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Readers will learn in the introduction to this volume that mathematicians owe a huge debt to R.A. Rankin and J.M. Whittaker for their efforts in preserving Ramanujan's "Lost Notebook." If it were not for them, Ramanujan's lost notebook likely would have been permanently lost. Rankin was born in Garlieston, Scotland, in October 1915 and died in Glasgow in January 2001. For several years he was professor of Mathematics at the University of Glasgow. An account of his life and work has been given by B.C. Berndt, W. Kohnen, and K. Ono in [79]. Whittaker was born in March 1905 in Cambridge and died in Sheffield in January 1984. At his retirement, he was vice-chancellor of Sheffield University. A description of Whittaker's life and work has been written by W.K. Hayman [150].

Through long lapse of time,
This knowledge was lost.
But now, as you are devoted to truth,
I will reveal the supreme secret.

Bhagavad Gita, IV.2 & IV.3

Preface

This volume is the first of approximately four volumes devoted to the examination of all claims made by Srinivasa Ramanujan in *The Lost Notebook and Other Unpublished Papers*. This publication contains Ramanujan's famous lost notebook; copies of unpublished manuscripts in the Oxford library, in particular, his famous unpublished manuscript on the partition function and the tau-function; fragments of both published and unpublished papers; miscellaneous sheets; and Ramanujan's letters to G.H. Hardy, written from nursing homes during Ramanujan's final two years in England. This volume contains accounts of 442 entries (counting multiplicities) made by Ramanujan in the aforementioned publication. The present authors have organized these claims into eighteen chapters, containing anywhere from two entries in Chapter 13 to sixty-one entries in Chapter 17.

Contents

Introduction

Finding the Lost Notebook

In the spring of 1976, G.E. Andrews visited Trinity College Library at Cambridge University. Dr. Lucy Slater had suggested to him that there were materials deposited there from the estate of the late G.N. Watson that might contain some work on q-series. In one box of materials from Watson's estate, Andrews found several items written by Srinivasa Ramanujan. The most interesting item in this box was a manuscript of more than one hundred pages written on 138 sides in Ramanujan's distinctive handwriting. The sheets contained over six hundred mathematical formulas listed consecutively without proofs. Although technically not a notebook, and although technically not "lost," as we shall see later, it was natural in view of the fame of Ramanujan's notebooks [227] to name this manuscript *Ramanujan's lost notebook.* Almost surely, this manuscript, or at least most of it, was written during the last year of Ramanujan's life, after his return to India from England. We do not possess a bona fide proof of this claim, but we shall later present considerable evidence for it.

The manuscript contains no introduction or covering letter. In fact, there are hardly any words in the manuscript. There are a few marks evidently made by a cataloguer, and there are also a few remarks in the handwriting of G.H. Hardy. Undoubtedly, the most famous objects examined in the lost notebook are the *mock theta functions*, about which more will be said later. Concerning this manuscript, Ms. Rosemary Graham, manuscript cataloguer of the Trinity College Library, remarked, "...the notebook and other material was discovered among Watson's papers by Dr. J.M. Whittaker, who wrote the obituary of Professor Watson for the Royal Society. He passed the papers to Professor R.A. Rankin of Glasgow University, who, in December 1968, offered them to Trinity College so that they might join the other Ramanujan manuscripts already given to us by Professor Rankin on behalf of Professor Watson's widow." Since her late husband had been a fellow and scholar at Trinity College and had had an abiding, lifelong affection for Trinity College, Mrs. Watson agreed with Rankin's suggestion that the library at Trinity College would be the best place to preserve her husband's papers. Since Ramanujan had also been a fellow at Trinity College, Rankin's suggestion was even more appropriate.

The natural, burning question now is, *How did this manuscript of Ramanujan come into Watson's possession?* We think that the manuscript's history can be traced.

History of the Lost Notebook

After Ramanujan died on April 26, 1920, his notebooks and unpublished papers were given by his widow, Janaki, to the University of Madras. Also at that time, Hardy strongly advocated bringing together all of Ramanujan's manuscripts, both published and unpublished, for publication. On August 30, 1923, Francis Dewsbury, the registrar at the University of Madras, wrote to Hardy informing him that [81, p. 266]:

> I have the honour to advise despatch to-day to your address per registered and insured parcel post of the four manuscript note-books referred to in my letter No. 6796 of the 2nd idem.
> I also forward a packet of miscellaneous papers which have not been copied. It is left to you to decide whether any or all of them should find a place in the proposed memorial volume. Kindly preserve them for ultimate return to this office.

(The notebooks were returned to Madras, but Hardy evidently kept all the miscellaneous papers.) Although no accurate record of this material exists, the amount sent to Hardy was doubtless substantial. It is therefore highly likely that this "packet of miscellaneous papers" contained the aforementioned "lost notebook." Rankin, in fact, opines [230], [82, p. 124]:

> It is clear that the long MS represents work of Ramanujan subsequent to January 1920 and there can therefore be little doubt that it constitutes the whole or part of the miscellaneous papers dispatched to Hardy from Madras on 30 August 1923.

Further details can be found in Rankin's accounts of Ramanujan's unpublished manuscripts [230], [81, pp. 120–123], [82, pp. 117–142].

In 1934, Hardy passed on to Watson a considerable amount of his material on Ramanujan. However, it appears that either Watson did not possess the "lost" notebook in 1936 and 1937 when he published his papers [289], [290] on mock theta functions, or he had not examined it thoroughly. In any event, Watson [289, p. 61], [81, p. 330] writes that he believes that Ramanujan was unaware of certain third order mock theta functions and their transformation formulas. But, in his lost notebook, Ramanujan did indeed examine these functions and their transformation formulas. Watson's interest in Ramanujan's mathematics waned in the late 1930s, and Hardy died in 1947. In conclusion, sometime between 1934 and 1947 and probably closer to 1947, Hardy gave Watson the manuscript we now call the "lost notebook." More will be said in the sequel about further contents of the lost notebook.

