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Bruno Benedetti
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Combinatorial Methods in Topology and Algebra

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Bruno Benedetti • Emanuele Delucchi • Luca Moci
Editors

Combinatorial Methods in Topology and Algebra

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Editors
Bruno Benedetti
Department of Mathematics
University of Miami
Coral Gables
Florida, USA

Emanuele Delucchi
Département de Mathématiques
Université de Fribourg
Fribourg, Switzerland

Luca Moci
IMJ-PRG
Université Paris-Diderot - Paris 7
Paris, France

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Contents

Part II

Introduction

Bruno Benedetti, Emanuele Delucchi, and Luca Moci

Combinatorics and discrete geometry have been studied since the beginning of mathematics. Yet it is only in the last 50 years that combinatorics has flourished, with striking structural developments and a growing field of applications. Part of the reason for this blossoming may lie in the startling developments of computer science, which have taught us to look at mathematics with algorithmic eyes.

Moreover, many new connections between combinatorics and classical areas of mathematics, such as algebra and geometry, have emerged since the 70s. With no claim of completeness, let us provide (without references) five examples.

Hyperplane Arrangements A finite collection of linear one-codimensional subspaces in a complex vector space V is called an arrangement of hyperplanes. The intersection pattern of these hyperplanes gives rise to a rich combinatorial structure (see below under "Matroid") bearing a subtle relationship with the topology of the space obtained by removing the hyperplanes from V. Classical objects such as configuration spaces arise as special instances of these spaces which, in general, enjoy some nice topological properties—for instance, they are *minimal* (e.g., they

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B. Benedetti (✉)
Department of Mathematics, University of Miami, Coral Gables, Florida, USA
e-mail: bruno@math.miami.edu

E. Delucchi
Université de Fribourg, Fribourg, Switzerland
e-mail: emanuele.delucchi@unifr.ch

L. Moci
IMJ-PRG, Université Paris-Diderot - Paris 7, Paris, France
e-mail: lucamoci@hotmail.com

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have the homotopy type of CW complexes where, in every dimension, the number of cells equals the rank of the homology).

From the point of view of algebraic geometry, an arrangement is defined by a product $p(z)$ of degree-one homogeneous polynomials. One of the main topics of current research in this field is the study of the *Milnor fiber* $p^{-1}(1)$ of the arrangement.

Coxeter Groups A Coxeter group W is any group presented as

$$\langle s_1, \ldots, s_n \mid \underbrace{s_i s_j s_i \ldots}_{m_{ij}\text{ terms}} = \underbrace{s_j s_i s_j \ldots}_{m_{ij}\text{ terms}} \rangle$$

with $m_{ii} = 1$ and $2 \leq m_{ij} \leq \infty$ for all $i \neq j$, where $1 \leq i, j \leq n$ and $m_{ij} = \infty$ means that no condition on $s_i s_j$ is imposed. Symmetric groups and dihedral groups are of this type: indeed, the name of these groups reveals their origin in the study of regular polytopes by H.S.M. Coxeter. The combinatorics of Coxeter groups is very rich and deeply connected with the representation theory of Lie algebras, the algebraic geometry of flag varieties, and the topology of (real) reflection arrangements. With every pair of elements in W one can associate a *Kazhdan-Lusztig polynomial*. The coefficients of these polynomials are non-negative, and (when W is finite) they can be expressed in terms of the intersection cohomology Schubert varieties.

Removing the restriction $m_{ii} = 1$ from the presentation of a Coxeter group, we obtain the associated *Artin group*. A theorem of Deligne shows that the orbit space of the action of a finite-type Coxeter group on the complement of the hyperplane arrangement defined by (the complexification of) its reflection hyperplanes is a classifying space for the associated Artin group.

