Robert Gilmore, Marc Lefranc

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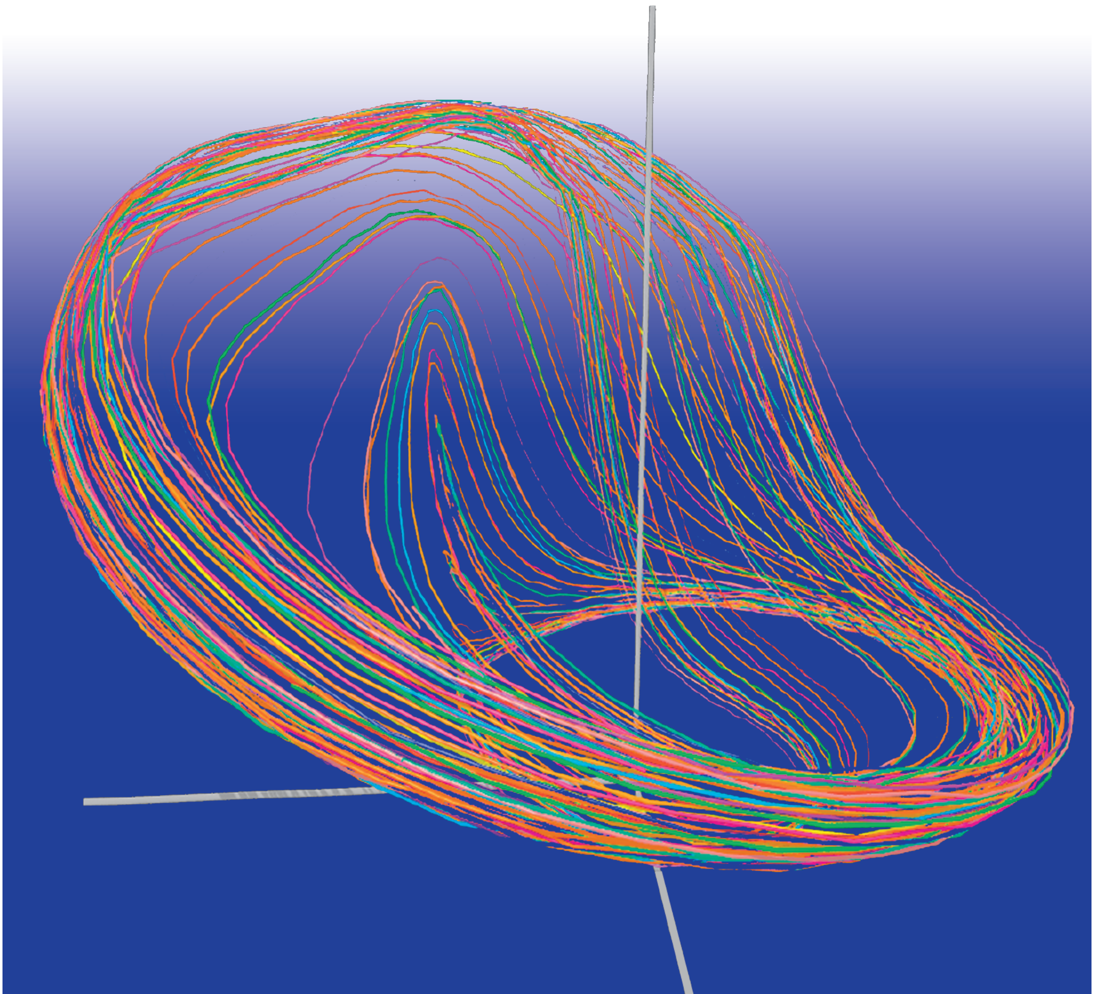
The Topology of Chaos

Alice in Stretch and Squeezeland
Second Revised and Enlarged Edition



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Contents

Preface to Second Edition

Since the appearance of the First Edition a decade ago our understanding of the relation between Topology and Chaos has grown quite a bit. This growth is reflected in the increased size of the Second Edition. For the most part the additional material is present in Chapters 11 and 12, and Appendix B of the current version. The two new chapters have been inserted between the first ten and the last two chapters of the First Edition.