Watson devoted about 10 to 15 years of his research to Ramanujan's work, with over 30 papers having their genesis in Ramanujan's mathematics, in particular, his notebooks and the letters he wrote to Hardy from India. Watson was Mason professor of pure mathematics at the University of Birmingham for most of his career, retiring in 1951. He died in 1965 at the age of 79. Rankin, who succeeded Watson as Mason professor of pure mathematics in Birmingham but who had since become professor of mathematics at the University of Glasgow, was asked to write an obituary of Watson for the London Mathematical Society. Rankin writes [230], [82, p. 120]:

> For this purpose I visited Mrs Watson on 12 July 1965 and was shown into a fair-sized room devoid of furniture and almost knee-deep in manuscripts covering the floor area. In the space of one day I had time only to make a somewhat cursory examination, but discovered a number of interesting items. Apart from Watson's projected and incomplete revision of Whittaker and Watson's *Modern Analysis* in five or more volumes, and his monograph on *Three decades of midland railway locomotives*, there was a great deal of material relating to Ramanujan, including copies of Notebooks 1 and 2, his work with B.M. Wilson on the Notebooks and much other material. ...
> In November 19 1965 Dr J.M. Whittaker who had been asked by the Royal Society to prepare an obituary notice [293], paid a similar visit and unearthed a second batch of Ramanujan material. A further batch was given to me in April 1969 by Mrs Watson and her son George.

A more colorful rendition of Whittaker's visit with Mrs. Watson was described in a letter of August 15, 1979, to Andrews [81, p. 304]:

> When the Royal Society asked me to write G.N. Watson's obituary memoir I wrote to his widow to ask if I could examine his papers. She kindly invited me to lunch and afterwards her son took me upstairs to see them. They covered the floor of a fair sized room to a depth of about a foot, all jumbled together, and were to be incinerated in a few days. One could only make lucky dips and, as Watson never threw away anything, the result might be a sheet of mathematics but more probably a receipted bill or a draft of his income tax return for 1923. By an extraordinary stroke of luck one of my dips brought up the Ramanujan material which Hardy must have passed on to him when he proposed to edit the earlier notebooks.

(That Watson's papers "were to be incinerated in a few days" seems fanciful.) Rankin dispatched Watson's and Ramanujan's papers to Trinity College in three batches on November 2, 1965; December 26, 1968; and December 30, 1969, with the Ramanujan papers being in the second shipment. Rankin did not realize the importance of Ramanujan's papers, and so when he wrote Watson's obituary [229] for the *Journal of the London Mathematical Society*, he did not mention any of Ramanujan's manuscripts. Thus, for almost eight years, Ramanujan's "lost notebook" and some fragments of papers by Ramanujan lay in the library at Trinity College, known only to a few of the library's cataloguers, Rankin, Mrs. Watson, Whittaker, and perhaps a few others. The 138-page manuscript waited there until Andrews found it and brought it before the mathematical public in the spring of 1976. It was not until the centenary of Ramanujan's birth on December 22, 1987, that Narosa Publishing House in New Delhi published in photocopy form Ramanujan's lost notebook and his other unpublished papers [228].

The Origin of the Lost Notebook

Having detailed the probable history of Ramanujan's lost notebook, we return now to our earlier claim that the lost notebook emanates from the last year of Ramanujan's life. On February 17, 1919, Ramanujan returned to India after almost five years in England, the last two being confined to nursing homes. Despite the weakening effects of his debilitating illness, Ramanujan continued to work on mathematics. Of this intense mathematical activity, up to the discovery of the lost notebook, the mathematical community knew only of the mock theta functions. These functions were described in Ramanujan's last letter to Hardy, dated January 12, 1920 [226, pp. xxix–xxx, 354–355], [81, pp. 220–223], where he wrote:

> I am extremely sorry for not writing you a single letter up to now .... I discovered very interesting functions recently which I call "Mock" ϑ-functions. Unlike the "False" ϑ-functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples.

In this letter, Ramanujan defines four third order mock theta functions, ten fifth order functions, and three seventh order functions. He also includes three identities satisfied by the third order functions and five identities satisfied by his first five fifth order functions. He states that the other five fifth order functions also satisfy similar identities. In addition to the definitions and formulas stated by Ramanujan in his last letter to Hardy, the lost notebook contains further discoveries of Ramanujan about mock theta functions. In particular, it contains the five identities for the second family of fifth order functions that were only mentioned but not stated in the letter.

We hope that we have made the case for our assertion that the lost notebook was composed during the last year of Ramanujan's life, when, by his own words, he discovered the mock theta functions. In fact, only a fraction (perhaps 5%) of the notebook is devoted to the mock theta functions themselves.

The Content of the Lost Notebook

The next fundamental question is, *What is in Ramanujan's lost notebook besides mock theta functions?* A majority of the results fall under the purview of q-series. These include mock theta functions, theta functions, partial theta function expansions, false theta functions, identities connected with the Rogers–Fine identity, several results in the theory of partitions, Eisenstein series, modular equations, the Rogers–Ramanujan continued fraction, other q-continued fractions, asymptotic expansions of q-series and q-continued fractions, integrals of theta functions, integrals of q-products, and incomplete elliptic integrals. Other continued fractions, other integrals, infinite series identities, Dirichlet series, approximations, arithmetic functions, numerical calculations, Diophantine equations, and elementary mathematics are some of the further topics examined by Ramanujan in his lost notebook.

The Narosa edition [228] contains further unpublished manuscripts, fragments of both published and unpublished papers, letters to Hardy written from nursing homes, and scattered sheets and fragments. The three most famous of these unpublished manuscripts are those on the partition function and Ramanujan's tau function, forty identities for the Rogers–Ramanujan functions, and the unpublished remainder of Ramanujan's published paper on highly composite numbers [222], [226, pp. 78–128].