Matroids Matroids are certain types of set systems whose study was initiated by H. Whitney in the 1930s as an abstract common generalization of properties of linear algebra and graph theory. One possible definition is the following. A *matroid* on a finite set E is a nonempty collection $\mathfrak{B}$ of subsets of E such that the *exchange axiom* holds:

- If $A, B \in \mathfrak{B}$, for any $a \in A \setminus B$ there exists an element $b \in B \setminus A$ such that $A \setminus \{a\} \cup \{b\}$ is in $\mathfrak{B}$.

The elements of $\mathfrak{B}$ are called *bases*, and from this definition the connection to linear algebra should be apparent: if E is any finite subset of a $\mathbb{K}$-vector space V, the maximal linearly independent subsets of E form a matroid, which in this case is said to be *realizable over* $\mathbb{K}$. However, not all matroids arise in this way: some matroids are realizable only over some field, and some are not realizable over any field. Characterization of realizability is one of the main areas of research in matroid theory. The connection with graphs is as follows: if E is the edge set of a connected graph, the set of all edge sets of spanning trees satisfies the above definition and thus forms a matroid on E.

The interplay of matroid theory with algebraic geometry and commutative algebra has undergone thriving development in recent years, one of the main bridges being the language of *tropical geometry*.

Stanley–Reisner Ideals Let Δ be a simplicial complex on n vertices. The *Stanley–Reisner ideal* I_Δ is defined by

$$I_\Delta \overset{\text{def}}{=} \bigcap_{F \text{ facet of } \Delta} (x_i \,:\, i \notin F).$$

Since the ideals on the right-hand side are monomial and prime, I_Δ is monomial and radical. The uniqueness of prime decompositions of ideals implies that I_Δ determines Δ uniquely. Interestingly, every radical monomial ideal I is of the form I_Δ for a suitable complex Δ. This shows that there is a one-to-one correspondence between simplicial complexes (on n vertices) and radical monomial ideals (in n variables)—they are thus essentially the same thing. This allows transfer of properties back and forth between the two worlds: for example, one can characterize topologically the simplicial complexes Δ for which the ring S/I_Δ is Cohen–Macaulay. A well studied combinatorial property implying Cohen–Macaulayness is, for example, *shellability*.

Face Vectors of Polytopes Given a simplicial complex Δ, we denote by f_i the number of i-dimensional faces; by convention $f_{-1} = 1$. The f-*vector* of C is the vector $(f_{-1}, f_0, \ldots, f_d)$. The h-vector $(h_0, \ldots, h_d)$ is defined by the polynomial equality

$$\sum_{i=0}^{d} h_i X^{d-i} = \sum_{i=0}^{d} f_{i-1}(X-1)^{d-i}.$$

A natural question is: What integer vectors can arise as f-vectors of triangulated spheres?

When the sphere is the boundary of a polytope, the question was settled by the so-called g-theorem, proved by Billera-Lee and by Stanley in 1979 using commutative algebra and toric varieties and thereby giving a new stimulus to these fields. In the general case, progress was made by S. Murai, who in 2007 proved that a large family of shellable spheres, most of which are non-polytopal, satisfies the “Hard Lefschetz property”, proved by Stanley for polytopes as a crucial step in his contribution to the g-theorem.

A consequence of the g-theorem is that h-numbers form a *unimodal* sequence, i.e., $h_0 \leq h_1 \leq \ldots \leq h_{\lfloor d/2 \rfloor} \geq \ldots \geq h_d$. Recently, Murai and Nevo proved the *generalized lower bound conjecture*, which claims the following: if in the h-vector of a d-polytope P one sees $h_{k-1} = h_k$ for some $k \leq d/2$, then there is a triangulation of P without simplices of dimension $\leq d-k$.

The present volume arises from a workshop on these interactions that was especially devoted to promoting outstanding young researchers. This INdAM Conference, entitled “Combinatorial Methods in Topology and Algebra” (or CoMeTA for short) took place in Cortona in September 2013. The detailed program of the conference is available at the website www.cometa2013.org.