We have made small changes in the first ten chapters. The principal change can be seen in Chapter One. This Chapter has been largely rewritten to make the entire work more accessible to someone first coming to the field. Chapter 2 contains two new short sections about homoclinic tangles and annulus maps. A brief discussion of embeddings has been largely expanded and relocated from Chapter 6 to Appendix B of the present work. Some recent beautiful experimental work done in Zaragoza, Spain, has explored the perestroikas that branched manifolds can undergo. This work has been included in Chapter 7. In Chapter 9, a new section summarizes the essential mathematical aspects of orbit forcing and topological entropy, including train track algorithms, and points to the relevant litterature. How knots can be used to compute entropy in real systems is also illustrated in a fluid experiment and in an optical system.

Unfortunately there has been little progress in our understanding of flows in higher dimensions (Chapter 11, First Edition) and too little development in the program for dynamical systems (Chapter 12, First Edition). These appear essentially unchanged as Chapters 13 and 14 in the present edition. Still, a new section in Chapter 13 presents an interesting proposal to generalize braids to dynamical triangulations of periodic points. This approach appears to be equivalent to the conventional one in three dimensions, and adapts naturally to phase spaces of any dimension.

Bounding Tori are introduced in Chapter 11. These two-dimensional surfaces enclose three-dimensional strange attractors. They have an elegant classification that goes back more than two centuries to the earlier great topologists (Euler). These structures serve to place yet another identifying tag on low-dimensional strange attractors.

There is a 'Representation Theory' for strange attractors that is similar in spirit, if not in detail, to the 'representation theory' for groups and algebras developed over a century ago. The representation theory for three-dimensional dynamical systems is now complete and presented in Chapter 12. For higher-dimensional dynamical systems the path has been blazed but not yet traversed. The representation theory provides a satisfying answer to the troubling question: "When you analyze an embedding of data generated by a chaotic dynamical system, what do you learn about the dynamical system and what do you learn about the embedding?"

The new Appendix B is devoted to the black magic of Embeddings. There are procedures for attempting to create embeddings and there are several different types of tests to assay whether an embedding has in fact been achieved. Some procedures are more reliable, others less so. These and other questions are explored in this Appendix.

We have taken this opportunity to correct mistakes that have crept into the First Edition. Hopefully, there are none in this version, but corrections and suggestions are welcomed and will be available online at the book's website: http://www.thetopologyofchaos.net/. We would like to thank friends and colleagues for pointing out mistakes (with a special mention to Michel Nizette and Mihir Khadilkar), pointing to places where our writing could have/should have been clearer, and most important, for their support and encouragement during the preparation of the Second Edition.

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Preface to the First Edition

Before the 1970s opportunities sometimes arose for physicists to study nonlinear systems. This was especially true in fields like fluid dynamics and plasma physics, where the fundamental equations are nonlinear and these nonlinearities masked (and still mask) the full spectrum of spectacularly rich behavior. When possible, we avoided being dragged into the study of abstract nonlinear systems. For we believed, to paraphrase a beautiful generalization of Tolstoy, that

All linear systems are the same.
Each nonlinear system is nonlinear in its own way.

At that time we believed that one could spend a whole lifetime studying the nonlinearities of the van der Pol oscillator [1, 2] and wind up knowing next to nothing about the behavior of the Duffing oscillator.

Nevertheless, other intrepid researchers had been making an assault on the complexities of nonlinear systems. Smale [3] described a mechanism responsible for generating a great deal of the chaotic behavior that has been studied up to the present time. Lorenz, studying a drastic truncation of the Navier–Stokes equation, discovered and described "sensitive dependence on initial conditions" (1963). The rigid order in which periodic orbits are created in the bifurcation set of the logistic map, and in fact any unimodal map of an interval to itself, was described by May [4] and by Metropolis, Stein, and Stein [5].

Still, there was a reluctance on the part of most scientists to indulge in the study of nonlinear systems. This all changed with Feigenbaum's discoveries (1978). He showed that scaling invariance in period-doubling cascades leads to quantitative (later, qualitative) predictions. These are the scaling ratios:

$$\delta = 4.669\,201\,609\,102\,9\ldots \quad \text{control parameter space}$$
$$\alpha = -2.502\,907\,875\,095\,9\ldots \quad \text{state variable space}$$

that are eigenvalues of a renormalization transformation. The transformation in the attitude of scientists is summarized by Gleick's [6] statement:

"It was a very happy and shocking discovery that there were structures in nonlinear systems that are always the same if you looked at them the right way."

