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Yield Design

Jean Salençon

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Preface

> "One of the principal objects of theoretical research in any department of knowledge is to find the point of view from which the subject appears in its greatest simplicity"[1]

This book originates from the lecture notes for a course on yield design taught at Hong Kong City University during recent years. It is presented in the form of a survey of the theory of yield design, which brings together and summarizes the books and lecture notes I published in French on that topic when teaching at École Nationale des Ponts et Chausssées and École Polytechnique (Paris, France).

The terminology "yield design" has been chosen as a counterpart and translation of the French "Calcul à la rupture" or "Analyse à la rupture" which has been used for a long time by civil engineers and others to refer to stability analyses of structures where only the concepts of equilibrium and resistance are taken into account.

In an explicit form, such analyses have been carried out for nearly four centuries, if we take Galileo's *Discorsi* as a starting point of the story, but they were overshadowed by the achievements of the theory of elasticity in the 19th Century.

1 GIBBS J.W., *Proceedings of the American Academy of Arts and Sciences*, May 1880 – June 1881, XVI, VIII, Boston, University Press/John Wilson & Co., pp. 420–421, 1881.

To make a long story short, we may jump to the mid-20th Century when we observe a renewal of interest in the yield design methods with the development of the theory of plasticity. At that time, within the framework of the perfectly plastic model with associated flow rule for the constituent material, the lower and upper bound theorems of limit analysis and the theory of limit loads were established, which provide the traditional yield design approaches with sound theoretical bases. In particular, the upper bound theorem of limit analysis refers to the kinematic approach, where the rate of work by the external forces is compared with the plastic dissipation rate. Also, after several unconvincing attempts based on the concept of a rigid perfectly plastic material, the status of limit loads was definitely settled in the 1970s through the mathematical theorem of existence and uniqueness of the solution to the elastoplastic evolution problem: under the assumption of elastic and perfectly plastic behavior with associated flow rule, these loads are the maximum loads that can actually be sustained by the system considered in a given geometry.

This is a happy ending to the story from the theoretical point of view but, since it is dependent on the assumption of a perfectly plastic behavior with associated flow rule, it may appear as substantiating the idea that the yield design approach loses all interest when this assumption is not valid (which is often the case for practical problems, e.g. stability analyses of earth structures in civil engineering).

As a matter of fact, the lower and upper bound theorems are only the consequences of the sole assumption that the resistance of the constituent material is defined by a convex domain assigned to the internal forces. In particular, the upper bound theorem is derived from the dual definition of this domain without referring to a flow rule or constitutive equation. Therefore, these theorems hold as the lower and upper bound theorems for the extreme loads in the yield design theory, encompassing the many aspects of its implementation to various stability analysis problems. From the theoretical viewpoint, the status of the extreme loads is now restricted to that of upper bounds for the stability or load carrying capacity of the system. This does not make any difference in what concerns the application of the method to practice since practical validation is the general rule come what may.

Therefore, the purpose of this book is to present a theory of yield design within the original "equilibrium/resistance" framework without referring to the theories of plasticity or limit analysis. The general theory is developed for the three-dimensional continuum model in a versatile form based on simple arguments from the mathematical theory of convexity. It is then straightforwardly transposed to the one-dimensional curvilinear continuum, for the yield design analysis of beams, and to the two-dimensional continuum model of plates and thin slabs subjected to bending.

The book is structured as follows:

– Chapter 1 gives an introduction of the concept of yield design, starting from historical landmarks and based on field and laboratory observations of the collapse of mechanical systems. Compatibility between the equilibrium of the considered system subjected to prescribed loads and the resistance of its constituent material is set as the cornerstone of yield design analyses as is apparent in recent construction codes implementing the ultimate limit state design (ULSD) philosophy.

– Chapter 2 presents the simple example of a truss structure in order to give an outline of the method introducing the concept of potential stability.

– Since the general theory will be developed within the continuum mechanics framework, Chapter 3 recalls the fundamentals of this model in its primal formulation, leading to the classical equilibrium equations, and its dual formulation with the theorem/principle of virtual (rate of) work.

Chapters 4 – 6 present the core of the theory:

– In Chapter 4, after defining the concept of multi-parameter loading mode, the compatibility between equilibrium and resistance is first expressed in its primal form, on the basis of the equilibrium equations and the strength domain of the material defined by a convex strength condition. The definition of the domain of potentially safe loads follows from the mathematical compatibility between the equilibrium equations and the strength condition. As a consequence of the convexity of the strength condition, the domain of potentially safe loads is convex, which makes it possible to obtain convenient interior estimates through the construction of statically admissible stress fields that comply with the strength condition.

