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M. Ram Murty
Michael Dewar
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Problems in the Theory of Modular Forms

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A great discovery solves a great problem but there is a grain of discovery in the solution of any problem.

- George Pólya, How to solve it.

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Part II Solutions

Acknowledgements

This book is based on an upper level undergraduate course given at Queen's University in the fall semester of 2011. The class consisted of primarily undergraduates, several graduate students, and a few post-doctoral fellows. Since the topic of modular forms has become central not only in number theory, but also other parts of mathematics (the most spectacular being theoretical physics), I felt an urgency to deliver this beautiful theory to the sincere student as quickly and painlessly as possible. As the topic of modular forms can be approached from various directions, some balance had to be achieved where only the most minimal background is assumed. The great nineteenth century philosopher Vivekananda has said that "it is practice first, and knowledge afterwards." Somehow, knowledge is gained through doing, by applying our knowledge, though it may be incomplete and our understanding imperfect. From this perspective, I have always felt that the best way to learn anything is via a "hands on" approach where the student learns the nuances of the theory through practical problem solving.

My two earlier books "Problems in Analytic Number Theory" and "Problems in Algebraic Number Theory" published by Springer were written with this underlying philosophy. Judging from the feedback I have received from students and researchers around the world, this teaching philosophy is well-founded and I hope the same is true with this book.

I would like to thank my two post-doctoral fellows, Michael Dewar and Hester Graves, in assisting me in this endeavour. They not only transferred my hand-written notes into LaTeX, but added substantial sections to amplify and elucidate the material. I also thank V. Kumar Murty, Sanoli Gun, Purusottam Rath, Alia Hamieh, and Kannappan Sampath for their valuable comments on earlier versions of this book. I especially thank Kannappan for his help in the production of LaTeX diagrams included in this book and the referees for their helpful comments.

Ram Murty
Kingston, Ontario

Preface

The earliest murmurs of a theory of modular forms can be traced back to the work of Jacobi in 1829 when he wrote his famous treatise *Fundamenta Nova Theoriae Functionum Ellipticarum* dealing with q-series and elliptic functions. In some parenthetic sense, this was further developed by Riemann, Hurwitz, Dedekind, Eisenstein, and Kronecker. However, it is in the work of Ramanujan, in his celebrated paper [29] of 1916 in which he introduced the τ-function, where we find the seeds of a comprehensive theory. There, Ramanujan studied the infinite product (in the variable q) given by

$$q \prod_{n=1}^{\infty} (1 - q^n)^{24} \tag{0.1}$$

which he expands as an infinite series and writes

$$\sum_{n=1}^{\infty} \tau(n) q^n.$$

Ramanujan computes by hand many of these coefficients and makes the following empirical observations about the integers $\tau(n)$:

(i) if $(m, n) = 1$ then $\tau(mn) = \tau(m)\tau(n)$,
(ii) if p is prime and $\alpha \geq 0$ then $\tau(p^{\alpha+2}) = \tau(p)\tau(p^{\alpha+1}) - p^{11}\tau(p^{\alpha})$,
(iii) if p is prime then $|\tau(p)| \leq 2p^{11/2}$.

Items (i) and (ii) were proved by Mordell in 1917 [23], a year after Ramanujan's paper appeared. But conjecture (iii) defied attempts by many eminent mathematicians until 1974, when Deligne [8] resolved it as a corollary of his solution to the Weil conjectures. In fact, only after Ramanujan's conjecture was reformulated in the context of algebraic geometry and its connection to the Weil conjectures made explicit, did mathematicians realize its central place in number theory, and mathematics in general.

At first, Ramanujan's conjectures are amazing. Why should coefficents of the power series defined by the infinite product in (0.1) show such structure? Why should they be multiplicative?

These questions were first addressed in a serious and fundamental way by Erich Hecke, who in the 1930's wrote a sequence of papers enunciating what we now call the Hecke theory of modular forms. For the most part, his theory was quite satisfactory in the sense that it explained conceptually why (i) and (ii) should be true. However, when it came to (iii), the theory suggested only a general conjecture but could not "explain" why it should be true. Such an explanation had to await further developments in algebraic geometry, largely due to Weil and Grothendieck.