This discovery launched an avalanche of work on nonlinear dynamical systems. Old experiments, buried and forgotten because of instabilities or unrepeatability due to incompetent graduate students (in their advisors' opinions) were resurrected and pushed as groundbreaking experiments exhibiting "first observations" of chaotic behavior (by these same advisors). And many new experiments were carried out, at first to test Feigenbaum's scaling predictions, then to test other quantitative predictions, then just to see what would happen.

Some of the earliest experiments were done on fluids, since the fundamental equations were known and are nonlinear. However, these experiments often suffered from the long time scales (days, weeks, or months) required to record a decent data set. Oscillating chemical reactions (e.g., the Belousov–Zhabotinskii reaction) yielded a wide spectrum of periodic and chaotic behavior that was relatively easy to control and to tune. These data sets could be generated in hours or days. Nonlinear electric circuits were also extensively studied, although there was (and still is) a prejudice to regard them with a jaundiced eye as little more than analog computers. Such data sets could be generated very quickly (seconds to minutes) – almost as fast as numerical simulations. Finally, laser laboratories contributed in a substantial way to very quickly (milliseconds to minutes) building up extensive and widely varying banks of chaotic data.

It was at this time (1988), about 10 years into the "nonlinear science" revolution, that one of the authors (R.G.) was approached by his colleague (J.R. Tredicce, then at Drexel, now at the Institut Non Linéaire de Nice) with the proposition: "Bob, can you help me explain my data?" (Chapter 1). So we swept the accumulated clutter off my desk and deposited his data. We looked, pushed, probed, discussed, studied, etc. for quite a while. Finally, I replied: "No." Tredicce left with his data. But he is very smart (he is an experimentalist!) and returned the following day with the same pile of stuff. The conversation was short and effective: "Bob," (still my name), "I'll bet that you *can't* explain my data." (Bob sees red!) We sat down and discussed further. At the time two tools were available for studying chaotic data. These involved estimating Lyapunov exponents (dynamical stability) and estimating fractal dimensions (geometry). Both required lots of very clean data and long calculations. They provided real number(s) with no convincing error bars, no underlying statistical theory, and no independent way to verify these guesses. And at the end of the day neither provided any information on "how to model the dynamics."

Even worse: Before doing an analysis I would like to know what I am looking for, or at least know what the spectrum of possible results looks like. For example, when we analyze chemical elements or radionuclides, there is a periodic table of the chemical elements and another for the atomic nuclei that accommodate any such analyses. At that time, no classification theory existed for strange attractors.

In response to Tredicce's dare, I promised to (try to) analyze his data. But I pointed out that a serious analysis couldn't be done until we first had some handle on the classification of strange attractors. This could take a long time. Tredicce promised to be patient. And he was.

Our first step was to consider the wisdom of Poincaré, who had suggested about a century earlier that one could learn a great deal about the behavior of nonlinear systems by studying their unstable periodic orbits, which

> "... yield us the solutions so precious, that is to say, they are the only breach through which we can penetrate into a place which up to now has been reputed to be inaccessible."

This observation was compatible with what we learned from experimental data: the most important features that governed the behavior of a system, and especially that governed the perestroikas of such systems (i.e., changes as control parameters are changed) are the features that you can't see – the unstable periodic orbits.

Accordingly, my colleagues and I studied the invariants of periodic orbits, their (Gauss) linking numbers. We also introduced a refined topological invariant based on periodic orbits – the relative rotation rates (Chapter 4). Finally, we used these invariants to identify topological structures (branched manifolds or templates, Chapter 5), which we used to classify strange attractors "in the large." The result was that "low-dimensional" strange attractors (i.e., those that could be embedded in three-dimensional spaces) could be classified. This classification depends on the periodic orbits "in" the strange attractor, in particular, on their organization as elicited by their invariants. The classification is topological. That is, it is given by a set of integers (also by very informative pictures). Not only that, these integers can be extracted from experimental data. The data sets do not have to be particularly long or particularly clean – especially by fractal dimension calculation standards. Further, there are built-in internal self-consistency checks. That is, the topological analysis algorithm (Chapter 6) comes with reject/fail to reject test criteria. This is the first – and remains the only – chaotic data analysis procedure with rejection criteria.