– Chapters 5 and 6 discuss the dual approach of the domain of potentially safe loads. Through the theorem/principle of virtual (rate of) work, it is possible to derive a necessary condition to be satisfied by the potentially safe loads, which does not refer to any stress field but uses kinematically admissible virtual velocity fields as test functions. This leads to the kinematic exterior approach of the domain of potentially safe loads, where the material strength condition is expressed in its mathematical dual formulation of maximum resisting (rate of) work. It is essential to keep in mind that this formulation does not imply any constitutive law and is just the mathematical dualization of the primal one.

– Chapter 7 is a kind of a return to Chapter 1 since it highlights the role played implicitly by the theory of yield design as the fundamental basis of the ULSD philosophy. It appears that the fundamental inequality of the kinematic exterior approach makes it possible to give an unambiguous quantified meaning to the symbolic inequality of ULSD.

– Chapter 8, with the explicit introduction of resistance parameters, takes advantage of the symmetric roles played by the loads applied to a system on the one side and the resistance of its constituent materials on the other in the equations to be satisfied for potential stability. It introduces the concept of potentially safe dimensioning of a system under a given set of prescribed loads as the counterpart of potentially safe loads when the dimensioning of the system is given. Potentially safe dimensioning generates a convex domain for which interior and kinematic exterior approaches are derived from the general theory. Optimal dimensioning of the system results in minimizing a given objective function. Also it is possible to account for the variability of the prescribed loads and for the physical scattering of the resistance parameters by giving a stochastic character to these data. From the definition of the domains of potentially safe loads and potentially safe dimensionings, there is no ambiguity in defining the concept of probability of stability of a system. Again, the interior approach and, essentially, the kinematic exterior approach provide lower and upper bound estimates for this probability.

– Chapter 9 looks at the yield design of structures. The curvilinear one-dimensional continuum model is first recalled with the concepts of wrench of forces and velocity distributor. The implementation of the yield design theory is straightforward, provided that the strength criteria of the constitutive elements, the joints and supports of the structure are correctly written.

– To conclude with a concise presentation of the yield design analysis of plates and thin slabs, Chapter 10 analyzes the construction of the corresponding two-dimensional model. The kinematics is defined by velocity distributor fields. The external forces are represented by force and moment densities and the internal forces are modeled by tensorial wrench fields.

– Chapter 11 presents the implementation of the yield design theory to metal plates and reinforced concrete slabs subjected to pure bending with strength criteria depending only on the internal moment tensor. The kinematic exterior approach appears as the most popular method, especially with relevant virtual motions based on the concept of *hinge lines*.

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Jean SALENÇON
March 2013

Chapter 1

Origins and Topicality of a Concept

Limit state design is, to some extent, a familiar terminology within the syllabuses of civil engineers' education, as it appears explicitly in the stability analyses of various types of structures or is present "anonymously" in the methods used for such analyses. Nevertheless, the variety of the corresponding approaches often makes it difficult to recognize that they proceed from the same fundamental principles, which are now the basis of the ultimate limit state design (ULSD) approach to the safety analysis of structures. As an introduction to the theory, this chapter will both present some famous historical milestones and the topicality of the subject referring to the principles of ULSD.

1.1. Historical milestones

1.1.1. *Dialogs concerning two new sciences*

The fundamental concept to be acknowledged first is that of *yield strength* as introduced by Galileo in his *Discorsi* [GAL 38a] on the simple experiment of a specimen in pure tension (Figure 1.1).