Afterwards, in the late 1940's, Hecke's theory was further developed on the one hand by Maass, who noticed the existence of a "real-analytic" counterpart, and by Rankin and Selberg, who developed a theory of L-series and obtained significant results towards conjecture (iii).

In the 1950's and 1960's, Harish-Chandra and, subsequently, Langlands reformulated the notion of a modular form in the larger framework of representation theory of Lie groups. This opened up a new universe linking number theory and representation theory, and subsequently led to the development of the Langlands program in the theory of automorphic representations. For a readable account of this connection, we refer the reader to Kumar Murty's article "Ramanujan and Harish-Chandra" [26].

In the 1960's and 1970's, Hecke's theory was extended to "higher levels," notably by Atkin and Lehner. At the same time, its connection to the theory of elliptic curves, and more generally abelian varieties, forged the link to arithmetic, algebraic geometry, and more precisely to the theory of Galois representations. Perhaps the most exciting event in this context was the insight of Hellegouarch and Frey relating Fermat's Last Theorem and modular forms. This led Serre to formulate precise conjectures that paved the way for a method of attack on Fermat's Last Theorem. The turning point was when Ribet showed that if every elliptic curve "arose from a modular form" (Taniyama's conjecture) then Fermat's Last Theorem follows.

Andrew Wiles recalls that when he heard this result, he set himself the task of proving Taniyama's conjecture. Wiles completed his proof in 1995. His solution of Fermat's Last Theorem required the full force of number theory, algebraic geometry, and representation theory.

Only a special case of Serre's conjecture was needed for Fermat's Last Theorem, but other cases had important consequences. A landmark theorem is the two-dimensional reciprocity law for Galois representations. In 2008, Khare and Wintenberger showed that every two-dimensional "odd" Galois representation "arises from a modular form."

This theorem can be viewed as a generalization of Wiles' work and at the same time, as the methods are different, it offers yet another (although equally difficult) resolution to Fermat's Last Theorem, which is only one of many conjectures to which the theory of modular forms has been applied.

A conjecture or a problem in mathematics acts like a muse that inspires further developments, new concepts, and a rich tapestry of fundamental ideas. The theory of modular forms is an essential part of mathematics and it is becoming increasingly clear that it will play a central role in the development of twenty-first century mathematics. This is our main motivation for the writing of this book. It is to acquaint the graduate student in a painless manner to the essential ideas of the theory. At the same time, as theory is sterile without practice, we have tried to invite and engage the student in this topic through the problem-solving approach. Along with my other two books, "Problems in Algebraic Number Theory" and 'Problems in Analytic Number Theory, " this book should serve as a practical guide for the serious student to teach herself or himself the rudiments of number theory and to embark in the exciting pursuit of research work in this area. At the end, I have listed some references for further study to assist the student in this lofty endeavour.

Kingston, Ontario *Ram Murty*
August 2014

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Part I
Problems

Chapter 1
Jacobi's *q*-series

1.1 The *q*-exponential function

It is not clear exactly how to define a q-series. Some experts humorously suggest that it is any power series in q. To some extent this may be true. However, one can say that part of the theory is connected with modular forms and another part with combinatorics. Euler, Jacobi, Ramanujan and many others have made expert use of q-series to derive remarkable number theoretic results ranging from the study of the partition function, the number of representations of natural numbers as sums of squares to the development of exotic continued fractions as is evidenced by the recent book [7] on the subject. During the last fifty years, more connections have arisen, most notably with Lie theory and representation theory as well as theoretical physics. Since the subject is gaining prominence and significance, pointing to a parallel world of mathematics in which q-analogues of classical theories exist, and since the prerequisites for the theory are minimal, we found it fitting to introduce the reader to the world of modular forms through q-series. Already, one sees several classical results such as Jacobi's triple product identity and his celebrated formula for the number of ways of writing a natural number as a sum of four squares as immediate consequences. Moreover, as the problem of representing a natural number as a sum of k squares (with k even) is intimately tied to the theory of modular forms of integral weight, it seems fitting to begin with a study of how q-series can be used to study this problem. The number of ways of writing a number as a sum of k squares with k odd is related to the study of modular forms of half-integral weight, which is beyond the scope of this book.

