5.6.2	Lorenz System	188
5.6.3	Duffing System	189
5.6.4	Van der Pol System	192
5.7	Uniqueness and Nonuniqueness	194
5.7.1	Local Moves	195
5.7.2	Global Moves	197
5.8	Standard Form	200
5.9	Topological Invariants	201
5.9.1	Kneading Theory	202
5.9.2	Linking Numbers	205
5.9.3	Relative Rotation Rates	207
5.10	Additional Properties	207
5.10.1	Period as Linking Number	208
5.10.2	EBK-Like Expression for Periods	208
5.10.3	Poincaré Section	209
5.10.4	Blow-Up of Branched Manifolds	210
5.10.5	Branched-Manifold Singularities	211
5.10.6	Constructing a Branched Manifold from a Map	212
5.10.7	Topological Entropy	213
5.11	Subtemplates	216
5.11.1	Two Alternatives	216
5.11.2	A Choice	218
5.11.3	Topological Entropy	219
5.11.4	Subtemplates of the Smale Horseshoe	221

5.11.5	Subtemplates Involving Tongues	222
5.12	Summary	224
6	Topological Analysis Program	227
6.1	Brief Summary of the Topological Analysis Program	227
6.2	Overview of the Topological Analysis Program	228
6.2.1	Find Periodic Orbits	228
6.2.2	Embed in R^3	229
6.2.3	Compute Topological Invariants	230
6.2.4	Identify Template	230
6.2.5	Verify Template	231
6.2.6	Model Dynamics	232
6.2.7	Validate Model	233
6.3	Data	234
6.3.1	Data Requirements	235
6.3.2	Processing in the Time Domain	236
6.3.3	Processing in the Frequency Domain	238
6.3.3.1	High-Frequency Filter	238
6.3.3.2	Low-Frequency Filter	238
6.3.3.3	Derivatives and Integrals	239
6.3.3.4	Hilbert Transforms	240
6.3.3.5	Fourier Interpolation	241
6.3.3.6	Transform and Interpolation	242
6.4	Embeddings	243
6.4.1	Embeddings for Periodically Driven Systems	244
6.4.2	Differential Embeddings	244
6.4.3	Differential–Integral Embeddings	247
6.4.4	Embeddings with Symmetry	248
6.4.5	Time-Delay Embeddings	249
6.4.6	Coupled-Oscillator Embeddings	251
6.4.7	SVD Projections	252
6.4.8	SVD Embeddings	254
6.4.9	Embedding Theorems	254
6.5	Periodic Orbits	256
6.5.1	Close Returns Plots for Flows	256
6.5.1.1	Close Returns Histograms	258
6.5.1.2	Tests for Chaos	258
6.5.2	Close Returns in Maps	259
6.5.2.1	First Return Map	259
6.5.2.2	p th Return Map	260
6.5.3	Metric Methods	261
6.6	Computation of Topological Invariants	262
6.6.1	Embed Orbits	262
6.6.2	Linking Numbers and Relative Rotation Rates	262
6.6.3	Label Orbits	263

6.7	Identify Template	263
6.7.1	Period-1 and Period-2 Orbits	263
6.7.2	Missing Orbits	264
6.7.3	More Complicated Branched Manifolds	264
6.8	Validate Template	264
6.8.1	Predict Additional Topological Invariants	265
6.8.2	Compare	265
6.8.3	Global Problem	265
6.9	Model Dynamics	265
6.10	Validate Model	268
6.10.1	Qualitative Validation	269
6.10.2	Quantitative Validation	269
6.11	Summary	270
7	Folding Mechanisms: A_2	271
7.1	Belousov–Zhabotinskii Chemical Reaction	272
7.1.1	Location of Periodic Orbits	273
7.1.2	Embedding Attempts	274
7.1.3	Topological Invariants	278
7.1.4	Template	281
7.1.5	Dynamical Properties	281
7.1.6	Models	283
7.1.7	Model Verification	283
7.2	Laser with Saturable Absorber	285
7.3	Stringed Instrument	288
7.3.1	Experimental Arrangement	288
7.3.2	Flow Models	290
7.3.3	Dynamical Tests	291
7.3.4	Topological Analysis	291
7.4	Lasers with Low-Intensity Signals	294
7.4.1	SVD Embedding	295
7.4.2	Template Identification	296
7.4.3	Results of the Analysis	297
7.5	The Lasers in Lille	297
7.5.1	Class B Laser Model	298
7.5.2	CO ₂ Laser with Modulated Losses	304
7.5.3	Nd-Doped YAG Laser	308
7.5.4	Nd-Doped Fiber Laser	311
7.5.5	Synthesis of Results	318
7.6	The Laser in Zaragoza	322
7.7	Neuron with Subthreshold Oscillations	328
7.8	Summary	334
8	Tearing Mechanisms: A_3	337
8.1	Lorenz Equations	337
8.1.1	Fixed Points	338