This Volume on the Lost Notebook

This volume is the first of approximately four volumes devoted to providing statements, proofs, and discussions of all the claims made by Ramanujan in his lost notebook and all his other manuscripts and letters published with the lost notebook in [228]. For simplicity, we shall sometimes refer to the entire volume [228] as the lost notebook, even though only 138 pages of this work constitute what was originally the lost notebook. We have attempted to arrange all this disparate material into chapters. Doubtless, we have inadvertently misplaced entries.

With the statement of each entry from Ramanujan's lost notebook, we provide the page number(s) in the lost notebook where the entry can be found. Almost all of Ramanujan's claims are given the designation "Entry," although a few of them have the appellation "Corollary." Results in this volume named theorems, corollaries (except in the aforementioned few cases), and lemmas are not due to Ramanujan. We emphasize that Ramanujan's claims always have page numbers from the lost notebook attached to them.

However, the format of Chapter 10, in which Ramanujan's empirical evidence for the Rogers–Ramanujan identities is discussed, is different. Here we quote Ramanujan from pages 358–361 in the lost notebook and then prove and discuss his claims.

So that readers can more readily find where a certain entry is discussed, we place at the conclusion of each volume a *Location Guide* to where entries can be found in that particular volume. Thus, if a reader wants to know whether a certain identity on page 172 of the Narosa edition [228] can be found in a particular volume, she can turn to this index and determine where in that volume identities on page 172 are discussed.

Following the Location Guide, we provide a *Provenance* indicating the sources from which we have drawn in preparing significant portions of the given chapters. We emphasize that in the Provenance we do not list all papers in which results from a given chapter are established. For example, the content of Chapter 6 has generated dozens of papers. In the chapter itself we have attempted to cite all relevant papers known to us, but in the Provenance we list only those papers from which we have drawn our exposition. On the other hand, almost all chapters contain material previously unpublished. For example, except for the combinatorial proofs, none of the material in Chapter 9 has been previously published.

We now describe the contents of each of the eighteen chapters constituting this first volume. Most, but not all, of the results have been established earlier in the literature, often by Andrews; or Berndt, usually in collaboration with some of his former or current graduate students; or other mathematicians, including the aforementioned students.

An enormous amount of material in the lost notebook is on the Rogers–Ramanujan continued fraction, $R(q)$, clearly one of Ramanujan's favorite functions. From (1.1.2) of Chapter 1, we observe that the Rogers–Ramanujan continued fraction can be represented as a quotient of theta functions. Hence, $R(q)$ lives in the realms of elliptic functions and modular forms, and so the vast machineries of these two fruitful fields can be employed to produce a plethora of theorems. Chapter 1 focuses on identities, modular equations, and representations for $R(q)$ arising from the theory of theta functions and modular equations. Ramanujan evaluated in closed form $R(\pm e^{-\pi\sqrt{n}})$, for certain rational values of n, with many of these values found in his lost notebook. However, in several cases, Ramanujan indicated only that he could find certain values without explicitly providing them. Chapter 2 is devoted to explicit evaluations of $R(\pm e^{-\pi\sqrt{n}})$. Published with the lost notebook is a fragment summarizing some of Ramanujan's findings on the Rogers–Ramanujan continued fraction and on his cubic continued fraction; this brief fragment is examined in Chapter 3. Partition-theoretic implications of the Rogers–Ramanujan continued fraction are contained in Chapter 4. Ramanujan obtained several interesting series representations for $R(q)$, especially one for $R^3(q)$, all of which can also be found in Chapter 4. Chapter 5 is devoted to finite Rogers–Ramanujan continued fractions and other finite continued fractions of the same sort. Some are connected with class invariants.

After these five chapters on the Rogers–Ramanujan continued fraction, we examine other q-continued fractions. Chapter 6 contains some beautiful general theorems followed by many elegant special cases found by Ramanujan. Chapter 7 is in a different vein and is devoted to some asymptotic formulas for continued fractions. One of Ramanujan's most engaging continued fractions is his continued fraction for $(q^2;q^3)_\infty/(q;q^3)_\infty$, the topic of Chapter 8. In contrast to the Rogers–Ramanujan continued fraction, which arises as a special case of general theorems in Chapter 6, this continued fraction does not. One of Ramanujan's most fascinating theorems in the lost notebook is the seemingly enigmatic formula (8.1.2) arising out of the theory of $(q^2;q^3)_\infty/(q;q^3)_\infty$, a theory much different from that of $R(q)$.

The Rogers–Fine identity is one of the most useful theorems in the subject of q-series. Although not explicitly given in his notebooks or lost notebook, Ramanujan clearly was familiar with it and found many applications for it in the lost notebook. More than two dozen identities associated with the Rogers–Fine identity are proved in Chapter 9, some by combinatorial means.

The Rogers–Ramanujan continued fraction is intimately associated with the Rogers–Ramanujan identities, which appear at various places in the first five chapters. In Chapter 10, we examine a fragment on these identities giving empirical evidence for the truth of the identities, and so evidently written before Ramanujan found proofs for them. This chapter is followed by a chapter on other identities of this sort.

Although mock theta functions will not be examined until a further volume, certain partial fraction expansions, the topic of Chapter 12, have intimate associations with mock theta functions.

Chapter 13 is devoted to the study of two of the most enigmatic formulas in the lost notebook. Both are product expansions. One is for a function prominent in the theory of the Rogers–Ramanujan identities. The other is for a quasi-theta function and so can be considered to be an analogue of the Jacobi triple product identity. Although some elements of our proofs might reflect Ramanujan's thinking, we are clearly in the dark about what led Ramanujan ever to think that such formulas might even exist.

One of the most intriguing identities in the lost notebook is a formula relating a character analogue of the Dedekind eta function, an integral of eta functions, and a value of a Dirichlet L-series. This wonderful formula and other integrals of theta functions are the subject of Chapter 14. In Chapter 15, we again examine integrals of eta functions, but these are much different and are related to incomplete elliptic integrals of the first kind. As with so much of the work in Ramanujan's lost notebook, there are no other results of this kind in the literature. The brief Chapter 16 is devoted to five integrals of q-products.