To ease the reader into this fascinating world, we begin our study with the function

$$\mathcal{E}_q(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{(q^n - 1)(q^{n-1} - 1) \cdots (q - 1)},$$

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and view it as the q-analog of the classical exponential function. We work with $\mathcal{E}_q(x)$ as a formal Laurent series. If we define the empty product as 1, then the series can be written as

$$\mathcal{E}_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q^n-1)(q^{n-1}-1)\cdots(q-1)},$$

It may not be clear to the reader why this should be considered as an analog of the exponential function. The underlying "philosophy" of moving from the world of natural numbers to the q-world seems to be partially based on the observation that

$$\lim_{q\to 1} \frac{q^n-1}{q-1} = n,$$

so that one views q^n-1 (or more precisely $(q^n-1)/(q-1)$) as the q-analog of the natural number n. Once this is understood, many of the functions in the q-world become meaningful and exhibit remarkable structural properties.

Exercise 1.1.1. *Show that*

$$\mathcal{E}_q(x) - \mathcal{E}_q\left(\frac{x}{q}\right) = \frac{x}{q}\mathcal{E}_q\left(\frac{x}{q}\right).$$

Exercise 1.1.2. *Prove that if* $|q|>1$ *then*

$$\mathcal{E}_q(x) = \prod_{n=1}^{\infty}\left(1+\frac{x}{q^n}\right).$$

Exercise 1.1.3. *Show that if* $|q|<1$ *then*

$$\prod_{n=0}^{\infty}(1+q^n x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(1-q)\cdots(1-q^n)}.$$

Exercise 1.1.4. *Prove that if* $|q|>1$ *then*

$$(1+x)\mathcal{E}_{\frac{1}{q}}(x) = \frac{1}{\mathcal{E}_q(x)}.$$

Deduce that if $|q|<1$ *then*

$$\prod_{n=0}^{\infty}(1+q^n x)^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{(q^n-1)\cdots(q-1)}.$$

1.2 Jacobi's Triple Product Identity

We now give a simple proof of the celebrated identity of Jacobi:

Theorem 1.2.1 (Jacobi Triple Product). *If* $|q| < 1$ *and* $x \neq 0$ *then*

$$\prod_{n=0}^{\infty}(1-q^{2n+2})(1+q^{2n+1}x)\left(1+\frac{q^{2n+1}}{x}\right) = \sum_{n=-\infty}^{\infty} q^{n^2}x^n. \tag{1.1}$$

Remark. According to Askey, this was also contained in some unpublished work of Gauss (see [3]). In our proof of (1.1), we follow Andrews (see [1]).

Proof. By Exercise 1.1.3 we have, upon replacing q by q^2 and x by xq,

$$\begin{aligned}\prod_{n=0}^{\infty}(1+xq^{2n+1}) &= \sum_{n=0}^{\infty}\frac{q^{n^2}x^n}{(1-q^2)\cdots(1-q^{2n})}\\ &= \sum_{n=0}^{\infty} q^{n^2}x^n\frac{\prod_{j=0}^{\infty}(1-q^{2n+2j+2})}{\prod_{j=0}^{\infty}(1-q^{2j+2})}\\ &= \prod_{j=0}^{\infty}(1-q^{2j+2})^{-1}\sum_{n=-\infty}^{\infty} q^{n^2}x^n\prod_{j=0}^{\infty}(1-q^{2n+2j+2}),\end{aligned}$$

since for negative n, the product inside the sum is zero. Again by Exercise 1.1.3, replacing q by q^2 and replacing x by $-q^{2n+2}$, we can write the product inside the summation as

$$\sum_{m=0}^{\infty}\frac{(-1)^m q^{m^2+m+2nm}}{(1-q^2)\cdots(1-q^{2m})}.$$

The sum under consideration becomes

$$\sum_{n=-\infty}^{\infty} q^{n^2}x^n\sum_{m=0}^{\infty}\frac{(-1)^m q^{m^2+m+2nm}}{(1-q^2)\cdots(1-q^{2m})}.$$

