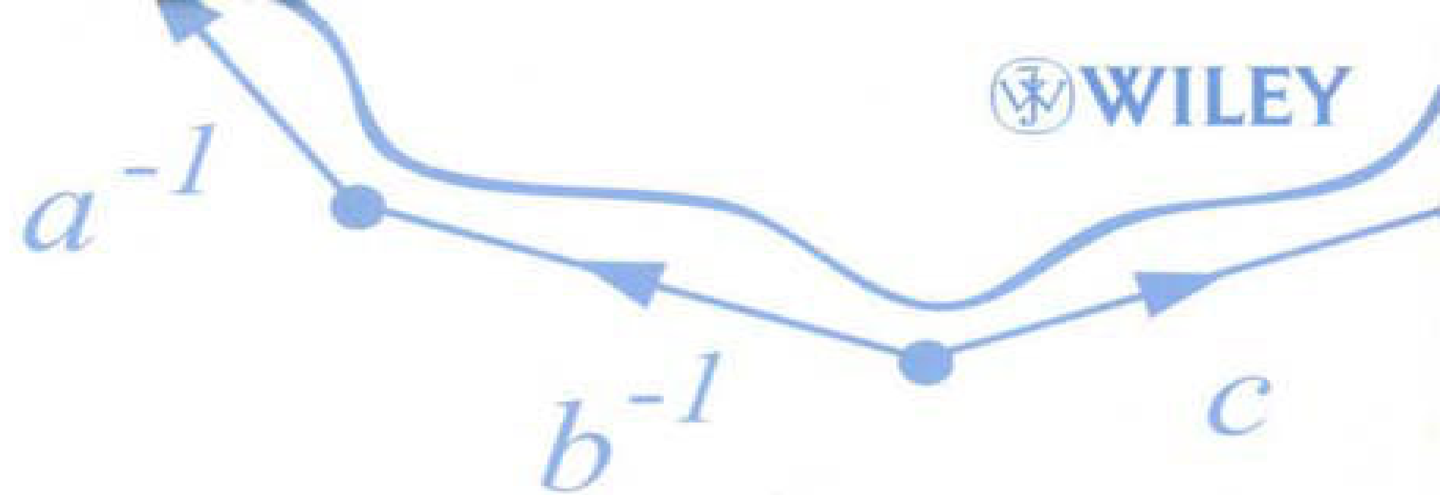

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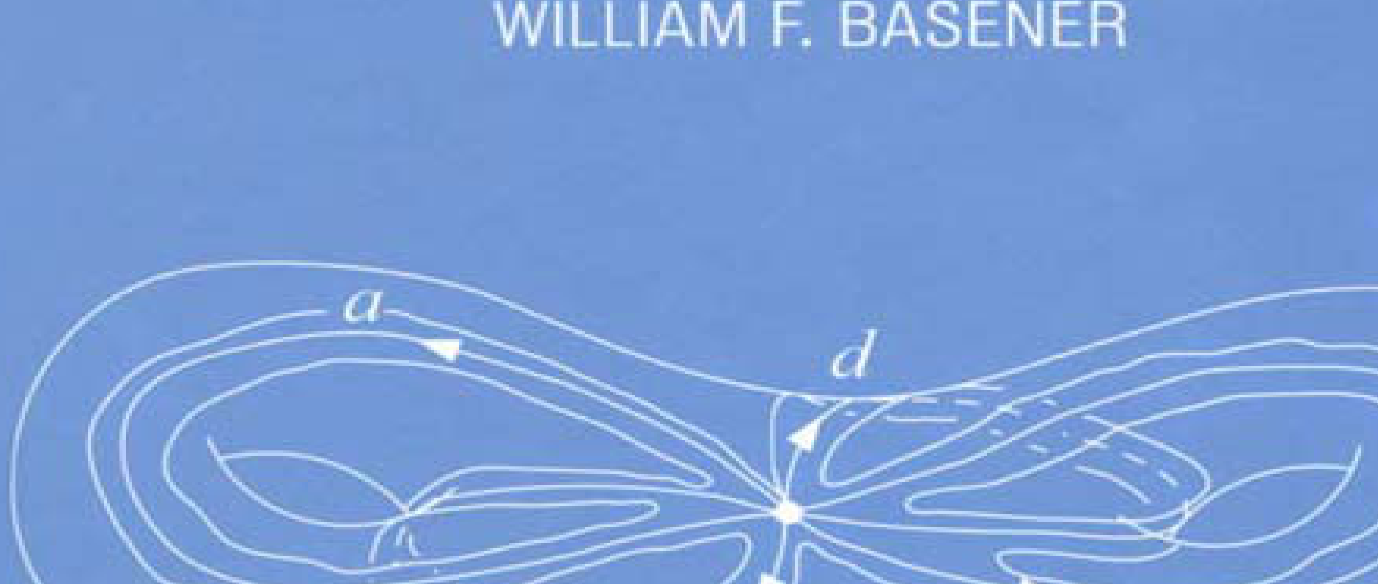
TOPOLOGY