It is difficult to organize Ramanujan's modular equations into one chapter, because they are frequently employed to prove other entries; for example, many new modular equations can be found in Chapter 1. Consigned to Chapter 17 are discussions of one page in the lost notebook and two fragments published with the lost notebook on modular equations.

The last chapter, Chapter 18, is devoted to two fragments on Lambert series, which are also prominent in Chapter 4.

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1

The Rogers–Ramanujan Continued Fraction and Its Modular Properties

1.1 Introduction

The Rogers–Ramanujan continued fraction, defined by

$$R(q) := \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \cdots, \qquad |q| < 1, \tag{1.1.1}$$

first appeared in a paper by L.J. Rogers [234] in 1894. Using the Rogers–Ramanujan identities, established for the first time in [234], Rogers proved that

$$R(q) = q^{1/5} \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{1.1.2}$$

Here and in the sequel we employ the customary q-product notation. Thus, set $(a)_0 := (a;q)_0 := 1$, and, for $n \geq 1$, let

$$(a)_n := (a;q)_n := \prod_{k=0}^{n-1} (1 - aq^k). \tag{1.1.3}$$

Furthermore, set

$$(a)_\infty := (a;q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k), \quad |q| < 1.$$

If the base q is understood, we use $(a)_n$ and $(a)_\infty$ instead of $(a;q)_n$ and $(a;q)_\infty$, respectively.

In his first two letters to G.H. Hardy [226, pp. xxvii, xxviii], [81, pp. 29, 57], Ramanujan communicated several theorems on $R(q)$. He also briefly mentioned the more general continued fraction

$$R(a,q) := \frac{1}{1} \, \frac{}{+} \, \frac{aq}{1} \, \frac{}{+} \, \frac{aq^2}{1} \, \frac{}{+} \, \frac{aq^3}{1} \, \frac{}{+} \cdots, \qquad |q| < 1, \tag{1.1.4}$$

now called the generalized Rogers–Ramanujan continued fraction , and further generalizations. Hardy was intrigued by Ramanujan's theorems on this continued fraction, and on 26 March 1913 (the day on which Paul Erdős was born) wrote [81, pp. 77–78]:

> What I should like above all is a definite proof of some of your results concerning continued fractions of the type
>
> $$\frac{x}{1}+\frac{x^2}{1}+\frac{x^3}{1}+\cdots;$$
>
> and I am quite sure that the wisest thing you can do, in your own interests, is to let me have one as soon as possible.

Later, in another letter, probably written on 24 December 1913, Hardy further exhorted [81, p. 87]

> If you will send me your proof written out carefully (so that it is easy to follow), I will (assuming that I agree with it—of which I have very little doubt) try to get it published for you in England. Write it in the form of a paper "On the continued fraction
>
> $$\frac{x}{1}+\frac{x^2}{1}+\frac{x^3}{1}+\cdots,"$$
>
> giving a full proof of the principal and most remarkable theorem, viz. that the fraction can be expressed in finite terms when $x=e^{-\pi\sqrt{n}}$, when $\underline{n}$ is rational.

However, Ramanujan never followed Hardy's advice.

In his notebooks [227], Ramanujan offered many beautiful theorems on $R(q)$. In particular, see (1.1.10) and (1.1.11) below, K.G. Ramanathan's papers [215]–[218], the *Memoir* by Andrews, Berndt, L. Jacobsen, and R.L. Lamphere [39], and Berndt's book [63, Chapter 32].

Ramanujan's lost notebook [228] contains a large number of beautiful, surprising, and remarkable results on the Rogers–Ramanujan continued fraction. In this opening chapter, we prove many theorems arising from modular properties of the Rogers–Ramanujan continued fraction. Papers containing proofs of results proved in this opening chapter include those by Berndt, S.–S. Huang, J. Sohn, and S.H. Son [78], S.–Y. Kang [171], [172], Ramanathan [215], Sohn [253], and Son [254]. But as we emphasized in the Introduction, succeeding chapters also contain theorems about the Rogers–Ramanujan continued fraction. Chapter 2 contains explicit evaluations of $R(q)$ found in the lost notebook. Chapter 3 focuses on a fragment on the Rogers–Ramanujan continued fraction and the cubic continued fraction, which is not found in the lost notebook but was published with the lost notebook. Chapter 4 is devoted to relations connecting $R(q)$ with Lambert series and partitions. Finite Rogers–Ramanujan continued fractions are featured in Chapter 5. Chapter 6 contains theorems in the lost notebook on generalizations (such as (1.1.4)), various analogues, and other q-continued fractions. A survey describing many of Ramanujan's discoveries about the Rogers–Ramanujan continued fraction, especially those found in the lost notebook, can be found in [71].

We now provide notation that will be used throughout the chapter. Recall Ramanujan's general theta function $f(a,b)$, namely,

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1. \tag{1.1.5}$$

The most important special cases of $f(a,b)$ are defined by (in Ramanujan's notation)

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \quad |q| < 1, \tag{1.1.6}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \qquad |q| < 1, \tag{1.1.7}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty, \qquad |q| < 1, \tag{1.1.8}$$

where the latter equality is Euler's pentagonal number theorem. The product representations in (1.1.6)–(1.1.8) follow from Jacobi's triple product identity, given in Lemma 1.2.2 below. Lastly, define

$$\chi(-q) := (q;q^2)_\infty. \tag{1.1.9}$$

Two of the most important formulas for $R(q)$ are given by

$$\frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5} f(-q^5)} \tag{1.1.10}$$

and

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{q f^6(-q^5)}. \tag{1.1.11}$$

These equalities were found by G.N. Watson [286], [287] in Ramanujan's notebooks and proved by him [286] in order to establish claims about the Rogers–Ramanujan continued fraction communicated by Ramanujan in the aforementioned two letters to Hardy. The proof of (1.1.10) given by Watson [286] is identical to the one given by Ramanujan in his unpublished manuscript on the partition and tau functions, which was published with his lost notebook [228, pp. 135–177, 238–243]; in particular, see page 238. The manuscript was published with proofs and commentary by Berndt and K. Ono [80]. With revised and more extensive commentary, the manuscript will be reproduced in the present authors' third volume on the lost notebook [38]. Different proofs of (1.1.10) and (1.1.11) can be found in Berndt's book [61, pp. 265–267].