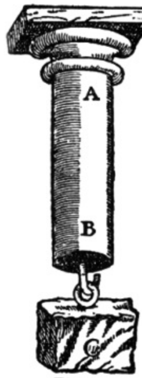


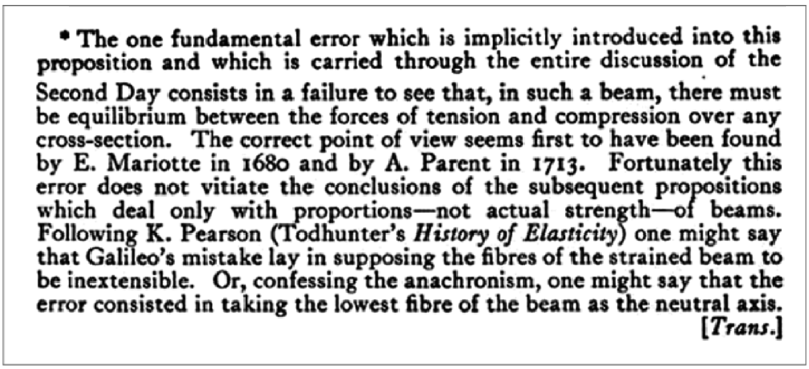
Fig. 1

Figure 1.1. *Longitudinal pull test (Galileo, Discorsi, 1st day [GAL 38a])*

Galileo uses this first characterization of the tenacity and coherence (*tenacità e coerenza*) of the material to explain the difficulty he finds in breaking a rod or a beam in tension while it is far easier to break it in bending: "*A prism or solid cylinder of glass, steel, wood or other breakable material which is capable of sustaining a very heavy weight when applied longitudinally is, as previously remarked, easily broken by the transverse application of a weight which may be much smaller in proportion as the length of the cylinder exceeds its thickness*". Considering a cantilever beam (Figure 1.2) built in a wall (section *AB*) and subjected to a weight applied at the other extremity (section *CD*), he first defines the "*absolute resistance to fracture as that offered to a longitudinal pull*". Then, he assumes that this resistance to tension will be localized in the section of the beam where it is fastened to the wall and that "*this resistance opposes the separation of the part BD lying outside the wall, from that portion lying inside*". The reasoning follows "*it is clear that if the cylinder breaks, fracture will occur at the point B where the edge of the mortise acts as a fulcrum for the lever BC*" [GAL 38a]. Introducing the second fundamental concept of the yield design approach, namely *equilibrium*, by writing the balance equation for the lever about *B*, Galileo finally relates the "*absolute resistance of the prism BD*" to its "*absolute resistance to fracture*" through the ratio of the short lever arm *BA/*2 to the long lever arm *BC*.

Figure 1.2. *Prism subjected to the transverse application of a weight (Galileo, Discorsi, 2nd day)*

Galileo's reasoning has been criticized, as shown in Figure 1.3, on the basis that the equilibrium of the cross-section *BA* is not satisfied.



* The one fundamental error which is implicitly introduced into this proposition and which is carried through the entire discussion of the Second Day consists in a failure to see that, in such a beam, there must be equilibrium between the forces of tension and compression over any cross-section. The correct point of view seems first to have been found by E. Mariotte in 1680 and by A. Parent in 1713. Fortunately this error does not vitiate the conclusions of the subsequent propositions which deal only with proportions—not actual strength—of beams. Following K. Pearson (Todhunter's *History of Elasticity*) one might say that Galileo's mistake lay in supposing the fibres of the strained beam to be inextensible. Or, confessing the anachronism, one might say that the error consisted in taking the lowest fibre of the beam as the neutral axis. [*Trans.*]

Figure 1.3. *Translator note [GAL 54, p. 115]*

Staying within the framework of the yield design approach for the beam, the criticism amounts to pointing out that the global equilibrium equation for the horizontal resultant force has not been taken into consideration. As a matter of fact, by focusing his attention only on the moment equation for the global equilibrium of the beam, Galileo obtains a necessary condition for the beam to sustain the load in a model where the constituent material is considered at the mesoscale of the section, with its resistance determined through the longitudinal pull test, and not at a more local level such as the longitudinal fibers as the criticism in Figure 1.3 would require: this is consistent with the fact that resistance to compression is never referred.

1.1.2. *Note on an application of the rules of maximum and minimum to some statical problems, relevant to architecture*

The appearance of soil mechanics as an engineering science is often associated with Coulomb’s memoir [COU 73] presented to the French Academy of sciences in 1773 after Coulomb returned from his eight year period in Martinique as a lieutenant in the French military corps of engineers. This *Essay* was devoted to various problems that he had encountered when building the “Fort Bourbon”: stability of pillars, arches and vaults, calculation of earth pressure on retaining walls, etc. (Figure 1.4).