Ultimately we discovered, through analysis of experimental data, that there is a secondary, more refined classification for strange attractors. This depends on a "basis set of orbits" that describes the spectrum of all the unstable periodic orbits "in" a strange attractor (Chapter 9).

The ultimate result is a doubly discrete classification of strange attractors. Both parts of this doubly discrete classification depend on unstable periodic orbits. The classification depends on identifying:

- A *branched manifold* – which describes the stretching and squeezing mechanisms that operate repetitively on a flow in phase space to build up a (hyperbolic) strange attractor and to organize all the unstable periodic orbits in the strange attractor in a unique way. The branched manifold is identified by the spectrum of the invariants of the periodic orbits that it supports.
- A *basis set of orbits* – which describes the spectrum of unstable periodic orbits in a (nonhyperbolic) strange attractor.

The perestroikas of branched manifolds and of basis sets of orbits in this doubly discrete classification obey well-defined topological constraints. These constraints provide both a rigidity and a flexibility for the evolution of strange attractors as control parameters are varied.
Along the way we discovered that dynamical systems with symmetry could be related to dynamical systems without symmetry in very specific ways (Chapter 10). As usual, these relations involve both a rigidity and a flexibility that are as surprising as they are delightful.

Many of these insights are described in the paper [7], which forms the basis for part of this book. We thank the editors of this journal for their policy of encouraging the transformation of research articles into a longer book format.

The encounter (falling in love?) of the other author (M.L.) with topological analysis dates back to 1991, when he was a Ph.D. student at the University of Lille, struggling to extract information from the very same type of chaotic laser that Tredicce was using. At that time, Marc was computing estimates of fractal dimensions for his laser. But the estimates depended very much on the coordinate system used and gave no insight into the mechanisms responsible for chaotic behavior, even less into the succession of the different behaviors observed. This was very frustrating. There had been this very intringuing paper in *Physical Review Letters* about a "characterization of strange attractors by integers," with appealing ideas and nice pictures. But as with many short papers, it was difficult to understand how you should proceed when faced with a real experimental system. Topological analysis struck back when Pierre Glorieux, then Marc's advisor, came back to Lille from a stay in Philadephia and handed him a preprint from the Drexel team, saying, "You should have a look at this stuff." The preprint was about topological analysis of the Belousov–Zhabotinskii reaction, a real-life system. It was the Rosetta Stone that helped put pieces together. Soon after, pictures of braids constructed from laser signals were piling up on his desk. They were absolutely identical to those extracted from the Belousov–Zhabotinskii data and described in the preprint. There was universality in chaos if you looked at it with the right tools. Eventually, the system that had motivated topological analysis in Philadephia, the CO_2 laser with modulated losses, was characterized in Lille and shown to be described by a horseshoe template. Indeed, Tredicce's laser could not be characterized by topological analysis because of long periods of zero output intensity that prevented invariants from being reliably estimated. The high signal-to-noise ratio of the laser in Lille allowed us to use a logarithmic amplifier and to resolve the structure of trajectories in the zero intensity region.

But a classification is only useful if there exist different classes. Thus, one of the early goals was to find experimental evidence of a topological organization that would differ from the standard Smale horseshoe. At that time, some regimes of the modulated CO_2 laser could not be analyzed for lack of a suitable symbolic encoding. The corresponding Poincaré sections had peculiar structures that, depending on the observer's mood, suggested a doubly iterated horsehoe or an underlying three-branch manifold. Since the complete analysis could not be carried out, much time was spent on trying to find at least one orbit that could not fit the horseshoe template. The result was extremely disappointing: For every orbit detected, there was at least one horseshoe orbit with identical invariants. One of the most important lessons of Judo is that if you experience resistance when pushing, you should pull (and vice versa). Similarly, this failed attempt to find a nonhorseshoe template turned into techniques to determine underlying templates when no symbolic coding is available and to construct such codings using the information extracted from topological invariants.