AND ITS APPLICATIONS

WILLIAM F. BASENER



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Topology and Its Applications

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To my wonderful wife, Amber, without whose support this endeavor would have been impossible.
And to our beautiful children, Abigail, Wesley, Lila and J. T.

“The heavens tell of the glory of God.
The skies display his marvelous craftsmanship”
Psalm 19:1

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Preface

This is a textbook for a first course in either topology beginning in Chapter 1 or geometric topology beginning in Chapter 3. Our goal is to present the essentials of topology that underpin mathematics while quickly moving to the most interesting and useful topics. The framework of this text is rigorous theorems and proofs. We have the philosophy that a good proof should be clean and elegant, and that clear and complete logic elucidates the heart of a matter more than does a long intuitive discussion. However, we are generous with exposition outside of the proofs, and we introduce geometric examples and interesting applications as early as possible. We hope that the reader gains intuition early in the text and appreciates the beauty of topology as well as its importance to mathematics and science.

The range of topics is distributed among the topological subfields of point set topology, combinatorial topology, differential topology, geometric topology, and algebraic topology, while offering a broad variety of examples and applications. Choices in subject matter reflect the desire to present the elegant and complete theory of topology, with numerous examples and figures, while leaving time in a course for applications. Applied examples investigate the use of topology in physics, computer graphics, condensed matter, economics, chemistry, robotics, cosmology, dynamical systems, modeling, groups, and other mathematical and scientific fields. However, our presentation is planned around the theoretic framework of topology, and the applications are used to add intuition and utility to the subject.

Applications of topology are different from applications of other areas of mathematics. The utility of topology comes from its ability to categorize and count objects using qualitative “approximate” information as opposed to exact values. Our primary criteria in choosing applications is to look for questions from outside of topology whose solution involved topology and would have been either significantly more difficult or impossible without topology. (Farmers might use calculus to optimize their fence planning, but do not need the Jordan curve theorem to determine whether their chickens can escape from a fenced-in area!) This criterion was suggested informally by Jeff Weeks.

In most applications the topology is employed out of a need to handle the qualitative information. In condensed-matter physics, for example, a main goal is to determine the emergent behavior of a very large number of interacting molecules. Because the exact positions of all individual molecules cannot be determined practically, and because of the nature of the interactions, understanding the topological qualitative properties of the interactions is an essential part of determining the properties of materials such as superconductors (Section 5.7). A primary goal in cosmology is to determine the topology, or “shape,” of the universe as a 3–manifold. This shape of the universe determines, among other things, whether the universe is destined to eventually collapse in on itself in a “big crunch.” (Section 3.7) A primary goal in dynamical systems, discussed throughout this text, is to use qualitative statements about a model to make qualitative, although certain, predictions about the resulting behavior. Qualitative properties of interactions in game theory discussed in Section 4.7 result in Nash equilibria, which govern many important interactions in economics. The basic principle in dynamical systems and much of game theory is that governing laws, especially those involving social or biological interactions, can be known only approximately. Moreover, even when precise laws are known, chaotic interactions can make the resulting behavior too complicated for precise predictions to be useful. Topology enables us to handle qualitative laws and determine qualitative, but provable, resulting behavior.