8.1.2	Stability of Fixed Points	339
8.1.3	Bifurcation Diagram	339
8.1.4	Templates	341
8.1.5	Shimizu–Morioka Equations	343
8.2	Optically Pumped Molecular Laser	343
8.2.1	Models	344
8.2.2	Amplitudes	346
8.2.3	Template	346
8.2.4	Orbits	347
8.2.5	Intensities	350
8.3	Fluid Experiments	352
8.3.1	Data	352
8.3.2	Template	353
8.4	Why A_3 ?	354
8.5	Summary	354
9	Unfoldings	357
9.1	Catastrophe Theory as a Model	357
9.1.1	Overview	357
9.1.2	Example	358
9.1.3	Reduction to a Germ	359
9.1.4	Unfolding the Germ	361
9.1.5	Summary of Concepts	362
9.2	Unfolding of Branched Manifolds: Branched Manifolds as Germs	362
9.2.1	Unfolding of Folds	362
9.2.2	Unfolding of Tears	363
9.3	Unfolding within Branched Manifolds: Unfolding of the Horseshoe	365
9.3.1	Topology of Forcing: Maps	365
9.3.2	Topology of Forcing: Flows	366
9.3.3	Forcing Diagrams	369
9.3.3.1	Orbits with Zero Entropy	371
9.3.3.2	Orbits with Positive Entropy	372
9.3.3.3	Additional Comments	372
9.3.4	Basis Sets of Orbits	374
9.3.5	Coexisting Basins	375
9.4	Missing Orbits	375
9.5	Routes to Chaos	377
9.6	Orbit Forcing and Topological Entropy: Mathematical Aspects	378
9.6.1	General Outline	378
9.6.2	Basic Mathematical Concepts	379
9.6.2.1	Braids and Braid Types	379
9.6.2.2	Braids and Surface Homeomorphisms	380
9.6.2.3	Nielsen–Thurston Classification	381
9.6.2.4	Application to Periodic Orbits and Braid Types	382
9.7	Topological Measures of Chaos in Experiments	383

9.7.1	Mixing in Fluids	383
9.7.2	Chaos in an Optical Parametric Oscillator	385
9.8	Summary	389
10	Symmetry	391
10.1	Information Loss and Gain	391
10.1.1	Information Loss	391
10.1.2	Exchange of Symmetry	392
10.1.3	Information Gain	392
10.1.4	Symmetries of the Standard Systems	392
10.2	Cover and Image Relations	393
10.2.1	General Setup	393
10.3	Rotation Symmetry 1: Images	394
10.3.1	Image Equations and Flows	394
10.3.2	Image of Branched Manifolds	396
10.3.3	Image of Periodic Orbits	398
10.4	Rotation Symmetry 2: Covers	400
10.4.1	Topological Index	401
10.4.2	Covers of Branched Manifolds	402
10.4.3	Covers of Periodic Orbits	403
10.5	Peeling: a New Global Bifurcation	404
10.5.1	Orbit Perestroika	405
10.5.2	Covering Equations	405
10.6	Inversion Symmetry: Driven Oscillators	407
10.7	Duffing Oscillator	409
10.8	Van der Pol Oscillator	413
10.9	Summary	418
11	Bounding Tori	419
11.1	Stretching & Folding vs. Tearing & Squeezing	420
11.2	Inflation	421
11.3	Boundary of Inflation	422
11.4	Index	423
11.5	Projection	424
11.6	Nature of Singularities	426
11.7	Trinions	427
11.8	Poincaré Surface of Section	429
11.9	Construction of Canonical Forms	429
11.10	Perestroikas	432
11.10.1	Enlarging Branches	433
11.10.2	Starving Branches	433
11.11	Summary	435
12	Representation Theory for Strange Attractors	437
12.1	Embeddings, Representations, Equivalence	438
12.2	Simplest Class of Strange Attractors	439

12.3	Representation Labels	440
12.3.1	Parity	440
12.3.2	Global Torsion	441
12.3.3	Knot Type	445
12.4	Equivalence of Representations with Increasing Dimension	446
12.4.1	Parity	447
12.4.2	Knot Type	447
12.4.3	Global Torsion	448
12.5	Genus- g Attractors	450
12.6	Representation Labels	451
12.6.1	Parity	451
12.6.2	Multitorsion Index	451
12.6.3	Knot Type	452
12.7	Equivalence in Increasing Dimension	453
12.7.1	Parity and Knot Type	453
12.7.2	Multitorsion Index	453
12.8	Summary	455
13	Flows in Higher Dimensions	457
13.1	Review of Classification Theory in R^3	457
13.2	General Setup	459
13.2.1	Spectrum of Lyapunov Exponents	459
13.2.2	Double Projection	461
13.3	Flows in R^4	462
13.3.1	Cyclic Phase Spaces	462
13.3.2	Floppiness and Rigidity	462
13.3.3	Singularities in Return Maps	463
13.4	Cusps in Weakly Coupled, Strongly Dissipative Chaotic Systems	466
13.4.1	Coupled Logistic Maps	466
13.4.2	Coupled Diode Resonators	469
13.5	Cusp Bifurcation Diagrams	470
13.5.1	Cusp Return Maps	472
13.5.2	Structure in the Control Plane	472
13.5.3	Comparison with the Fold	474
13.6	Nonlocal Singularities	475
13.6.1	Multiple Cusps	475
13.7	Global Boundary Conditions	477
13.8	From Braids to Triangulations: toward a Kinematics in Higher Dimensions	481
13.8.1	Knot Theory in Three Dimensions and Beyond	481
13.8.2	From Nonintersection to Orientation Preservation	482
13.8.3	Singularities in Higher Dimensions	490
13.9	Summary	490
14	Program for Dynamical Systems Theory	493
14.1	Reduction of Dimension	494