We now briefly describe some of the results proved in this chapter.

Our first theorem is remarkable. Ramanujan found three related identities in two variables, two of which contain (1.1.10) and (1.1.11) as special cases. Section 1.2 is devoted to Son's elegant proofs [254].

On page 48 in his lost notebook, Ramanujan offers two further formulas akin to (1.1.10) and (1.1.11). These formulas are "between" (1.1.10) and (1.1.11) in that they involve $R^2(q)$ and $R^3(q)$. Statements and proofs of these identities can be found in Section 1.3.

On the other hand, on page 206 in his lost notebook, Ramanujan claims that (1.1.10) and (1.1.11) can be refined by factoring each side into two factors and then equating appropriate factors on each side, giving four equalities. It is amazing that factoring in this way actually leads to identities, which are proved in Section 1.4.

In his first letter to Hardy [226, p. xxvii], [81, p. 29], Ramanujan claimed that $R^5(q)$ is a particular quotient of quartic polynomials in $R(q^5)$. This was first proved in print by Rogers [236] in 1920, while Watson [286] gave another proof nine years later. At scattered places in his notebooks [227], Ramanujan also gave modular equations relating $R(q)$ with $R(-q)$, $R(q^2)$, $R(q^3)$, and $R(q^4)$. In the publication of his lost notebook [228], these results are conveniently summarized by Ramanujan on page 365; in this book they can be found in Chapter 3. Proofs of most of these modular relations can be found in the *Memoir* [39, Entries 6, 20, 21, 24–26, pp. 11, 27, 28, 31–37], and in Berndt's book [63, Chapter 32, Entries 1–6]. Rogers [236] found modular equations relating $R(q)$ with $R(q^n)$, for $n = 2$, 3, 5, and 11; the latter equation is not found in Ramanujan's work. J. Yi [299] has found a modular equation for $n = 7$, while also devising simpler proofs for degrees 3 and 11. H.H. Chan and V. Tan [118] discovered a modular equation of degree 19 and devised another proof of Rogers's modular equation of degree 11. On page 205 in his lost notebook [228], Ramanujan offers two modular equations relating the Rogers–Ramanujan continued fraction at *three* arguments. These are proved in Section 1.5. The results described in the last three sections were first proved in the paper by Berndt, Huang, Sohn, and Son [78].

In the next four sections we establish several beautiful identities involving the Rogers–Ramanujan continued fraction and some elegant associated theta-function identities. These results were first proved by Kang [171]. In Section 1.6 we prove some theta-function identities of degree 5, in other words, modular equations of degree 5. In the following Section 1.7, we first establish some factorizations, which involve $R(q)$, of the identities in Section 1.6. The next theorem also provides factorizations, and these are in the same spirit as the factorizations of (1.1.10) and (1.1.11) in Section 1.4. In the following Section 1.8, we introduce Ramanujan's parameters $k := R(q)R^2(q^2)$, $\mu := R(q)R(q^4)$, and $\nu := R^2(q^{1/2})R(q)/R(q^2)$, and prove several elegant identities for $R(q)$, $\varphi(q)$, and $\psi(q)$ in terms of these parameters. Section 1.9 gives further identities arising from the parameter k.

In Section 1.10, we prove some formulas for $R(q)$, $R(q^2)$, and $R(q^3)$, each in terms of one of the others, arising from (1.1.11). These proofs are published here for the first time and are taken from Sohn's doctoral thesis [253].

1.2 Two-Variable Generalizations of (1.1.10) and (1.1.11)

On page 207 in his lost notebook [228], Ramanujan listed three identities,

$$P - Q = 1 + \frac{f(-q^{1/5}, -\lambda q^{2/5})}{q^{1/5} f(-\lambda^{10} q^5, -\lambda^{15} q^{10})}, \tag{1.2.1}$$

$$PQ = 1 - \frac{f(-\lambda, -\lambda^4 q^3) f(-\lambda^2 q, -\lambda^3 q^2)}{f^2(-\lambda^{10} q^5, -\lambda^{15} q^{10})}, \tag{1.2.2}$$

and

$$P^5 - Q^5 = 1 + 5PQ + 5P^2Q^2 + \frac{f(-q, -\lambda^5 q^2) f^5(-\lambda^2 q, -\lambda^3 q^2)}{q\, f^6(-\lambda^{10} q^5, -\lambda^{15} q^{10})}, \tag{1.2.3}$$

without specifying the functions P and Q. In this section, the functions P and Q are determined, and the identities, which are remarkable generalizations of (1.1.10) and (1.1.11), are proved.

We shall need several lemmas.

Lemma 1.2.1. *We have*

$$f(-1, a) = 0 \tag{1.2.4}$$

and, if n is an integer,

$$f(a, b) = a^{n(n+1)/2} b^{n(n-1)/2} f\left(a(ab)^n, b(ab)^{-n}\right). \tag{1.2.5}$$

For proofs of these elementary properties, see [61, p. 34, Entry 18].

Lemma 1.2.2 (Jacobi's Triple Product Identity). *If $f(a, b)$ is defined by* (1.1.5)*, then*

$$f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty.$$

For a proof, see [61, p. 35, Entry 19].

Corollary 1.2.1.

$$f(-q, -q^4) f(-q^2, -q^3) = f(-q) f(-q^5).$$

This follows immediately from Lemma 1.2.2 and (1.1.8). See also [61, p. 44, Corollary].