Most of the applications appear in separate sections. This provides the reader (or instructor) with flexibility, choosing the applications that are most relevant. This format also provides ample room for background exposition with each application. Instructors may choose to cover any variety of the applications, or may assign them as reading for the students. One possible format, which has proved useful, is to have students read the applied sections and give presentations on applications, teaching each other.

Every scientific discipline has its own jargon, its own set of goals, and its own way of viewing the world. Thus, in each applied section there is a balancing act between presenting the material from the point of view of the applied field and presenting it in a manner consistent with the theory of topology. The result, due mostly to the background of the author, is a presentation of the applied topology from the perspective of a mathematician with all possible respect for the applied field. For a thorough treatment of the applied field the reader should consult the references cited in the sections.

One other unique feature of this book is the occasional “core intuition” segments. These short paragraphs explain the basic intuition for some of the topics. Hopefully, this will aid the reader encountering the theory of topology for the first time. One has to take great care, of course, to avoid depending too heavily on intuition. Like a magician in front of an audience, theory can play tricks on us when we look only for what we want to see.

A good student will learn to read the text with a pencil and paper in hand. Questions should be asked about all definitions: Can I think of examples? Can I create an equivalent formulation of the definition? Can I draw the picture of an example? What are each of the parts of the definition there for? Similar questions should be considered when encountering a theorem: Does the theorem make intuitive sense? Does it look similar to another theorem I know? How would I begin to prove it? Do I recall all terms used in the theorem? Can I think of an example? Can I think of a counterexample? (Probably not, but trying to beat the theorem often gives insights as to why it is true!) Can I draw a picture of it? Is it true if I remove some of the conditions? Can I generalize it or think of a specific simple case? A proof should be read not only step by step to see its logical progression, but as a whole. It is often helpful to try to summarize the proof in a single sentence.

The most important logical prerequisite is a standard sequence in calculus. Some of the material, particularly the sections on topological groups, the fundamental group, and homology, involves the algebra of groups. Chapter B provides the basic theory. One recurring theme is the demonstration of connections between topology and topics from mathematics and science. In most cases no previous experience is assumed. For example, Chapter 1 begins with coverage of the ε, δ definition of continuity and we prove that the open set definition of continuity is a generalization. No prior exposure to the ε, δ definition is assumed.

The chapters are organized to be covered in order. However, Chapter 6 does not rely on Chapter 5, with the exception of Section 6.7. So it is possible to skip some or all of Chapter 5. This allows an instructor to cover the basics both the fundamental group (Chapter 5) and the basics of Homology (Chapter 4) in a course with limited time.

The author is honored to thank a number of people who helped create this book. George Thurston was a great help at every stage of writing, suggesting many of the applications in quantum physics and thermodynamics. George also proofread much of the book and made numerous helpful suggestions about pedagogy. Tamas Wiandt read the book in its entirety and made too many good suggestions to count. Glenn R. Hall and Bob Devaney both assisted with the sections on the history and notions of chaos (Sections 1.7.1 and 1.7.3). Robert Ghrist provided guidance in the section on topology of robot coordination (Section 3.7.1). Jeff Weeks was a great help with the section of topology in cosmology (Section 3.7.3), as was the NASA WMAP team. Nicolas Ray was very helpful with the section on index theory in computer graphics (Section 4.9). James Sethna greatly improved the section on condensed matter (Section 5.7.1). Afra Zomorodian assisted with the section of computational topology (Section 6.9). Denis Blackmore also provided help with the section on computational topology. Bernie Brooks, Matthew Coppenbarger, Doug Meadows, and Joel Zablow each read significant portions of the book and provided helpful feedback.

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Of course, all errors are the responsibility of the author alone. All comments and suggestions about this work are

encouraged, and can be emailed to the author.

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