About this Book

In the first part we have collected short surveys, which may be viewed as written and expanded versions of the talks given by the various speakers. Since the quality level of the lectures was very high, we believe that the inclusion of such material may be of great help for future studies. These surveys cover various topics:

(i) Hyperplane arrangements.
(ii) Matroids.
(iii) Polytopes and geometric combinatorics.
(iv) f-vectors of cellular complexes and triangulations.
(v) Combinatorial commutative algebra.
(vi) Coxeter groups and Kazhdan–Lusztig and Eulerian polynomials.
(vii) Combinatorial approaches to physics and analysis.

The second part consists of three peer-reviewed full research papers.

- The first sheds new light on positive sum systems. If $x_1, \dots, x_n$ are real numbers summing to zero, consider the family P^+ of all subsets $J \subseteq [n] := \{1, 2, \dots, n\}$ such that $\sum_{j \in J} x_j > 0$. Björner proves that the order complex of P^+, viewed as a poset under set containment, triangulates a shellable ball, whose f-vector depends only on n, and whose h-polynomial is the classical Eulerian polynomial.
- The second investigates an unexpected action by the group S_{n+1} on the minimal projective De Concini–Procesi model associated to the braid arrangements of type A_{n-1}. The action naturally arises from the fact that this model is isomorphic to the moduli space $\overline{M}_{0,n+1}$ of genus 0 stable curves with $n + 1$ marked points.
- The third contribution focuses on Stanley's 1977 conjecture that the h-vectors of matroids are pure O-sequences. The conjecture is shown to hold in a few special cases, for example when the Cohen-Macaulay type is less than or equal to 3.

Part I

Extremal Graph Theory and Face Numbers of Flag Triangulations of Manifolds

Michał Adamaszek

Abstract We indicate how tools of extremal graph theory, mainly the stability method for Turán graphs, can be applied to derive upper bounds for face numbers of flag triangulations of spheres and manifolds.

1 Introduction

If $G = (V, E)$ is an arbitrary simple, finite, undirected graph, we denote by $\mathrm{Cl}(G)$ the *clique complex* of G, which is a simplicial complex defined as follows. The vertices of $\mathrm{Cl}(G)$ are the vertices of G and the faces of $\mathrm{Cl}(G)$ are those vertex sets which induce a clique (a complete subgraph) of G. The simplicial complexes which arise in this way are also known as *flag complexes*. This family includes for instance order complexes of posets and it appears in Gromov's theory of non-positive simplicial curvature [3].

A typical problem studied in enumerative combinatorics is to describe the f-vectors of interesting families of simplicial complexes. The f*-vector* $(f_0(K), \dots, f_d(K))$ of a d-dimensional complex K has as its ith entry, $f_i(K)$, the number of i-dimensional faces of K. The full classification of f-vectors of all flag complexes is probably impossible, although they are known to satisfy a number of non-trivial constraints [6]. We study this problem for the family of flag complexes which triangulate spheres and, more generally, homology manifolds and pseudomanifolds.

Note that G is the 1-skeleton of $\mathrm{Cl}(G)$. If we denote by $c_i(G)$ the number of cliques of cardinality i in G, then $f_i(\mathrm{Cl}(G)) = c_{i+1}(G)$, in particular $f_0(\mathrm{Cl}(G)) = |V(G)|$ and $f_1(\mathrm{Cl}(G)) = |E(G)|$. Our problem is thus equivalent to asking for the relations between clique numbers of graphs which satisfy some topological hypotheses.

M. Adamaszek (✉)
Fachbereich Mathematik, Universität Bremen, Bibliothekstraße 1, 28359 Bremen, Germany
e-mail: aszek@mimuw.edu.pl

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2 Upper Bounds

Let $K(n, s)$ denote the n-vertex balanced complete s-partite graph. It is uniquely determined by the requirement that the vertices can be split into s parts of sizes $\lfloor \frac{n}{s} \rfloor$ or $\lceil \frac{n}{s} \rceil$ each, with all possible edges between the parts and no edges within any part (see Fig. 1). By Turán's theorem this graph maximizes the number of edges among n-vertex graphs G which satisfy $c_{s+1}(G) = 0$.