14.2	Equivalence	496
14.3	Structure Theory	497
14.3.1	Reducibility of Dynamical Systems	497
14.4	Germes	498
14.5	Unfolding	500
14.6	Paths	502
14.7	Rank	502
14.7.1	Stretching and Squeezing	503
14.8	Complex Extensions	504
14.9	Coxeter–Dynkin Diagrams	504
14.10	Real Forms	506
14.11	Local vs. Global Classification	507
14.12	Cover–Image Relations	508
14.13	Symmetry Breaking and Restoration	508
14.13.1	Entrainment and Synchronization	509
14.14	Summary	511

Appendix A Determining Templates from Topological Invariants 513

A.1	The Fundamental Problem	513
A.2	From Template Matrices to Topological Invariants	515
A.2.1	Classification of Periodic Orbits by Symbolic Names	515
A.2.2	Algebraic Description of a Template	516
A.2.3	Local Torsion	517
A.2.4	Relative Rotation Rates: Examples	517
A.2.5	Relative Rotation Rates: General Case	519
A.3	Identifying Templates from Invariants	523
A.3.1	Using an Independent Symbolic Coding	524
A.3.2	Simultaneous Determination of Symbolic Names and Template	527
A.4	Constructing Generating Partitions	531
A.4.1	Symbolic Encoding as an Interpolation Process	531
A.4.2	Generating Partitions for Experimental Data	535
A.4.3	Comparison with Methods Based on Homoclinic Tangencies	536
A.4.4	Symbolic Dynamics on Three Symbols	538
A.5	Summary	539

Appendix B Embeddings 541

B.1	Diffeomorphisms	541
B.2	Mappings of Data	543
B.2.1	Too Little Data	543
B.2.2	Too Much Data	545
B.2.3	Just the Right Amount of Data	547
B.3	Tests for Embeddings	547
B.4	Tests of Embedding Tests	549
B.4.1	Trial Data Set	549
B.5	Geometric Tests for Embeddings	550
B.5.1	Fractal Dimension Estimation	550

B.5.2	False Near Neighbor Estimates	553
B.6	Dynamical Tests for Embeddings	554
B.7	Topological Test for Embeddings	555
B.8	Postmortem on Embedding Tests	557
B.8.1	Generality	557
B.8.2	Computational Load	557
B.8.3	Variability	558
B.8.4	Statistics	559
B.8.5	Parameters	560
B.8.6	Noise	560
B.8.7	The Self-Intersection Problem	561
B.8.8	Reliability and Limitations	561
B.9	Stationarity	562
B.10	Beyond Embeddings	563
B.11	Summary	563

Appendix C Frequently Asked Questions 565

C.1	Is Template Analysis Valid for Non-Hyperbolic Systems?	565
C.2	Can Template Analysis Be Applied to Weakly Dissipative Systems?	566
C.3	What About Higher-Dimensional Systems?	567

References 569

Index 581

Preface to Second Edition

Since the appearance of the First Edition a decade ago our understanding of the relation between Topology and Chaos has grown quite a bit. This growth is reflected in the increased size of the Second Edition. For the most part the additional material is present in Chapters 11 and 12, and Appendix B of the current version. The two new chapters have been inserted between the first ten and the last two chapters of the First Edition.

We have made small changes in the first ten chapters. The principal change can be seen in Chapter One. This Chapter has been largely rewritten to make the entire work more accessible to someone first coming to the field. Chapter 2 contains two new short sections about homoclinic tangles and annulus maps. A brief discussion of embeddings has been largely expanded and relocated from Chapter 6 to Appendix B of the present work. Some recent beautiful experimental work done in Zaragoza, Spain, has explored the perestroikas that branched manifolds can undergo. This work has been included in Chapter 7. In Chapter 9, a new section summarizes the essential mathematical aspects of orbit forcing and topological entropy, including train track algorithms, and points to the relevant literature. How knots can be used to compute entropy in real systems is also illustrated in a fluid experiment and in an optical system.