Lemma 1.2.3. *Let* $U_n = a^{n(n+1)/2}b^{n(n-1)/2}$ *and* $V_n = a^{n(n-1)/2}b^{n(n+1)/2}$. *Then*

$$f(U_1, V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right).$$

For a proof of Lemma 1.2.3, see [61, p. 48, Entry 31].

The next entry is Ramanujan's version of the quintuple product identity, and it is found on page 207 of his lost notebook, the same page as the identities for P and Q given above. Although Ramanujan undoubtedly used the quintuple product many times in proving results offered in his notebooks, this is the only instance where he recorded the quintuple product identity. For a proof along the lines that Ramanujan might have used and for references to other proofs, see [61, pp. 80–83].

Entry 1.2.1 (Quintuple Product Identity; p. 207). *For* $|\lambda x^3| < 1$,

$$f(-\lambda^2 x^3, -\lambda x^6) + x f(-\lambda, -\lambda^2 x^9) = \frac{f(-x^2, -\lambda x) f(-\lambda x^3)}{f(-x, -\lambda x^2)}. \tag{1.2.6}$$

To prove (1.2.3), we need instances of the following general product formula, which is due to Son [254]. Special cases of this lemma can be found in Ramanujan's notebooks [227]; see Berndt's books [61, pp. 264, 307, 346, 348], [62, pp. 142, 145, 188, 192].

Lemma 1.2.4. *Let* $|ab| < 1$, *let* p *be an odd prime, let* j *and* k *be integers with* $(j,k) \not\equiv (0,0) \pmod{p}$, *let* $\zeta := \exp(2\pi i/p)$, *and let* $x = s$, $0 \le x < p$, *be the solution of*

$$(j+k)x + j \equiv 0 \pmod{p}$$

when p *does not divide* $j+k$. *Then*

$$\prod_{n=1}^{p} f(\zeta^{jn} a, \zeta^{kn} b) \tag{1.2.7}$$
$$= \begin{cases} \dfrac{f^p(a^{s+1}b^s, a^{p-s-1}b^{p-s}) f(a^p, b^p)}{f(a^{p(s+1)}b^{ps}, a^{p(p-s-1)}b^{p(p-s)})}, & \text{if} \quad j+k \not\equiv 0 \pmod{p}, \\ f^p(-ab)\dfrac{f(a^p, b^p)}{f(-a^p b^p)}, & \text{if} \quad j+k \equiv 0 \pmod{p}. \end{cases}$$

Proof. Let

$$C := \prod_{n=1}^{p} f(-\zeta^{jn} a, -\zeta^{kn} b).$$

By the Jacobi triple product identity, Lemma 1.2.2,

$$C = \prod_{n=1}^{p} (\zeta^{jn}a; \zeta^{(j+k)n}ab)_\infty (\zeta^{kn}b; \zeta^{(j+k)n}ab)_\infty (\zeta^{(j+k)n}ab; \zeta^{(j+k)n}ab)_\infty$$
$$= C_1C_2C_3, \tag{1.2.8}$$

where

$$C_1 := \prod_{\ell=1}^{p} (\zeta^{j\ell}a; \zeta^{(j+k)\ell}ab)_\infty,$$
$$C_2 := \prod_{\ell=1}^{p} (\zeta^{k\ell}b; \zeta^{(j+k)\ell}ab)_\infty,$$

and

$$C_3 := \prod_{\ell=1}^{p} (\zeta^{(j+k)\ell}ab; \zeta^{(j+k)\ell}ab)_\infty.$$

First suppose that $j + k \not\equiv 0 \pmod p$. Then

$$C_1 = \prod_{\substack{n=0 \\ n\equiv s \,(\mathrm{mod}\, p)}}^{\infty} \left(1 - a(ab)^n\right)^p \prod_{\substack{n=0 \\ n\not\equiv s \,(\mathrm{mod}\, p)}}^{\infty} \left(1 - a^p(ab)^{pn}\right)$$
$$= \prod_{n=0}^{\infty} \left(1 - a(ab)^{pn+s}\right)^p \prod_{n=0}^{\infty} \left(1 - a^p(ab)^{pn}\right) \Bigg/ \prod_{\substack{n=0 \\ n\equiv s \,(\mathrm{mod}\, p)}}^{\infty} \left(1 - a^p(ab)^{pn}\right)$$
$$= (a^{s+1}b^s; a^pb^p)_\infty^p \frac{(a^p; a^pb^p)_\infty}{(a^{p(s+1)}b^{ps}; a^{p^2}b^{p^2})_\infty}.$$

Similarly, since $p - s - 1$ is a solution of $(j + k)x + k \equiv 0 \pmod p$,

$$C_2 = (a^{p-s-1}b^{p-s}; a^pb^p)_\infty^p \frac{(b^p; a^pb^p)_\infty}{(a^{p(p-s-1)}b^{p(p-s)}; a^{p^2}b^{p^2})_\infty},$$

and since $p - 1$ is a solution of $(j + k)x + (j + k) \equiv 0 \pmod p$,

$$C_3 = (a^pb^p; a^pb^p)_\infty^p \frac{(a^pb^p; a^pb^p)_\infty}{(a^{p^2}b^{p^2}; a^{p^2}b^{p^2})_\infty}.$$

Hence, by (1.2.8) and the Jacobi triple product identity, Lemma 1.2.2,

$$C = C_1C_2C_3$$
$$= \left\{(a^{s+1}b^s; a^pb^p)_\infty (a^{p-s-1}b^{p-s}; a^pb^p)_\infty (a^pb^p; a^pb^p)_\infty\right\}^p$$
$$\times \frac{(a^p; a^pb^p)_\infty (b^p; a^pb^p)_\infty (a^pb^p; a^pb^p)_\infty}{(a^{p(s+1)}b^{ps}; a^{p^2}b^{p^2})_\infty (a^{p(p-s-1)}b^{p(p-s)}; a^{p^2}b^{p^2})_\infty (a^{p^2}b^{p^2}; a^{p^2}b^{p^2})_\infty}$$
$$= f^p(-a^{s+1}b^s, -a^{p-s-1}b^{p-s}) \frac{f(-a^p, -b^p)}{f(-a^{p(s+1)}b^{ps}, -a^{p(p-s-1)}b^{p(p-s)})},$$

which, after $-a$ and $-b$ are replaced by a and b, respectively, establishes Lemma 1.2.4 in the case that $j+k \not\equiv 0 \pmod p$.