Now let $J(n, s)$ be defined in the same way, except that we require each of the s parts to induce a cycle. We call this graph the *join* of s cycles of (almost) equal lengths. Note that $\mathrm{Cl}(J(n, s))$ is homeomorphic to the sphere S^{2s-1} (it is a topological join of s copies of S^1). The meta-statement we wish to advertise is that, provided n is large enough, $J(n, s)$ maximizes the number of edges among n-vertex graphs G for which $\mathrm{Cl}(G)$ is manifold-like in any reasonable sense.

More precisely, in [1] we show the following upper bound.

Theorem 1 *Let G be a graph with n vertices such that $\mathrm{Cl}(G)$ is a weak pseudomanifold of odd dimension $d = 2s - 1$, which satisfies the middle Dehn-Sommerville equation. If n is sufficiently large then we have*

$$|E(G)| \le \frac{s-1}{2s} n^2 + n \quad \big(\approx |E(J(n, s)|\big).$$

In particular, the conclusion holds when $\mathrm{Cl}(G)$ is a $(2s-1)$-dimensional homology manifold.

If $d = 3$ and $\mathrm{Cl}(G)$ is a homology 3-manifold the same bound was shown to hold for all n in [7]. A simplicial complex of dimension d is a *weak pseudomanifold* if every $(d-1)$-dimensional face belongs to exactly two facets. The graph G which achieves the upper bound of the theorem is $J(n, s)$. Moreover, one can expect this extremum to be stable, in the sense that the graphs for which $|E(G)|$ is close enough to the upper bound will be similar to $J(n, s)$. In [2] this was shown in dimension $d = 3$ in the form of the next theorem, which also contributes to the classification problem for f-vectors of flag 3-spheres.

Theorem 2 *Suppose G is a graph with n vertices such that $\mathrm{Cl}(G)$ is a 3-dimensional homology manifold and*

$$\frac{1}{4}n^2 + \frac{1}{2}n + \frac{17}{4} < |E(G)| \le \frac{1}{4}n^2 + n.$$

If n is sufficiently large then G is a join of two cycles of lengths $\frac{1}{2}n \pm O(\sqrt{n})$.

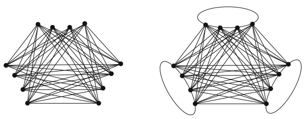

Fig. 1 Illustration for the definition of $K(n, s)$ (*left*) and $J(n, s)$ (*right*)

By Zykov's extension of Turán's theorem the graph $K(n, s)$ maximizes not only the number of edges, but in fact all clique numbers $c_2(G), \dots, c_s(G)$ among graphs G with n vertices and $c_{s+1}(G) = 0$. It is likely that the methods used for Theorems 1 and 2 can be extended to prove an analogous statement about maximality of higher face numbers of $\mathrm{Cl}(J(n, s))$. At the time of writing it appears that the following conjecture (also stated in [8]) can be turned into a theorem.

Conjecture 3 If G is a graph with n vertices such that $\mathrm{Cl}(G)$ is a homology manifold of odd dimension $d = 2s - 1$ and n is sufficiently large, then

$$c_k(G) \le c_k(J(n, s))$$

for all $1 \le k \le s$.

3 Proofs

The technique we use to study dense flag manifold triangulations was developed in [2]. It relies on the similarity between the graphs $K(n, s)$ and $J(n, s)$ coupled with the special role played by Turán's graphs $K(n, s)$ in extremal graph theory. A typical application of this technique goes along the following lines.