Unfortunately there has been little progress in our understanding of flows in higher dimensions (Chapter 11, First Edition) and too little development in the program for dynamical systems (Chapter 12, First Edition). These appear essentially unchanged as Chapters 13 and 14 in the present edition. Still, a new section in Chapter 13 presents an interesting proposal to generalize braids to dynamical triangulations of periodic points. This approach appears to be equivalent to the conventional one in three dimensions, and adapts naturally to phase spaces of any dimension.

Bounding Tori are introduced in Chapter 11. These two-dimensional surfaces enclose three-dimensional strange attractors. They have an elegant classification that goes back more than two centuries to the earlier great topologists (Euler). These structures serve to place yet another identifying tag on low-dimensional strange attractors.

There is a ‘Representation Theory’ for strange attractors that is similar in spirit, if not in detail, to the ‘representation theory’ for groups and algebras developed over

a century ago. The representation theory for three-dimensional dynamical systems is now complete and presented in Chapter 12. For higher-dimensional dynamical systems the path has been blazed but not yet traversed. The representation theory provides a satisfying answer to the troubling question: “When you analyze an embedding of data generated by a chaotic dynamical system, what do you learn about the dynamical system and what do you learn about the embedding?”

The new Appendix B is devoted to the black magic of Embeddings. There are procedures for attempting to create embeddings and there are several different types of tests to assay whether an embedding has in fact been achieved. Some procedures are more reliable, others less so. These and other questions are explored in this Appendix.

We have taken this opportunity to correct mistakes that have crept into the First Edition. Hopefully, there are none in this version, but corrections and suggestions are welcomed and will be available online at the book’s website: <http://www.thetopologyofchaos.net/>. We would like to thank friends and colleagues for pointing out mistakes (with a special mention to Michel Nizette and Mihir Khadilkar), pointing to places where our writing could have/should have been clearer, and most important, for their support and encouragement during the preparation of the Second Edition.

This work was supported in part by the National Science Foundation under grant number NSF-0754081.

June 2011

Rouen, France
Lille, France

Robert Gilmore (robert.gilmore@drexel.edu)
Marc Lefranc (marc.lefranc@univ-lille1.fr)

Preface to the First Edition

Before the 1970s opportunities sometimes arose for physicists to study nonlinear systems. This was especially true in fields like fluid dynamics and plasma physics, where the fundamental equations are nonlinear and these nonlinearities masked (and still mask) the full spectrum of spectacularly rich behavior. When possible, we avoided being dragged into the study of abstract nonlinear systems. For we believed, to paraphrase a beautiful generalization of Tolstoy, that

All linear systems are the same.

Each nonlinear system is nonlinear in its own way.

At that time we believed that one could spend a whole lifetime studying the nonlinearities of the van der Pol oscillator [1, 2] and wind up knowing next to nothing about the behavior of the Duffing oscillator.

Nevertheless, other intrepid researchers had been making an assault on the complexities of nonlinear systems. Smale [3] described a mechanism responsible for generating a great deal of the chaotic behavior that has been studied up to the present time. Lorenz, studying a drastic truncation of the Navier–Stokes equation, discovered and described “sensitive dependence on initial conditions” (1963). The rigid order in which periodic orbits are created in the bifurcation set of the logistic map, and in fact any unimodal map of an interval to itself, was described by May [4] and by Metropolis, Stein, and Stein [5].

Still, there was a reluctance on the part of most scientists to indulge in the study of nonlinear systems. This all changed with Feigenbaum’s discoveries (1978). He showed that scaling invariance in period-doubling cascades leads to quantitative (later, qualitative) predictions. These are the scaling ratios:

$$\begin{aligned}\delta &= 4.669\,201\,609\,102\,9\dots && \text{control parameter space} \\ \alpha &= -2.502\,907\,875\,095\,9\dots && \text{state variable space}\end{aligned}$$

that are eigenvalues of a renormalization transformation. The transformation in the attitude of scientists is summarized by Gleick’s [6] statement:

“It was a very happy and shocking discovery that there were structures in nonlinear systems that are always the same if you looked at them the right way.”

This discovery launched an avalanche of work on nonlinear dynamical systems. Old experiments, buried and forgotten because of instabilities or unrepeatability due to incompetent graduate students (in their advisors' opinions) were resurrected and pushed as groundbreaking experiments exhibiting "first observations" of chaotic behavior (by these same advisors). And many new experiments were carried out, at first to test Feigenbaum's scaling predictions, then to test other quantitative predictions, then just to see what would happen.

Some of the earliest experiments were done on fluids, since the fundamental equations were known and are nonlinear. However, these experiments often suffered from the long time scales (days, weeks, or months) required to record a decent data set. Oscillating chemical reactions (e.g., the Belousov–Zhabotinskii reaction) yielded a wide spectrum of periodic and chaotic behavior that was relatively easy to control and to tune. These data sets could be generated in hours or days. Nonlinear electric circuits were also extensively studied, although there was (and still is) a prejudice to regard them with a jaundiced eye as little more than analog computers. Such data sets could be generated very quickly (seconds to minutes) – almost as fast as numerical simulations. Finally, laser laboratories contributed in a substantial way to very quickly (milliseconds to minutes) building up extensive and widely varying banks of chaotic data.