But the search for different templates was not over. Two of Marc's colleagues, Dominique Derozier and Serge Bielawski, proposed that he study a fiber-optic laser they had in their laboratory (that was the perfect system for studying knots). This system exhibits chaotic tongues when the modulation frequency is near a subharmonic of its relaxation frequency: It was tempting to check whether the topological structures in each tongue differed. That was indeed the case: The corresponding templates were basically horseshoe templates but with a global torsion increasing systematically from one tongue to the other. A Nd:YAG laser was also investigated. It showed similar behavior, until the day when Guillaume Boulant, the Ph.D. student working on the laser, came to Marc's office and said, "I have a weird data set." Chaotic attractors were absolutely normal, return maps resembled the logistic map very much, but the invariants were simply not what we were used to. This was the first evidence of a reverse horseshoe attractor. How topological organizations are modified as a control parameter is varied was the subject of many discussions in Lille in the following months; a rather accurate picture finally emerged, and papers began to be written. In the last stages, Marc did a bibliographic search just to clear his mind, and … a recent 22-page *Physical Review* paper, by McCallum and Gilmore, turned up. Even though it was devoted to the Duffing attractor, it described with great detail what was happening in our lasers as control parameters were modified. Every occurrence of "we conjecture that" in the papers was hastily replaced by "our experiments confirm the theoretical prediction …," and papers were sent to *Physical Review*. They were accepted 15 days later, with a very positive review. Soon after, the referee contacted us and proposed a joint effort on extensions of topological analysis. The referee was Bob, and this was the start of what we hope will be a long-lasting collaboration.

It would indeed be very nice if these techniques could be extended to the analysis of strange attractors in higher (than three) dimensions. Such an extension, if it is possible, cannot rely on the most powerful tools available in three dimensions. These are the topological invariants used to tease out information on how periodic orbits are organized in a strange attractor. We cannot use these tools (linking numbers, relative rotation rates) because knots "fall apart" in higher dimensions. We explore (Chapter 11) an inviting possibility for studying an important class of strange attractors in four dimensions. If a classification procedure based on these methods is successful, the door is opened to classifying strange attractors in R^n, $n > 3$. A number of ideas that may be useful in this effort have already proved useful in two closely related fields (Chapter 12): lie group theory and singularity theory.

Some of the highly technical details involved in extracting templates from data have been archived in the appendix. Other technical matters are archived at our web sites.[1)]

Much of the early work in this field was done in response to the challenge by J.R. Tredicce and carried out with my colleagues and close friends: H.G. Solari, G.B. Mindlin, N.B. Tufillaro, F. Papoff, and R. Lopez-Ruiz. Work on symmetries was done with C. Letellier. Part of the work carried out in this program has been supported by the National Science Foundation under grants NSF 8843235 and NSF 9987468. Similarly, Marc would like to thank colleagues and students with whom he enjoyed working and exchanging ideas about topological analysis: Pierre Glorieux, Ennio Arimondo, Francesco Papoff, Serge Bielawski, Dominique Derozier, Guillaume Boulant, and Jérôme Plumecoq. Bob's stays in Lille were partially funded by the University of Lille, the Centre National de la Recherche Scientifique, Drexel University under sabbatical leave, and by the NSF.

Last and most important, we thank our wives Claire and Catherine for their warm encouragement while physics danced in our heads, and our children, Marc and Keith, Clara and Martin, who competed with our research, demanded our attention, and, in doing so, kept us human.

Lille, France, January 2002 *Robert Gilmore and Marc Lefranc*

1) http://einstein.drexel.edu/directory/faculty/Gilmore.html and http://www.phlam.univ-lille1.fr/perso/lefranc.html. Last access: 1 January 2002.

1
Introduction

The subject of this book is chaos as seen through the filter of topology. The origin of this book lies in the analysis of data generated by a dynamical system operating in a chaotic regime. Throughout this book we develop topological tools for analyzing chaotic data and then show how they are applied to experimental data sets.

More specifically, we describe how to extract, from chaotic data, topological signatures that determine the stretching and squeezing mechanisms that act on flows in phase space and that are responsible for generating chaotic data.

In the first section of this introductory chapter we very briefly review some of the basic ideas from the field of nonlinear dynamics and chaos. This is done to make the work as self-contained as possible. More in-depth treatment of these ideas can be found in the references provided.