1. Suppose G is a graph with n vertices such that $\mathrm{Cl}(G)$ is a homology manifold of dimension $d = 2s - 1$. In the first step we use the middle Dehn-Sommerville equation for $\mathrm{Cl}(G)$ to conclude that $c_{s+1}(G)$ is a linear combination of the numbers $1, c_1(G), \dots, c_s(G)$. In particular $c_{s+1}(G) = O(n^s)$.
2. Now assume that G is as dense as $K(n, s)$, i.e. it has approximately $\frac{s-1}{2s}n^2$ or more edges. In a "typical" or "random" graph with this edge density a constant fraction of $(s + 1)$-tuples of vertices would span a clique. However, G has much fewer $(s+1)$-cliques, namely just $O(n^s)$. This has a consequence for the structure of G, which must be "similar" to $K(n, s)$, meaning that it can be obtained from $K(n, s)$ by adding or removing a relatively small number of edges (which can be $o(n^2)$ or even $O(n)$ depending on the specific problem). This step is known as the *stability method* in extremal graph theory, its origins going back to [5].
3. A graph which is similar to $K(n, s)$ is also similar to $J(n, s)$. The additional geometric properties of $\mathrm{Cl}(G)$, such as being a pseudomanifold or having homology spheres as face links, provide extra restrictions on the structure of G. They can now be used to rigidify G and conclude that it must be a join of cycles.

Let us mention that in the even dimensions $d = 2s$ this strategy is complicated by the fact that the extremal examples are (conjecturally) highly non-unique: one can take a join of $s - 1$ cycles with an *arbitrary* flag triangulation of S^2.

4 Conclusion

Much more is conjectured, than known, about the f-vectors of flag triangulations of spheres. In dimensions $d = 1$ and $d = 2$ they are easy to describe. In dimension $d = 3$ they can be classified up to, possibly, a finite number of exceptions, by the combined results of [2, 4, 7, 9]. The classification will be complete if Theorem 2 holds for all values of n, not just for sufficiently large ones. Note that already for $d = 3$ the classification relies on the deep result of Davis and Okun [4], namely the three-dimensional Charney-Davis conjecture. Full classification is available in dimension $d = 4$ (see [9]). In higher dimensions the only non-trivial restrictions known to hold are the upper bounds of Theorem 1. For more conjectures in this area see [7, 10].

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Combinatorial Stratifications and Minimality of Two-Arrangements

Karim A. Adiprasito

Abstract I present a result according to which the complement of any affine 2-arrangement in $\mathbb{R}^d$ is *minimal*, that is, it is homotopy equivalent to a cell complex with as many i-cells as its ith Betti number. To this end, we prove that the Björner–Ziegler complement complexes, induced by combinatorial stratifications of any essential 2-arrangement, admit perfect discrete Morse functions. This result extend previous work by Falk, Dimca–Papadima, Hattori, Randell, and Salvetti–Settepanella, among others.

A c-arrangement is a finite collection of distinct affine subspaces of $\mathbb{R}^d$, all of codimension c, with the property that the codimension of the non-empty intersection of any subset of $\mathfrak{A}$ is a multiple of c. For example, after identifying $\mathbb{C}$ with $\mathbb{R}^2$, any collection of hyperplanes in $\mathbb{C}^d$ can be viewed as a 2-arrangement in $\mathbb{R}^{2d}$. However, not all two-arrangements arise this way, cf. [10, Sect. III, 5.2] and [22]. In this paper, we study the complement $\mathfrak{A}^{\mathrm{c}} := \mathbb{R}^d \setminus \mathfrak{A}$ of any 2-arrangement $\mathfrak{A}$ in $\mathbb{R}^d$.

Subspace arrangements $\mathfrak{A}$ and their complements $\mathfrak{A}^{\mathrm{c}}$ have been extensively studied in several areas of mathematics. Thanks to the work by Goresky and MacPherson [10], the homology of $\mathfrak{A}^{\mathrm{c}}$ is well understood; it is determined by the *intersection poset* of the arrangement, which is the set of all nonempty intersections of its elements, ordered by reverse inclusion. In fact, the intersection poset determines even the homotopy type of the compactification of $\mathfrak{A}$ [23]. On the other hand, it does not determine the homotopy type of the complement of $\mathfrak{A}^{\mathrm{c}}$, even if we restrict ourselves to complex hyperplane arrangements [3, 16, 17], and understanding the homotopy type of $\mathfrak{A}^{\mathrm{c}}$ remains challenging.