It was at this time (1988), about 10 years into the "nonlinear science" revolution, that one of the authors (R.G.) was approached by his colleague (J.R. Tredicce, then at Drexel, now at the Institut Non Linéaire de Nice) with the proposition: "Bob, can you help me explain my data?" (Chapter 1). So we swept the accumulated clutter off my desk and deposited his data. We looked, pushed, probed, discussed, studied, etc. for quite a while. Finally, I replied: "No." Tredicce left with his data. But he is very smart (he is an experimentalist!) and returned the following day with the same pile of stuff. The conversation was short and effective: "Bob," (still my name), "I'll bet that you *can't* explain my data." (Bob sees red!) We sat down and discussed further. At the time two tools were available for studying chaotic data. These involved estimating Lyapunov exponents (dynamical stability) and estimating fractal dimensions (geometry). Both required lots of very clean data and long calculations. They provided real number(s) with no convincing error bars, no underlying statistical theory, and no independent way to verify these guesses. And at the end of the day neither provided any information on "how to model the dynamics."

Even worse: Before doing an analysis I would like to know what I am looking for, or at least know what the spectrum of possible results looks like. For example, when we analyze chemical elements or radionuclides, there is a periodic table of the chemical elements and another for the atomic nuclei that accommodate any such analyses. At that time, no classification theory existed for strange attractors.

In response to Tredicce's dare, I promised to (try to) analyze his data. But I pointed out that a serious analysis couldn't be done until we first had some handle on the classification of strange attractors. This could take a long time. Tredicce promised to be patient. And he was.

Our first step was to consider the wisdom of Poincaré, who had suggested about a century earlier that one could learn a great deal about the behavior of nonlinear systems by studying their unstable periodic orbits, which

“... yield us the solutions so precious, that is to say, they are the only breach through which we can penetrate into a place which up to now has been reputed to be inaccessible.”

This observation was compatible with what we learned from experimental data: the most important features that governed the behavior of a system, and especially that governed the perestroikas of such systems (i.e., changes as control parameters are changed) are the features that you can't see – the unstable periodic orbits.

Accordingly, my colleagues and I studied the invariants of periodic orbits, their (Gauss) linking numbers. We also introduced a refined topological invariant based on periodic orbits – the relative rotation rates (Chapter 4). Finally, we used these invariants to identify topological structures (branched manifolds or templates, Chapter 5), which we used to classify strange attractors “in the large.” The result was that “low-dimensional” strange attractors (i.e., those that could be embedded in three-dimensional spaces) could be classified. This classification depends on the periodic orbits “in” the strange attractor, in particular, on their organization as elicited by their invariants. The classification is topological. That is, it is given by a set of integers (also by very informative pictures). Not only that, these integers can be extracted from experimental data. The data sets do not have to be particularly long or particularly clean – especially by fractal dimension calculation standards. Further, there are built-in internal self-consistency checks. That is, the topological analysis algorithm (Chapter 6) comes with reject/fail to reject test criteria. This is the first – and remains the only – chaotic data analysis procedure with rejection criteria.

Ultimately we discovered, through analysis of experimental data, that there is a secondary, more refined classification for strange attractors. This depends on a “basis set of orbits” that describes the spectrum of all the unstable periodic orbits “in” a strange attractor (Chapter 9).

The ultimate result is a doubly discrete classification of strange attractors. Both parts of this doubly discrete classification depend on unstable periodic orbits. The classification depends on identifying:

- A *branched manifold* – which describes the stretching and squeezing mechanisms that operate repetitively on a flow in phase space to build up a (hyperbolic) strange attractor and to organize all the unstable periodic orbits in the strange attractor in a unique way. The branched manifold is identified by the spectrum of the invariants of the periodic orbits that it supports.
- A *basis set of orbits* – which describes the spectrum of unstable periodic orbits in a (nonhyperbolic) strange attractor.

The perestroikas of branched manifolds and of basis sets of orbits in this doubly discrete classification obey well-defined topological constraints. These constraints provide both a rigidity and a flexibility for the evolution of strange attractors as control parameters are varied.

Along the way we discovered that dynamical systems with symmetry could be related to dynamical systems without symmetry in very specific ways (Chapter 10). As usual, these relations involve both a rigidity and a flexibility that are as surprising as they are delightful.

Many of these insights are described in the paper [7], which forms the basis for part of this book. We thank the editors of this journal for their policy of encouraging the transformation of research articles into a longer book format.