Second, if $j+k \equiv 0 \pmod p$,

$$C_1 = \prod_{n=0}^{\infty} \left(1 - a^p (ab)^{pn}\right) = (a^p; a^p b^p)_\infty.$$

Similarly,

$$C_2 = (b^p; a^p b^p)_\infty,$$

and, by (1.1.8),

$$C_3 = (ab; ab)_\infty^p = f^p(-ab).$$

Hence, by (1.2.8) and the Jacobi triple product identity, Lemma 1.2.2, we deduce that

$$C = C_1 C_2 C_3 = f^p(-ab)(a^p; a^p b^p)_\infty (b^p; a^p b^p)_\infty = f^p(-ab) \frac{f(-a^p, -b^p)}{f(-a^p b^p)},$$

and so the proof is complete after $(-a, -b)$ is replaced by (a, b). □

We are now ready to give Son's proofs [254] of the mysterious identities on page 207 of the lost notebook [228].

Entry 1.2.2 (p. 207). *If*

$$P = \frac{f(-\lambda^{10} q^7, -\lambda^{15} q^8) + \lambda q f(-\lambda^5 q^2, -\lambda^{20} q^{13})}{q^{1/5} f(-\lambda^{10} q^5, -\lambda^{15} q^{10})} \tag{1.2.9}$$

and

$$Q = \frac{\lambda f(-\lambda^5 q^4, -\lambda^{20} q^{11}) - \lambda^3 q f(-q, -\lambda^{25} q^{14})}{q^{-1/5} f(-\lambda^{10} q^5, -\lambda^{15} q^{10})}, \tag{1.2.10}$$

then (1.2.1)*,* (1.2.2)*, and* (1.2.3) *hold.*

Proof. In Lemma 1.2.3, let $a = -q^{1/5}$, $b = -\lambda q^{2/5}$, and $n = 5$, and then employ Lemma 1.2.1 to obtain (1.2.1).

By (1.2.9) and (1.2.10), the identity (1.2.2) is equivalent to the identity,

$$\begin{aligned} S := {} & f(-\lambda, -\lambda^4 q^3) f(-\lambda^2 q, -\lambda^3 q^2) \\ = {} & f(-\lambda^{10} q^5, -\lambda^{15} q^{10}) f(-\lambda^{10} q^5, -\lambda^{15} q^{10}) \\ & - \lambda f(-\lambda^5 q^4, -\lambda^{20} q^{11}) f(-\lambda^{10} q^7, -\lambda^{15} q^8) \\ & - \lambda^2 q f(-\lambda^5 q^4, -\lambda^{20} q^{11}) f(-\lambda^5 q^2, -\lambda^{20} q^{13}) \\ & + \lambda^3 q f(-q, -\lambda^{25} q^{14}) f(-\lambda^{10} q^7, -\lambda^{15} q^8) \\ & + \lambda^4 q^2 f(-q, -\lambda^{25} q^{14}) f(-\lambda^5 q^2, -\lambda^{20} q^{13}). \end{aligned} \tag{1.2.11}$$

Then

$$S = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} h(u, v),$$

where

$$h(u, v) := (-1)^{u+v} \lambda^{(5u^2+5v^2-u-3v)/2} q^{(3u^2+3v^2-u-3v)/2}.$$

We now subdivide this sum into five sums according to

$$2u + v \equiv k \pmod 5, \qquad 0 \le k \le 4.$$

Then

$$5u = 2(2u + v) + (u - 2v) \equiv 0 (\text{mod } 5),$$

which implies that $u - 2v \equiv -2k \pmod 5$. Write

$$S = S_0 + S_1 + S_2 + S_3 + S_4, \tag{1.2.12}$$

where S_k denotes the sum for $2u+v \equiv k \pmod 5$, $0 \le k \le 4$. Let $2u+v = 5m$ and $u - 2v = -5n$. Then $u = 2m - n$, $v = m + 2n$, and

$$\begin{aligned} h(u, v) &= h(2m - n, m + 2n) \\ &= (-1)^{(3m+n)} \lambda^{5(5m^2+5n^2-m-n)/2} q^{5(3m^2+3n^2-m-n)/2}. \end{aligned}$$

Therefore,

$$\begin{aligned} S_0 &= \sum_{\substack{u,v \\ 2u+v\equiv 0 \pmod 5}} h(u, v) \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h(2m - n, m + 2n) \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{(3m+n)} \lambda^{5(5m^2+5n^2-m-n)/2} q^{5(3m^2+3n^2-m-n)/2} \\ &= \sum_{m=-\infty}^{\infty} (-1)^m (\lambda^{25} q^{15})^{m^2/2} (\lambda^{-5} q^{-5})^{m/2} \\ &\quad \times \sum_{n=-\infty}^{\infty} (-1)^n (\lambda^{25} q^{15})^{n^2/2} (\lambda^{-5} q^{-5})^{n/2} \\ &= f(-\lambda^{10} q^5, -\lambda^{15} q^{10}) f(-\lambda^{10} q^5, -\lambda^{15} q^{10}). \end{aligned} \tag{1.2.13}$$

Similarly,

$$S_1 = -\lambda f(-\lambda^5 q^4, -\lambda^{20} q^{11}) f(-\lambda^{10} q^7, -\lambda^{15} q^8), \tag{1.2.14}$$

$$S_2 = -\lambda^2 q f(-\lambda^5 q^4, -\lambda^{20} q^{11}) f(-\lambda^5 q^2, -\lambda^{20} q^{13}), \tag{1.2.15}$$

$$S_3 = \lambda^3 q f(-q, -\lambda^{25} q^{14}) f(-\lambda^{10} q^7, -\lambda^{15} q^8), \tag{1.2.16}$$

and

$$S_4 = \lambda^4 q^2 f(-q, -\lambda^{25} q^{14}) f(-\lambda^5 q^2, -\lambda^{20} q^{13}). \tag{1.2.17}$$

Substituting (1.2.13)–(1.2.17) in (1.2.12) and then using (1.2.11), we complete the proof of (1.2.2).