A standard approach to study the homotopy type of a topological space X is to find a *model* for it, that is, a CW complex homotopy equivalent to it. By cellular homology any model of a space X must use at least $\beta_i(X)$ i-cells for each i, where β_i is the ith (rational) Betti number. A natural question arises: Is the complement of an arrangement *minimal*, i.e., does it have a model with *exactly* $\beta_i(X)$ i-cells for all

K.A. Adiprasito (✉)
Einstein Institute for Mathematics, Hebrew University of Jerusalem, Edmond J. Safra Campus, Givat Ram, 91904 Jerusalem, Israel
e-mail: sunya00@gmail.com

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i? Studying minimality is not without its motivations; it appears, for instance, in the study of abelian covers of X [14].

Building on previous work by Hattori [12], Falk [8], and Cohen–Suciu [6], around 2000 Dimca–Papadima [7] and Randell [15] independently showed that the complement of any complex hyperplane arrangement is a minimal space. Roughly speaking, the idea is to consider the distance to a complex hyperplane in general position as a Morse function on the Milnor fiber to establish a Lefschetz-type hyperplane theorem for the complement of the arrangement. An elegant inductive argument completes their proof.

On the other hand, the complement of an arbitrary subspace arrangement is, in general, *not* minimal. In fact, complements of subspace arrangements might have arbitrary torsion in cohomology (cf. [10, Sect. III, Theorem. A]). This naturally leads to the following question:

Problem 1 (Minimality) Is the complement $\mathfrak{A}^{\mathrm{c}}$ of every c-arrangement $\mathfrak{A}$ minimal?

The interesting case is $c = 2$. In fact, if c is not 2, the complements of c-arrangements, and even c-arrangements of pseudospheres (cf. [5, Sects. 8 and 9]), are easily shown to be minimal. In 2007, Salvetti–Settepanella [19] proposed a combinatorial approach to Problem 1, based on Forman's discretization of Morse theory [9]. Discrete Morse functions are defined on regular CW complexes rather than on manifolds; instead of critical points, they have combinatorially-defined *critical faces*. Any discrete Morse function with c_i critical i-faces on a complex C yields a model for C with exactly c_i i-cells. Salvetti–Settepanella studied discrete Morse functions on the *Salvetti complexes* [18], which are models for complements of complexified real arrangements. Remarkably, they found that all Salvetti complexes admit *perfect* discrete Morse functions, that is, functions with exactly $\beta_i(\mathfrak{A}^{\mathrm{c}})$ critical i-faces. Formans's Theorem now yields the desired minimal models for $\mathfrak{A}^{\mathrm{c}}$.

This tactic does not extend to the generality of complex hyperplane arrangements. However, models for complex arrangements, and even for c-arrangements, have been introduced and studied by Björner and Ziegler [5]. In the case of complexified-real arrangements, their models contain the Salvetti complex as a special case. While our notion of the combinatorial stratification is slightly more restrictive than Björner–Ziegler's, it still includes most of the combinatorial stratifications studied in [5]. For example, we still recover the $\mathbf{s}^{(1)}$-stratification which gives rise to the Salvetti complex. With these tools at hand, we can tackle Problem 1 combinatorially:

Problem 2 (Optimality of Classical Models) Are there perfect discrete Morse functions on the Björner–Ziegler models for the complements of arbitrary two-arrangements?

We are motivated by the fact that discrete Morse theory provides a simple yet powerful tool to study stratified spaces. On the other hand, there are several difficulties to overcome. In fact, Problem 2 is more ambitious than Problem 1 in many respects:

- Few regular CW complexes, even among the minimal ones, admit perfect discrete Morse functions. For example, many 3-balls [4] and many contractible 2-complexes [21] are not collapsible.
- There are few results in the literature predicting the existence of perfect Morse functions. For example, it is not known whether any subdivision of the 4-simplex is collapsible, cf. [13, Problem. 5.5].
- Solving Problem 2 could help in obtaining a more explicit picture of the attaching maps for the minimal model; compare Salvetti–Settepanella [19] and Yoshinaga [20].