The encounter (falling in love?) of the other author (M.L.) with topological analysis dates back to 1991, when he was a Ph.D. student at the University of Lille, struggling to extract information from the very same type of chaotic laser that Tredicce was using. At that time, Marc was computing estimates of fractal dimensions for his laser. But the estimates depended very much on the coordinate system used and gave no insight into the mechanisms responsible for chaotic behavior, even less into the succession of the different behaviors observed. This was very frustrating. There had been this very intriguing paper in *Physical Review Letters* about a “characterization of strange attractors by integers,” with appealing ideas and nice pictures. But as with many short papers, it was difficult to understand how you should proceed when faced with a real experimental system. Topological analysis struck back when Pierre Glorieux, then Marc’s advisor, came back to Lille from a stay in Philadelphia and handed him a preprint from the Drexel team, saying, “You should have a look at this stuff.” The preprint was about topological analysis of the Belousov–Zhabotinskii reaction, a real-life system. It was the Rosetta Stone that helped put pieces together. Soon after, pictures of braids constructed from laser signals were piling up on his desk. They were absolutely identical to those extracted from the Belousov–Zhabotinskii data and described in the preprint. There was universality in chaos if you looked at it with the right tools. Eventually, the system that had motivated topological analysis in Philadelphia, the CO₂ laser with modulated losses, was characterized in Lille and shown to be described by a horseshoe template. Indeed, Tredicce’s laser could not be characterized by topological analysis because of long periods of zero output intensity that prevented invariants from being reliably estimated. The high signal-to-noise ratio of the laser in Lille allowed us to use a logarithmic amplifier and to resolve the structure of trajectories in the zero intensity region.

But a classification is only useful if there exist different classes. Thus, one of the early goals was to find experimental evidence of a topological organization that would differ from the standard Smale horseshoe. At that time, some regimes of the modulated CO₂ laser could not be analyzed for lack of a suitable symbolic encoding. The corresponding Poincaré sections had peculiar structures that, depending on the observer’s mood, suggested a doubly iterated horseshoe or an underlying three-branch manifold. Since the complete analysis could not be carried out, much time was spent on trying to find at least one orbit that could not fit the horseshoe

template. The result was extremely disappointing: For every orbit detected, there was at least one horseshoe orbit with identical invariants. One of the most important lessons of Judo is that if you experience resistance when pushing, you should pull (and vice versa). Similarly, this failed attempt to find a nonhorseshoe template turned into techniques to determine underlying templates when no symbolic coding is available and to construct such codings using the information extracted from topological invariants.

But the search for different templates was not over. Two of Marc's colleagues, Dominique Derozier and Serge Bielawski, proposed that he study a fiber-optic laser they had in their laboratory (that was the perfect system for studying knots). This system exhibits chaotic tongues when the modulation frequency is near a subharmonic of its relaxation frequency: It was tempting to check whether the topological structures in each tongue differed. That was indeed the case: The corresponding templates were basically horseshoe templates but with a global torsion increasing systematically from one tongue to the other. A Nd:YAG laser was also investigated. It showed similar behavior, until the day when Guillaume Boulant, the Ph.D. student working on the laser, came to Marc's office and said, "I have a weird data set." Chaotic attractors were absolutely normal, return maps resembled the logistic map very much, but the invariants were simply not what we were used to. This was the first evidence of a reverse horseshoe attractor. How topological organizations are modified as a control parameter is varied was the subject of many discussions in Lille in the following months; a rather accurate picture finally emerged, and papers began to be written. In the last stages, Marc did a bibliographic search just to clear his mind, and ... a recent 22-page *Physical Review* paper, by McCallum and Gilmore, turned up. Even though it was devoted to the Duffing attractor, it described with great detail what was happening in our lasers as control parameters were modified. Every occurrence of "we conjecture that" in the papers was hastily replaced by "our experiments confirm the theoretical prediction ...," and papers were sent to *Physical Review*. They were accepted 15 days later, with a very positive review. Soon after, the referee contacted us and proposed a joint effort on extensions of topological analysis. The referee was Bob, and this was the start of what we hope will be a long-lasting collaboration.

It would indeed be very nice if these techniques could be extended to the analysis of strange attractors in higher (than three) dimensions. Such an extension, if it is possible, cannot rely on the most powerful tools available in three dimensions. These are the topological invariants used to tease out information on how periodic orbits are organized in a strange attractor. We cannot use these tools (linking numbers, relative rotation rates) because knots "fall apart" in higher dimensions. We explore (Chapter 11) an inviting possibility for studying an important class of strange attractors in four dimensions. If a classification procedure based on these methods is successful, the door is opened to classifying strange attractors in R^n , $n > 3$. A number of ideas that may be useful in this effort have already proved useful in two closely related fields (Chapter 12): lie group theory and singularity theory.

Some of the highly technical details involved in extracting templates from data have been archived in the appendix. Other technical matters are archived at our web sites.¹⁾

Much of the early work in this field was done in response to the challenge by J.R. Tredicce and carried out with my colleagues and close friends: H.G. Solari, G.B. Mindlin, N.B. Tufillaro, F. Papoff, and R. Lopez-Ruiz. Work on symmetries was done with C. Letellier. Part of the work carried out in this program has been supported by the National Science Foundation under grants NSF 8843235 and NSF 9987468. Similarly, Marc would like to thank colleagues and students with whom he enjoyed working and exchanging ideas about topological analysis: Pierre Glorieux, Ennio Arimondo, Francesco Papoff, Serge Bielawski, Dominique Derozier, Guillaume Boulant, and Jérôme Plumecoq. Bob's stays in Lille were partially funded by the University of Lille, the Centre National de la Recherche Scientifique, Drexel University under sabbatical leave, and by the NSF.