Interchanging the sums, we get

$$\sum_{m=0}^{\infty}\frac{(-1)^m(\frac{q}{x})^m}{(1-q^2)\cdots(1-q^{2m})}\sum_{n=-\infty}^{\infty} q^{(m+n)^2}x^{n+m}.$$

The innermost sum is

$$\sum_{n=-\infty}^{\infty} q^{n^2}x^n$$

and can be factored out. The remaining sum is

$$\prod_{j=0}^{\infty}\left(1+\frac{q^{2j+1}}{x}\right)^{-1}$$

by Exercise 1.1.4 (where we have set q^2 for q and $\frac{q}{x}$ for x). Putting everything together gives the desired result. □

Exercise 1.2.2 (Euler's pentagonal number theorem). *Show that if* $|q| < 1$ *then*

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{k(3k-1)}{2}}.$$

Theorem 1.2.3 (Jacobi's formula). *If* $|q| < 1$ *then*

$$\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{k=0}^{\infty}(-1)^k(2k+1)q^{\frac{k(k+1)}{2}}.$$

Proof. From the triple product identity, with x replaced by $-x$, we see that

$$\prod_{n=1}^{\infty}(1-q^{2n})(1-xq^{2n-1})(1-x^{-1}q^{2n-1}) = \sum_{k=-\infty}^{\infty}(-x)^k q^{k^2}.$$

The left hand side has a factor $(1-xq)$ coming from $n=1$, and consequently vanishes when $x=\frac{1}{q}$. The same is true of the right hand side. Writing

$$\prod_{n=1}^{\infty}(1-xq^{2n-1}) = (1-xq)\prod_{n=1}^{\infty}(1-xq^{2n+1}),$$

we obtain

$$\prod_{n=1}^{\infty}(1-q^{2n})(1-xq^{2n+1})(1-x^{-1}q^{2n-1}) = \frac{1}{1-xq}\sum_{k=-\infty}^{\infty}(-x)^k q^{k^2}.$$

Putting $x=\frac{1}{q}$ in the left hand side gives

$$\prod_{n=1}^{\infty}(1-q^{2n})^3.$$

For the right hand side, we use l'Hopital's rule to take the limit as $x\to\frac{1}{q}$ to get

$$-\frac{1}{q}\sum_{k=-\infty}^{\infty}(-1)^k k q^{1-k+k^2} = -\sum_{k=-\infty}^{\infty}(-1)^k k q^{k^2-k}.$$

We observe that the function $f(k)=k^2-k$ has the property that $f(k)=f(-(k-1))$. Thus, pairing up k and $-(k-1)$ in the sum, we get that it is

$$-\sum_{k=1}^{\infty}\{(-1)^k k+(-1)^{k-1}(1-k)\}q^{k^2-k}=-\sum_{k=1}^{\infty}(-1)^k(2k-1)q^{k^2-k}.$$

We can rewrite this sum as

$$\sum_{k=1}^{\infty}(-1)^{k-1}(2(k-1)+1)q^{k(k-1)}=\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)}.$$

Changing q by $q^{\frac{1}{2}}$ gives the desired result. □

Exercise 1.2.4. *Prove that*

$$\sum_{n=-\infty}^{\infty}(4n+1)q^{2n^2+n}=\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{\frac{k(k+1)}{2}}.$$

1.3 Jacobi's two-square theorem

In this section, our goal is to obtain formulas for the number of ways a natural number m can be written as a sum of two squares. We recognize this as the computation of the m-th coefficient in the power series expansion

$$\left(\sum_{n=-\infty}^{\infty}q^{n^2}\right)^2.$$

We prove:

Theorem 1.3.1.

$$\left(\sum_{n=-\infty}^{\infty}q^{n^2}\right)^2=1+4\sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{1-q^{4n+1}}-\frac{q^{4n+3}}{1-q^{4n+3}}\right).$$

An immediate corollary is deduced by expanding the right hand side as a power series in q. Let $r_k(n)$ be the number of ways of writing n as a sum of k squares.