We answer both problems in the affirmative.

Theorem 3 ([1]) *Any complement complex of any* 2*-arrangement* $\mathfrak{A}$ *in* S^d *or* $\mathbb{R}^d$ *admits a perfect discrete Morse function.*

Corollary 4 ([1]) *The complement of any affine* 2*-arrangement in* $\mathbb{R}^d$*, and the complement of any* 2*-arrangement in* S^d*, is a minimal space.*

A crucial step on the way to the proof of Theorem 3 is the proof of a Lefschetz-type hyperplane theorem for the complements of two-arrangements. The lemma we actually need is a bit technical, but roughly speaking, the result can be phrased in the following way:

Theorem 5 ([1]) *Let* $\mathfrak{A}^c$ *denote the complement of any affine* 2*-arrangement* $\mathfrak{A}$ *in* $\mathbb{R}^d$*, and let* H *be any hyperplane in* $\mathbb{R}^d$ *in general position with respect to* $\mathfrak{A}$*. Then* $\mathfrak{A}^c$ *is homotopy equivalent to* $H \cap \mathfrak{A}^c$ *with finitely many* e*-cells attached, where* $e = \lceil d/2 \rceil = d - \lfloor d/2 \rfloor$.

An analogous theorem holds for complements of c-arrangements ($c \neq 2$, with $e = d - \lfloor d/c \rfloor$); it is an immediate consequence of the analogue of Corollary 4 for c-arrangements, $c \neq 2$. Theorem 5 extends a result on complex hyperplane arrangements, which follows the classical Lefschetz theorem, applied to the Milnor fiber [7, 11, 15]. The main ingredients to our study are:

- The formula to compute the homology of subspace arrangements in terms of the intersection lattice, due to Goresky and MacPherson [10].
- The study of combinatorial stratifications as initiated by Björner and Ziegler [5].
- The study of the collapsibility of complexes whose geometric realizations satisfy certain geometric constraints, as discussed previous work of Benedetti and Adiprasito, cf. [2].
- The idea of Alexander duality for Morse functions, in particular the elementary notion of "out-j collapse".
- The notion of (Poincaré) duality of discrete Morse functions, which goes back to Forman [9]. This is used to establish discrete Morse functions on complement complexes.

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Random Triangular Groups

Sylwia Antoniuk, Tomasz Łuczak, and Jacek Świątkowski

Abstract Let $\langle S|R\rangle$ denote a group presentation, where S is a set of n generators while R is a set of relations consisting of distinct cyclically reduced words of length three. The above presentation is called a triangular group presentation and the group it generates is called a triangular group. We study the following model $\Gamma(n,p)$ of a random triangular group. The set of relations R in $\Gamma(n,p)$ is chosen randomly, namely every relation is present in R independently with probability p. We study how certain properties of a random group $\Gamma(n,p)$ change with respect to the probability p. In particular, we show that there exist constants $c, C > 0$ such that if $p < \frac{c}{n^2}$, then a.a.s. a random group $\Gamma(n,p)$ is a free group and if $p > C\frac{\log n}{n^2}$, then a.a.s. this group has Kazhdan's property (T). What is more interesting, we show that there exist constants $c', C' > 0$ such that if $\frac{C'}{n} < p < c'\frac{\log n}{n^2}$, then a.a.s. a random group $\Gamma(n,p)$ is neither free, nor has Kazhdan's property (T). We prove the above statements using random graphs and random hypergraphs.

The notion of a random group goes back to Gromov [3], who studied groups given by random group presentations. Let $\langle S|R\rangle$ denote a group presentation with a set of generators S and a set of relations R consisting of distinct cyclically reduced words of length three, that is R consists of words of the form abc, where $a, b, c \in S \cup S^{-1}$ and $a \neq b^{-1}, b \neq c^{-1}, c \neq a^{-1}$. The above presentation is called a *triangular group presentation* and the group it generates is called a *triangular group*.