Last and most important, we thank our wives Claire and Catherine for their warm encouragement while physics danced in our heads, and our children, Marc and Keith, Clara and Martin, who competed with our research, demanded our attention, and, in doing so, kept us human.

Lille, France, January 2002

Robert Gilmore and Marc Lefranc

1) <http://einstein.drexel.edu/directory/faculty/Gilmore.html> and <http://www.phlam.univ-lille1.fr/perso/lefranc.html>. Last access: 1 January 2002.

1

Introduction

The subject of this book is chaos as seen through the filter of topology. The origin of this book lies in the analysis of data generated by a dynamical system operating in a chaotic regime. Throughout this book we develop topological tools for analyzing chaotic data and then show how they are applied to experimental data sets.

More specifically, we describe how to extract, from chaotic data, topological signatures that determine the stretching and squeezing mechanisms that act on flows in phase space and that are responsible for generating chaotic data.

In the first section of this introductory chapter we very briefly review some of the basic ideas from the field of nonlinear dynamics and chaos. This is done to make the work as self-contained as possible. More in-depth treatment of these ideas can be found in the references provided.

In the second section we describe, for purposes of motivation, a laser that has been operated under conditions in which it behaved chaotically. The topological methods of analysis that we describe in this book were developed in response to the challenge of analyzing chaotic data sets generated by this laser.

In the third section we list a number of questions we would like to be able to answer when analyzing a chaotic signal. None of these questions can be addressed by the older tools for analyzing chaotic data. The older methods involve estimates of the spectrum of Lyapunov exponents and estimates of the spectrum of fractal dimensions. The question that we would particularly like to be able to answer is this: How does one model the dynamics? To answer this question we must determine the stretching and squeezing mechanisms that operate together – repeatedly – to generate chaotic data. The stretching mechanism is responsible for *sensitivity to initial conditions* while the squeezing mechanism is responsible for *recurrent non-periodic behavior*. These two mechanisms operate repeatedly to generate a strange attractor with a self-similar structure.

A new analysis method, topological analysis, has been developed to respond to the fundamental question just stated [7, 8]. At the present time this method is suitable only for strange attractors that can be embedded in three-dimensional spaces. However, for such strange attractors it offers a complete and satisfying resolution to this question. The results are previewed in the fourth section of this chapter. In the final section we provide a brief overview of the organization of this book. In particular, we summarize the organization and content of the following chapters.

It is astonishing that the topological analysis tools that we describe have provided answers to more questions than we had originally asked. This analysis procedure has also raised more questions than we have answered. We hope that the interaction between experiment and theory and between old questions answered and new questions raised will hasten the evolution of the field of nonlinear dynamics.

1.1

Brief Review of Useful Concepts

There are a number of texts that can serve as excellent introductions to the study of nonlinear dynamics and chaos. These include [9–21]. Any one of these can be used to fill in details that we may pass by a little too quickly in our study. There are also many texts that serve as introductions to topology. All cover far more material than we use here. As a result, we do not recommend that a reader invest time in any one of these texts. We will introduce the topological concepts as needed as we proceed.

For now we review very briefly some of the foundational ideas of chaos.

What Is chaos? We take the following as a useful definition for chaos, or chaotic motion. Chaos is motion that is

1. deterministic
2. bounded
3. nonperiodic
4. sensitive to initial conditions.

Where does this motion take place? It is useful to describe the state of a physical system by a set of coordinates. The most convenient way to do this is to establish a phase space. A point in the phase space describes the state of the physical system. The “motion” described above is that of a point in the phase space. For example, the phase space needed to describe the motion of two particles in a plane is eight-dimensional: two coordinates and two velocity components are required to describe the state of each particle’s motion. We will work with smaller phase spaces.

What is “deterministic motion”? The motion of the coordinates x_i of a point in an N -dimensional phase space is governed by a set of N first-order ordinary differential equations:

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \dots, x_N; c). \quad (1.1)$$

The coordinates $x \in R^N$ are called *phase space coordinates* and the coordinates $c \in R^K$ are called *control parameters*. A set of N equations of this type is called a *dynamical system*.

What is “bounded” motion? The trajectory defined by the equations of motion (1.1) can be parameterized by the time coordinate: $x(t)$. *Bounded* means that the maximum distance between any two points on the trajectory over all times is less than infinity: $\text{Max}_{(t_1, t_2)} ||x(t_1) - x(t_2)|| < \infty$.