Corollary 1.3.2. *For $n \geq 1$, we have $r_2(n)=4(d_1(n)-d_3(n))$ where $d_i(n)$ is the number of divisors of n congruent to $i \pmod 4$.*

The following proof is due to M.D. Hirschhorn [15].

Proof of Theorem 1.3.1. In the Jacobi triple product identity, we put a^4q for x and q^2 for q to get

$$\sum_{n=-\infty}^{\infty}a^{4n}q^{2n^2+n}=\prod_{n=1}^{\infty}(1+a^4q^{4n-1})(1+a^{-4}q^{4n-3})(1-q^{4n}).$$

Multiply both sides by a and differentiate with respect to a, to obtain

$$\sum_{n=-\infty}^{\infty}(4n+1)a^{4n}q^{2n^2+n} = \prod_{n=1}^{\infty}(1+a^4q^{4n-1})(1+a^{-4}q^{4n-3})(1-q^{4n}) + a\frac{d}{da}\left\{\prod_{n=1}^{\infty}(1+a^4q^{4n-1})(1+a^{-4}q^{4n-3})(1-q^{4n})\right\}.$$

To differentiate the product, it is useful to observe that if

$$P(a) = \prod_{n=1}^{\infty} f_n(a),$$

then by taking the logarithmic derivative of both sides, we get

$$\frac{P'(a)}{P(a)} = \sum_{n=1}^{\infty}\frac{f_n'(a)}{f_n(a)}.$$

In our case, we get

$$\sum_{n=-\infty}^{\infty}(4n+1)a^{4n}q^{2n^2+n} = \prod_{n=1}^{\infty}(1+a^4q^{4n-1})(1+a^{-4}q^{4n-3})(1-q^{4n}) \times\left\{1+a\sum_{n=1}^{\infty}\frac{4a^3q^{4n-1}}{1+a^4q^{4n-1}} - \frac{4a^{-5}q^{4n-3}}{1+a^{-4}q^{4n-3}}\right\}.$$

We now put $a=1$ and get

$$\sum_{n=-\infty}^{\infty}(4n+1)q^{2n^2+n} = \left(\prod_{n=1}^{\infty}(1+q^{4n-1})(1+q^{4n-3})(1-q^{4n})\right) \times\left\{1-4\sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{1+q^{4n+1}} - \frac{q^{4n+3}}{1+q^{4n+3}}\right)\right\} \tag{1.2}$$

after changing the index of summation to start from $n=0$.

By Theorem 1.2.3 and Exercise 1.2.4, the sum on the left hand side is

$$\prod_{n=1}^{\infty}(1-q^n)^3.$$

This can be rewritten as

$$\prod_{n=1}^{\infty}(1-q^{2n-1})^3(1-q^{2n})^3.$$

The product on the right hand side of (1.2) can be written as

$$\prod_{n=1}^{\infty}\left(\frac{1-q^{4n-2}}{1-q^{2n-1}}\right)(1-q^{4n})=\prod_{n=1}^{\infty}\left(\frac{1-q^{2n}}{1-q^{2n-1}}\right),$$

so we now deduce that

$$\prod_{n=1}^{\infty}(1-q^{2n-1})^4(1-q^{2n})^2=1-4\sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{1+q^{4n+1}}-\frac{q^{4n+3}}{1+q^{4n+3}}\right).$$

But the left hand side is

$$\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}\right)^2$$

from the triple product formula. Changing q to $-q$ gives

$$\left(\sum_{n=-\infty}^{\infty}q^{n^2}\right)^2=1+4\sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{1-q^{4n+1}}-\frac{q^{4n+3}}{1-q^{4n+3}}\right),$$

which is the desired result. □

Exercise 1.3.3. *Let $x_4(n)$ be defined as follows:*

$$x_4(n)=\begin{cases}0 & \text{if } n \text{ is even}\\ +1 & \text{if } n\equiv 1 \pmod 4\\ -1 & \text{if } n\equiv 3 \pmod 4\end{cases}$$

Prove that $x_4(n)$ is completely multiplicative. That is, show that $x_4(mn)=x_4(m)x_4(n)$.