The subject of our interest is the following model of a random triangular group.

Definition 1 Let $\Gamma(n,p)$ denote a model of a random triangular group given by a random group presentation $\langle S|R\rangle$ with n generators, in which the set of relations R is chosen randomly in the following way: each cyclically reduced word of length three over the alphabet $S \cup S^{-1}$ is present in R independently with probability p.

S. Antoniuk (✉) • T. Łuczak
Faculty of Mathematics and Computer Science, Adam Mickiewicz University, ul. Umultowska 87, 61-614 Poznań, Poland
e-mail: antoniuk@amu.edu.pl; tomasz@amu.edu.pl

J. Świątkowski
Instytut Matematyczny, Uniwersytet Wrocławski, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
e-mail: swiatkow@math.uni.wroc.pl

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We are especially interested in the asymptotic properties of groups in the $\Gamma(n, p)$ model. In particular, we say that the random group $\Gamma(n, p)$, where $p = p(n)$ is some function of n, has a given property *asymptotically almost surely (a.a.s.)*, if the probability that $\Gamma(n, p)$ has this property tends to 1 as $n \to \infty$.

In [4] Żuk investigated threshold functions for specific and important properties of random triangular groups, such as Kazhdan's property (T), the property of being a free group or the property of being a trivial group. However, we should mention that Żuk studied a slightly different model of a random triangular group in which, rather than picking every relation independently, we choose uniformly at random the whole set of relations R among all the sets of prescribed size. Żuk's results stated for the $\Gamma(n, p)$ model read as follows.

Theorem 2 (Żuk [4]) *Let* $\epsilon > 0$.

1. *If* $p < n^{-2-\epsilon}$, *then a.a.s.* $\Gamma(n, p)$ *is a free group.*
2. *If* $n^{-2+\epsilon} < p < n^{-3/2-\epsilon}$, *then a.a.s.* $\Gamma(n, p)$ *is infinite, hyperbolic and has Kazhdan's property (T).*
3. *If* $p > n^{-3/2+\epsilon}$, *then a.a.s.* $\Gamma(n, p)$ *is trivial.*

In our work we managed to determine threshold functions more precisely than just up to the $n^{o(1)}$ factor. Our main results are captured in the following two theorems.

Theorem 3 (Antoniuk et al. [2])
There exist constants $c, c', C, C' > 0$, *such that:*

1. *If* $p < \frac{c}{n^2}$, *then a.a.s* $\Gamma(n, p)$ *is a free group.*
2. *If* $\frac{c'}{n^2} < p < \frac{C' \log n}{n^2}$, *then a.a.s.* $\Gamma(n, p)$ *is neither free, nor has Kazhdan's property (T).*
3. *If* $p > \frac{C \log n}{n^2}$, *then a.a.s.* $\Gamma(n, p)$ *has Kazhdan's property (T).*

Here, it is worth mentioning that we managed not only to improve bounds on the critical probability, but what is more interesting, we discovered a new period in the evolution of a random triangular group, in which a.a.s. this group is neither free, nor has Kazhdan's property (T).

Theorem 4 (Antoniuk et al. [1]) *There exists a constant* $C > 0$, *such that for* $p > Cn^{-3/2}$ *a.a.s.* $\Gamma(n, p)$ *is trivial.*

The proof of the first part of Theorem 3 relies on the fact that if p is small enough, then the expected number of relations in a random presentation $\langle S|R\rangle$ is also small and therefore we can find a generator $a \in S$ such that a and a^{-1} are present in at most one relation. Consequently, using Tietze movements we can eliminate generators from presentation one by one obtaining in the end a presentation without any relations.

For the second part of the proof of Theorem 3 we use the fact that if $p > \frac{c'}{n^2}$ for sufficiently large constant $c' > 0$, the presentation complex C_P of a random presentation $\langle S|R\rangle$ is a.a.s. aspherical and therefore it is the classifying space for