What is periodic motion? This is motion that returns to its starting point after an elapsed time T : $x(t) = x(t + T)$, for some $T > 0$. T is called the *period* if there is no smaller positive value (for example, $T/2$) for which this equation is true.

What is “sensitivity to initial conditions”? A point x_1 in phase space can serve as an initial condition for a trajectory through it: $x_1(t)$. Two nearby points can serve as initial conditions for two trajectories, one starting at each point. At first the two trajectories remain close to each other. If the distance between them grows exponentially with time, $||x_2(t) - x_1(t)|| \simeq e^{\lambda t}$ with $\lambda > 0$, then the system is said to exhibit sensitivity to initial conditions. This means that, although the evolution is deterministic, the future position of an initial condition becomes unpredictable after some time. The term λ is called a *Lyapunov exponent*.

How do you visualize chaotic motion? A very convenient way to do this is to view the trajectory of an initial condition in the phase space. It often happens that the trajectory “wanders around” in the phase space until it “settles down” onto, or is “attracted to,” a reasonably well-defined structure. This structure, when it exists, is called by mathematicians an Ω limit set and by physicists a strange attractor or a chaotic attractor. Plots of strange attractors appear liberally throughout this work: for example, see Figures 6.10 and 7.6.

A strange attractor contains no periodic orbits. However, buried in the strange attractor lies a host of unstable periodic orbits. It is these orbits that form the foundation of the topological analysis methods that are presented in this work. Good approximates to these orbits can be extracted from experimental data.

How do you search for chaotic behavior? A simple, very effective way to do this is to study the motion at a sequence of values of some control parameter. A suitable tool is called a bifurcation diagram. This amounts to a plot of one of the phase space coordinates as a function of one of the control parameters. Phase space plots of the logistic map $x' = \lambda x(1 - x)$ are presented in Figures 2.3 and 2.4. A practiced eye (training time ~ 2 s) can easily distinguish chaotic from nonchaotic behavior.

Is there a “smoking gun” for chaotic behavior? No. The original discovery [22] that launched a thousand studies was that a period-doubling cascade was a prelude to chaos and that a number of invariants were associated with such cascades (see Preface). The possibility of confirming these predictions, or showing that they were not correct, attracted a number of experimentalists into this field (cf. [10]). Having said that, not all “routes to chaos” go by the period-doubling pathway.

What topological tools will be used? We will study how the unstable periodic orbits that exist in plenty in a strange attractor are organized. To do this we will rely on the Gauss linking number of a pair of closed orbits as well as a closely related idea (relative rotation rates). We will also study braids and introduce cardboard-type structures that serve to hold all the periodic orbits in a strange attractor in a very simple way. Toward the end of this work we will introduce more venerable topological ideas: Euler characteristic, genus, isotopy, and so on.

1.2

Laser with Modulated Losses

The possibility of observing chaos in lasers was originally demonstrated by Arecchi *et al.* [23] and by Gioggia and Abraham [24]. The use of lasers as a testbed for generating deterministic chaotic signals has two major advantages over fluid and chemical systems, which until that time had been the principal sources for chaotic data:

1. Time scales intrinsic to a laser (10^{-7} – 10^{-3} s) are much shorter than time scales in fluid experiments and oscillating chemical reactions. This is important for experimentalists, since it is possible to explore a very large parameter range during a relatively short time.
2. Reliable laser models exist in terms of a small number of ordinary differential equations whose solutions show close qualitative similarity to the behavior of the lasers that are modeled [25, 26].

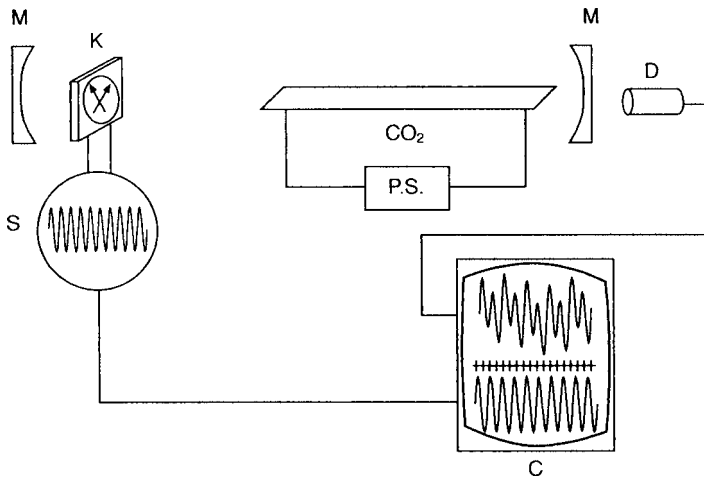


Figure 1.1 This schematic representation of a laser with modulated losses shows the carbon dioxide tube (CO₂); power source (P.S.); mirrors (M); Kerr cell (K); signal generator (S); detector (D); and computer, oscilloscope, and

recorder (C). A variable electric field across the Kerr cell rotates its polarization direction and modulates the electric field within the cavity.