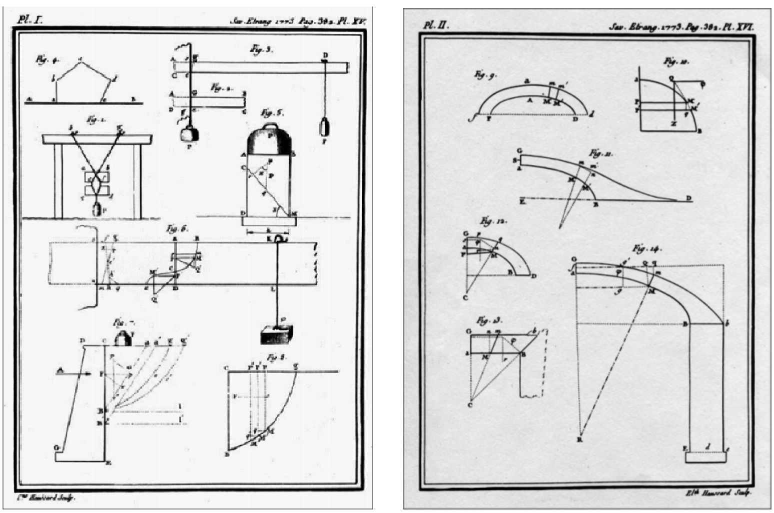


Figure 1.4. *Figure plates in Coulomb’s Essay [COU 73]*

The first guiding idea of Coulomb's rationale in tackling these problems is making a clear distinction between the *active forces*, which are the prescribed loads acting on the structure under consideration, and the characteristics of *resistance* of the material, which set the bounds to the "*coherence*" forces that can be mobilized (Figure 1.5).

I V.

Du Frottement.

Le frottement & la cohéſion ne ſont point des forces actives comme la gravité, qui exerce toujours ſon effet en entier, mais ſeulement des forces coërcitives; l'on eſtime ces deux forces par les limites de leur réſiſtance. Lorſqu'on dit, par exemple, que dans certains bois polis, le frottement ſur un plan horizontal d'un corps peſant neuf livres, eſt trois livres; c'eſt dire que toute force au-deſſous de trois livres ne troublera point ſon état de repos.

Je ſuppoſerai ici que la réſiſtance dûe au frottement eſt proportionnelle à la preſſion, comme l'a trouvé M. Amontons; quoique dans les groſſes maſſes le frottement ne ſuive pas exactement cette loi. D'après cette ſuppoſition, l'on trouve dans les briques le frottement, les trois quarts de la preſſion. Il ſera bon de faire des épreuves ſur les matériaux que l'on voudra employer. Il eſt impoſſible de fixer ici le frottement

X x ij

5

Figure 1.5. *Defining friction and cohesion in Coulomb's Essay [COU 73]*

The second guiding idea is that the resistance forces are exerted locally along an assumed failure surface, anticipating, to a certain extent, the concept of the stress vector to be introduced some 50 years later. In the simple case of a stone column under a compressive load (Figure 1.4), Coulomb explains the principles of the analysis: the active force on the assumed fracture surface must be balanced by the "coherence" force; the fracture surface will be determined through a minimization process.

On the basis of the same principle, Coulomb's stability analysis of a retaining wall is a fundamental landmark for the theory of yield design. Coulomb starts with the celebrated "Coulomb's wedge" reasoning (Figures 1.4, 1.6), where he assumes the failure surface to be plane and states a condition for stability that the active forces on the assumed fracture surface *Ba* must be balanced by the "coherence" forces, from which he derives, through minimization and maximization processes, two bounds for the horizontal force that can be applied to *CB* so that the wall be stable. Because of its simplicity, this reasoning is often presented as the Coulomb analysis of the stability of a retaining wall. In fact, Coulomb, after showing how the friction along the wall could be taken into account, states that, to be complete, the analysis should look for the curve that produces the highest pressure on *CB* and sketches the process for this determination.

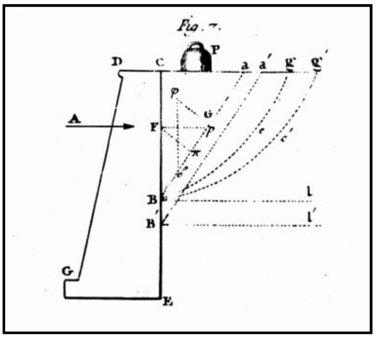


Figure 1.6. *Coulomb's wedge [COU 73]*

1.1.3. *Compatibility between equilibrium and resistance*

It is not difficult to point out the common features of the analyses that have been briefly presented here.

– First, the concept of *resistance* is introduced as a mechanical characteristic of the constituent material. After having been determined through a given simple experiment, it is used in any other circumstances and sets the *limits* to the resisting forces that can be actually mobilized.

– Then, the idea that the resistance of a given structure – a result at the *global* level – can be derived from the knowledge of the resistance of its constituent material(s), which is a property at the *local* level.