Exercise 1.3.4. *Show that*

$$r_2(n)=4\sum_{d|n}x_4(d).$$

1.4 Jacobi's four square theorem

Using results obtained for $r_2(n)$, we will derive an explicit formula for $r_4(n)$, the number of ways of writing n as a sum of four squares. This formula is due to Jacobi who derived it using the theory of elliptic functions. Here, we will follow a method due to Ramanujan that is completely elementary and based on the following exercise.

Exercise 1.4.1. *Show that*

(a) $\frac{1}{2}\left(\cot\frac{\theta}{2}\right)\sin n\theta = \frac{1}{2} + \cos\theta + \cos 2\theta + \cdots + \cos(n-1)\theta + \frac{1}{2}\cos n\theta,$

(b) $2(\sin m\theta)(\sin n\theta) = \cos(m-n)\theta - \cos(m+n)\theta.$

Exercise 1.4.2. *For* $|q| < 1$*, we let* $u_r = \frac{q^r}{1-q^r}$ *so that*

$$\frac{q^r}{(1-q^r)^2} = u_r(1+u_r).$$

Prove that

$$\sum_{m=1}^{\infty} u_m(1+u_m) = \sum_{n=1}^{\infty} nu_n.$$

Exercise 1.4.3. *With notation as in the previous exercise, show that*

$$\sum_{m=1}^{\infty}(-1)^{m-1}u_{2m}(1+u_{2m}) = \sum_{n=1}^{\infty}(2n-1)u_{4n-2}.$$

We are now ready to prove the following theorem due to Ramanujan.

Theorem 1.4.4. *Let* θ *be real and not a multiple of* π*. Set*

$$L = L(q,\theta) = \frac{1}{4}\cot\frac{\theta}{2} + \sum_{n=1}^{\infty} u_n \sin n\theta,$$

$$T_1 = T_1(q,\theta) = \left(\frac{1}{4}\cot\frac{\theta}{2}\right)^2 + \sum_{n=1}^{\infty} u_n(1+u_n)\cos n\theta,$$

$$T_2 = T_2(q,\theta) = \frac{1}{2}\sum_{n=1}^{\infty} nu_n(1-\cos n\theta),$$

where $u_n = \frac{q^n}{1-q^n}$*. Then* $L^2 = T_1 + T_2$*.*

Before we begin the proof of this theorem, we indicate how it implies Jacobi's theorem on representing a natural number as a sum of four squares.

Corollary 1.4.5.

$$\left\{1 + 4\sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{1-q^{4n+1}} - \frac{q^{4n+3}}{1-q^{4n+3}}\right)\right\}^2 = 1 + 8\sum_{\substack{n=1 \\ n\not\equiv 0 \pmod 4}}^{\infty}\frac{nq^n}{1-q^n}.$$

Proof. Put $\theta = \frac{\pi}{2}$ in Theorem 1.4.4. Then

$$L = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} \sin n\frac{\pi}{2}.$$

If n is even, $\sin n\frac{\pi}{2} = 0$ so the right hand side is

$$\frac{1}{4} + \sum_{n=0}^{\infty} \frac{q^{2n+1}}{1-q^{2n+1}} \sin(2n+1)\frac{\pi}{2}.$$

Now, if n is even, $\sin(2n+1)\frac{\pi}{2} = 1$ and if n is odd, $\sin(2n+1)\frac{\pi}{2} = -1$. In other words,

$$4L = 1 + 4\sum_{n=0}^{\infty} \left(\frac{q^{4n+1}}{1-q^{4n+1}} - \frac{q^{4n+3}}{1-q^{4n+3}} \right).$$

Since $\cos n\frac{\pi}{2} = 0$ if n is odd and $(-1)^{\frac{n}{2}}$ if n is even, computing T_1 and T_2 in Theorem 1.4.4 gives

$$16T_1 = 1 - 16\sum_{m=1}^{\infty} (-1)^{m-1} u_{2m}(1+u_{2m}),$$

$$16T_2 = 8\sum_{m=1}^{\infty} (2m-1)u_{2m-1} + 32\sum_{m=1}^{\infty} (2m-1)u_{4m-2}.$$

Putting everything together and using Exercise 1.4.3 gives

$$16L^2 = 16(T_1+T_2) = 1 + 8\sum_{\substack{n=1 \\ n\not\equiv 0 \pmod 4}}^{\infty} nu_n,$$

which is the desired result. □

Corollary 1.4.6. *Let $r_4(n)$ be the number of ways of writing n as a sum of four squares. Then*

$$r_4(n) = 8\sum_{\substack{d|n \\ d\not\equiv 0 \pmod 4}} d.$$

In particular, $r_4(n) > 0$ for all $n \geq 1$.