In the second section we describe, for purposes of motivation, a laser that has been operated under conditions in which it behaved chaotically. The topological methods of analysis that we describe in this book were developed in response to the challenge of analyzing chaotic data sets generated by this laser.

In the third section we list a number of questions we would like to be able to answer when analyzing a chaotic signal. None of these questions can be addressed by the older tools for analyzing chaotic data. The older methods involve estimates of the spectrum of Lyapunov exponents and estimates of the spectrum of fractal dimensions. The question that we would particularly like to be able to answer is this: How does one model the dynamics? To answer this question we must determine the stretching and squeezing mechanisms that operate together – repeatedly – to generate chaotic data. The stretching mechanism is responsible for *sensitivity to initial conditions* while the squeezing mechanism is responsible for *recurrent non-periodic behavior*. These two mechanisms operate repeatedly to generate a strange attractor with a self-similar structure.

A new analysis method, topological analysis, has been developed to respond to the fundamental question just stated [7, 8]. At the present time this method is suitable only for strange attractors that can be embedded in three-dimensional spaces. However, for such strange attractors it offers a complete and satisfying resolution to this question. The results are previewed in the fourth section of this chapter. In the final section we provide a brief overview of the organization of this book. In particular, we summarize the organization and content of the following chapters.

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It is astonishing that the topological analysis tools that we describe have provided answers to more questions than we had originally asked. This analysis procedure has also raised more questions than we have answered. We hope that the interaction between experiment and theory and between old questions answered and new questions raised will hasten the evolution of the field of nonlinear dynamics.

1.1
Brief Review of Useful Concepts

There are a number of texts that can serve as excellent introductions to the study of nonlinear dynamics and chaos. These include [9–21]. Any one of these can be used to fill in details that we may pass by a little too quickly in our study. There are also many texts that serve as introductions to topology. All cover far more material than we use here. As a result, we do not recommend that a reader invest time in any one of these texts. We will introduce the topological concepts as needed as we proceed.

For now we review very briefly some of the foundational ideas of chaos.

What Is chaos? We take the following as a useful definition for chaos, or chaotic motion. Chaos is motion that is

1. deterministic
2. bounded
3. nonperiodic
4. sensitive to initial conditions.

Where does this motion take place? It is useful to describe the state of a physical system by a set of coordinates. The most convenient way to do this is to establish a phase space. A point in the phase space describes the state of the physical system. The "motion" described above is that of a point in the phase space. For example, the phase space needed to describe the motion of two particles in a plane is eight-dimensional: two coordinates and two velocity components are required to describe the state of each particle's motion. We will work with smaller phase spaces.

What is "deterministic motion"? The motion of the coordinates x_i of a point in an N-dimensional phase space is governed by a set of N first-order ordinary differential equations:

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_N; c). \tag{1.1}$$

The coordinates $x \in R^N$ are called *phase space coordinates* and the coordinates $c \in R^K$ are called *control parameters*. A set of N equations of this type is called a *dynamical system*.

What is "bounded" motion? The trajectory defined by the equations of motion (1.1) can be parameterized by the time coordinate: $x(t)$. *Bounded* means that the maximum distance between any two points on the trajectory over all times is less than infinity: $\text{Max}_{(t_1,t_2)}||x(t_1) - x(t_2)|| < \infty$.

What is periodic motion? This is motion that returns to its starting point after an elapsed time T: $x(t) = x(t + T)$, for some $T > 0$. T is called the *period* if there is no smaller positive value (for example, $T/2$) for which this equation is true.

What is "sensitivity to initial conditions"? A point x_1 in phase space can serve as an initial condition for a trajectory through it: $x_1(t)$. Two nearby points can serve as initial conditions for two trajectories, one starting at each point. At first the two trajectories remain close to each other. If the distance between them grows exponentially with time, $||x_2(t) - x_1(t)|| \simeq e^{\lambda t}$ with $\lambda > 0$, then the system is said to exhibit sensitivity to initial conditions. This means that, although the evolution is deterministic, the future position of an initial condition becomes unpredictable after some time. The term λ is called a *Lyapunov exponent*.