– For this determination, the rationale is based upon the statement that equilibrium equations of the structure must be satisfied while complying with the limits imposed by the resistance of the constituent material(s). In other words, *equilibrium and resistance must be mathematically compatible*.

– The practical implementation of this statement is made through the choice or the assumption of some particularly crucial zone in the structure (cross-section in the first case and failure surface in the second case), where it is anticipated that compatibility between equilibrium and resistance should be checked.

As it is shown in Figure 1.3 in the case of Galileo's analysis, it may be objected that such approaches do not take into account the behavior of the material, that is the fact that the material deforms under the forces it is subjected to. But it must be recalled that although the concept of linear elasticity was first introduced by Hooke in the 1660s, it was only in 1807 that Young's recognized shear as an elastic deformation; three-dimensional linear elasticity itself was only really formalized in the 1820s (Navier, Cauchy and others) at the same time as the concept of the stress tensor. As noted before, the yield design approach implicitly embodies an anticipation of the concept of internal forces. This is not surprising since the intuition of internal forces is primarily linked to that of rupture being localized on surfaces or lines as observed on full-, reduced- or small-scale experiments (Figures 1.7 and 1.8).

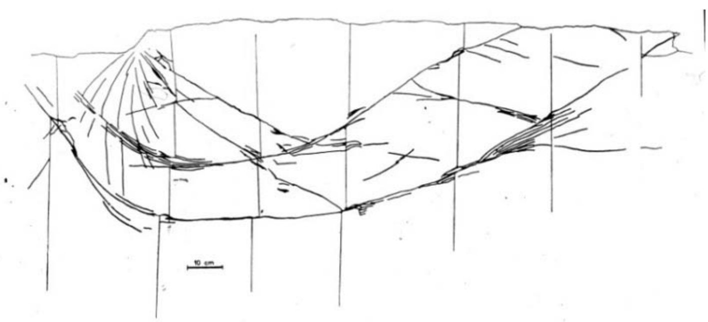


Figure 1.7. *"Slip line" pattern under a foundation in a purely cohesive material (medium-scale experiment) [HAB 84]*

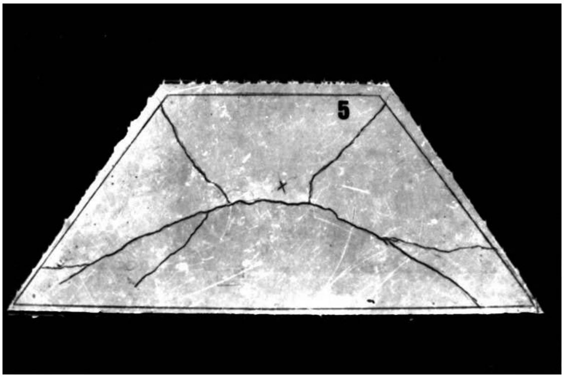


Figure 1.8. *Bending of a reinforced plaster slab: evidence of hinge curves (M. Milicevic)*

1.2. Topicality of the yield design approach

1.2.1. *The Coulomb's Essay legacy*

Coulomb's *Memoir* was at the origin of many methods used by engineers for the stability analyses of various types of structures. In the case of masonry vaults, the works by Méry [MER 40] and Durand-Claye [DUR 67, DUR 80] have been extensively studied by Heyman [HEY 66, HEY 69, HEY 72, HEY 80, HEY 82, HEY 98] and Delbecq [DEL 81, DEL 82]: it is interesting to note that they often combined Coulomb's original reasoning with elastic arguments, thus losing its original theoretical meaning without any damage from the practical point of view.

Soil mechanics, which is sometimes considered as having found its very origin in Coulomb's *Memoir*, exhibits numerous methods clearly related to it for the stability analysis of slopes, retaining walls, fills and earth dams or for the calculation of the bearing capacity of the surface foundations [BER 52, BIS 54, BØN 77, BRI 53, BU 93, CHA 07, CHE 69a, CHE 69b, CHE 70a, CHE 70b, CHE 73a, CHE 75a, CHE 75b, COU 79, JOS 80, DRU 52, GRE 49, HIL 50, HOU 82, KÖT 03, KÖT 09, LAU 11, MAN 72, MAR 05, MAR 09, MAS 99, MAT 79, MEY 51, MEY 53, MEY 63, MIC 98, MIC 09, PRA 55, REN 35, SAL 74, SAL 76, SAL 82, SAL 85, SAL 95a,