In (1.2.1), replace $q^{1/5}$ by $\zeta^n q^{1/5}$, where ζ is a primitive fifth root of unity and $n = 1, 2, 3, 4, 5$, and then multiply the five identities. Thus, we find that

$$\prod_{n=1}^{5} \left(\frac{P}{\zeta^n} - \zeta^n Q - 1 \right) = \frac{1}{q f^5(-\lambda^{10} q^5, -\lambda^{15} q^{10})} \prod_{n=1}^{5} f(-\zeta^n q^{1/5}, -\zeta^{2n} \lambda q^{2/5}). \tag{1.2.18}$$

Simplifying the left side of (1.2.18) yields

$$P^5 - Q^5 - 1 - 5PQ - 5P^2 Q^2. \tag{1.2.19}$$

Now in Lemma 1.2.4, let $a = -q^{1/5}$, $b = -\lambda q^{2/5}$, $p = 5$, $j = 1$, and $k = 2$. Then $s = 3$ is a solution of $3x + 1 \equiv 0 \pmod 5$, and so

$$\prod_{n=1}^{5} f(-\zeta^n q^{1/5}, -\zeta^{2n} \lambda q^{2/5}) = \frac{f(-q, -\lambda^5 q^2) f^5(-\lambda^2 q, -\lambda^3 q^2)}{f(-\lambda^{10} q^5, -\lambda^{15} q^{10})}. \tag{1.2.20}$$

Using (1.2.19) and (1.2.20) in (1.2.18), we finish the proof of (1.2.3). □

Now we shall show that (1.1.10) and (1.1.11) are special cases of (1.2.1) and (1.2.3).

Proof of (1.1.10) *and* (1.1.11). Let $\lambda = 1$ in (1.2.1) and (1.2.3). Then by applying the quintuple product identity, Entry 1.2.1, with $(x, \lambda) = (q, q^2)$ and (q^2, q^{-1}), respectively, we see that by Lemma 1.2.1, Lemma 1.2.2, and (1.1.2),

$$P = \frac{f(-q^7, -q^8) + q f(-q^2, -q^{13})}{q^{1/5} f(-q^5)} = \frac{f(-q^2, -q^3)}{q^{1/5} f(-q, -q^4)} = \frac{1}{R(q)} \tag{1.2.21}$$

and

$$Q = \frac{f(-q^4, -q^{11}) - q f(-q, -q^{14})}{q^{-1/5} f(-q^5)} = \frac{q^{1/5} f(-q, -q^4)}{f(-q^2, -q^3)} = R(q). \tag{1.2.22}$$

Since $PQ = 1$, (1.2.1) and (1.2.3) reduce to (1.1.10) and (1.1.11), respectively. □

1.3 Hybrids of (1.1.10) and (1.1.11)

Entry 1.3.1 (p. 48). *If* $f(-q)$ *is defined by* (1.1.8), *then*

$$\sum_{n=-\infty}^{\infty}(-1)^n(10n+3)q^{(5n+3)n/2}=\left(\frac{3}{R^2(q)}+R^3(q)\right)q^{2/5}f^3(-q^5) \quad (1.3.1)$$

and

$$\sum_{n=-\infty}^{\infty}(-1)^n(10n+1)q^{(5n+1)n/2}=\left(\frac{1}{R^3(q)}-3R^2(q)\right)q^{3/5}f^3(-q^5). \quad (1.3.2)$$

Proof. The key to our proofs is Jacobi's identity [61, p. 39, Entry 24(ii)],

$$f^3(-q)=\sum_{n=-\infty}^{\infty}(-1)^n nq^{n(n+1)/2}. \quad (1.3.3)$$

By (1.1.10),

$$\left(\frac{1}{R(q)}-1-R(q)\right)^3=\frac{f^3(-q^{1/5})}{q^{3/5}f^3(-q^5)},$$

from which it follows that

$$q^{3/5}f^3(-q^5)\left\{5-\left(\frac{3}{R^2(q)}+R^3(q)\right)+\left(\frac{1}{R^3(q)}-3R^2(q)\right)\right\}=f^3(-q^{1/5}). \quad (1.3.4)$$

If we expand the left side of (1.3.4) as a power series in q, we find that the exponents of q in

$$5q^{3/5}f^3(-q^5) \quad (1.3.5)$$

are congruent to $\frac{3}{5}\pmod{1}$, the exponents in

$$-q^{3/5}f^3(-q^5)\left(\frac{3}{R^2(q)}+R^3(q)\right) \quad (1.3.6)$$

are congruent to $\frac{1}{5}\pmod{1}$, and the exponents in

$$q^{3/5}f^3(-q^5)\left(\frac{1}{R^3(q)}-3R^2(q)\right) \quad (1.3.7)$$

are integers.

By Jacobi's identity (1.3.3),

$$\begin{aligned}f^3(-q^{1/5})&=\sum_{n=-\infty}^{\infty}(-1)^n nq^{n(n+1)/10} \quad (1.3.8)\\&=\sum_{n=-\infty}^{\infty}(-1)^{5n}(5n)q^{5n(5n+1)/10}\\&\quad+\sum_{n=-\infty}^{\infty}(-1)^{5n+1}(5n+1)q^{(5n+1)(5n+2)/10}\end{aligned}$$