How do you visualize chaotic motion? A very convenient way to do this is to view the trajectory of an initial condition in the phase space. It often happens that the trajectory "wanders around" in the phase space until it "settles down" onto, or is "attracted to," a reasonably well-defined structure. This structure, when it exists, is called by mathematicians an Ω limit set and by physicists a strange attractor or a chaotic attractor. Plots of strange attractors appear liberally throughout this work: for example, see Figures 6.10 and 7.6.

A strange attractor contains no periodic orbits. However, buried in the strange attractor lies a host of unstable periodic orbits. It is these orbits that form the foundation of the topological analysis methods that are presented in this work. Good approximates to these orbits can be extracted from experimental data.

How do you search for chaotic behavior? A simple, very effective way to do this is to study the motion at a sequence of values of some control parameter. A suitable tool is called a bifurcation diagram. This amounts to a plot of one of the phase space coordinates as a function of one of the control parameters. Phase space plots of the logistic map $x' = \lambda x(1 - x)$ are presented in Figures 2.3 and 2.4. A practiced eye (training time $\sim$ 2 s) can easily distinguish chaotic from nonchaotic behavior.

Is there a "smoking gun" for chaotic behavior? No. The original discovery [22] that launched a thousand studies was that a period-doubling cascade was a prelude to chaos and that a number of invariants were associated with such cascades (see Preface). The possibility of confirming these predictions, or showing that they were not correct, attracted a number of experimentalists into this field (cf. [10]). Having said that, not all "routes to chaos" go by the period-doubling pathway.

What topological tools will be used? We will study how the unstable periodic orbits that exist in plenty in a strange attractor are organized. To do this we will rely on the Gauss linking number of a pair of closed orbits as well as a closely related idea (relative rotation rates). We will also study braids and introduce cardboard-type structures that serve to hold all the periodic orbits in a strange attractor in a very simple way. Toward the end of this work we will introduce more venerable topological ideas: Euler characteristic, genus, isotopy, and so on.

1.2 Laser with Modulated Losses

The possibility of observing chaos in lasers was originally demonstrated by Arecchi *et al.* [23] and by Gioggia and Abraham [24]. The use of lasers as a testbed for generating deterministic chaotic signals has two major advantages over fluid and chemical systems, which until that time had been the principal sources for chaotic data:

1. Time scales intrinsic to a laser (10^{-7}–10^{-3} s) are much shorter than time scales in fluid experiments and oscillating chemical reactions. This is important for experimentalists, since it is possible to explore a very large parameter range during a relatively short time.
2. Reliable laser models exist in terms of a small number of ordinary differential equations whose solutions show close qualitative similarity to the behavior of the lasers that are modeled [25, 26].

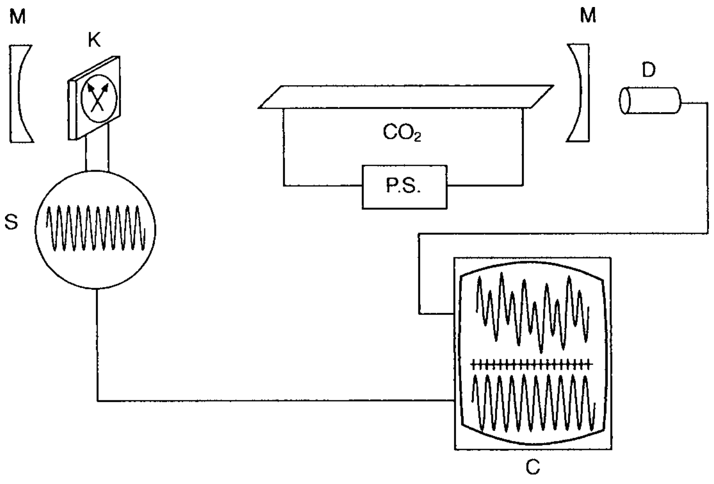

Figure 1.1 This schematic representation of a laser with modulated losses shows the carbon dioxide tube (CO_2); power source (P.S.); mirrors (M); Kerr cell (K); signal generator (S); detector (D); and computer, oscilloscope, and recorder (C). A variable electric field across the Kerr cell rotates its polarization direction and modulates the electric field within the cavity.