Proof. We need only invoke Theorem 1.3.1 and Corollary 1.4.5 to deduce that

$$\sum_{n=0}^{\infty} r_4(n)q^n = \left(\sum_{n=-\infty}^{\infty} q^{n^2} \right)^4 = 1 + 8\sum_{\substack{n=1 \\ n\not\equiv 0 \pmod 4}}^{\infty} \frac{nq^n}{1-q^n}.$$

We write the right hand side as

$$1 + 8 \sum_{\substack{n=1 \\ n \not\equiv 0 \pmod 4}}^{\infty} n \sum_{m=1}^{\infty} q^{nm} = 1 + 8 \sum_{n=1}^{\infty} q^n \left(\sum_{\substack{d|n \\ d \not\equiv 0 \pmod 4}} d \right),$$

from which we deduce the desired formula for $r_4(n)$. □

Proof of Theorem 1.4.4. Squaring L, we obtain

$$\begin{aligned} L^2 &= \left\{ \frac{1}{4} \cot \frac{\theta}{2} + \sum_{n=1}^{\infty} u_n \sin n\theta \right\}^2 \\ &= \left(\frac{1}{4} \cot \frac{\theta}{2} \right)^2 + \frac{1}{2} \left(\cot \frac{\theta}{2} \right) \sum_{n=1}^{\infty} u_n \sin n\theta + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_m u_n \sin m\theta \sin n\theta \\ &= \left(\frac{1}{4} \cot \frac{\theta}{2} \right)^2 + S_1 + S_2, \end{aligned}$$

where

$$S_1 = \frac{1}{2} \left(\cot \frac{\theta}{2} \right) \sum_{n=1}^{\infty} u_n \sin n\theta$$

and

$$S_2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_m u_n \sin m\theta \sin n\theta.$$

By Exercise 1.4.1,

$$\frac{1}{2} \left(\cot \frac{\theta}{2} \right) \sin n\theta = \frac{1}{2} + \cos \theta + \cos 2\theta + \cdots + \cos(n-1)\theta + \frac{1}{2} \cos n\theta$$

and

$$2 (\sin m\theta) (\sin n\theta) = \cos(m-n)\theta - \cos(m+n)\theta,$$

so that

$$S_1 = \sum_{n=1}^{\infty} u_n \left\{ \frac{1}{2} + \cos \theta + \cos 2\theta + \cdots + \cos(n-1)\theta + \frac{1}{2} \cos n\theta \right\}$$

and

$$S_2 = \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_m u_n \left\{ \cos(m-n)\theta - \cos(m+n)\theta \right\}.$$

In other words,

$$L^2 = \left(\frac{1}{4} \cot \frac{\theta}{2} \right)^2 + c_0 + \sum_{k=1}^{\infty} c_k \cos k\theta$$

for certain constants c_k, after re-arranging $S_1 + S_2$ as a cosine series. Now

$$c_0 = \frac{1}{2}\sum_{n=1}^{\infty}\left(u_n + u_n^2\right) = \frac{1}{2}\sum_{n=1}^{\infty} n u_n,$$

by Exercise 1.4.2. For $k \geq 1$, we have

$$c_k = \frac{1}{2}u_k + \sum_{n=k+1}^{\infty} u_n + \frac{1}{2}\sum_{m-n=k} u_m u_n + \frac{1}{2}\sum_{n-m=k} u_m u_n - \frac{1}{2}\sum_{m+n=k} u_m u_n,$$

where $m, n \geq 1$ in all of the summations. We may rewrite this as

$$c_k = \frac{1}{2}u_k + \sum_{j=1}^{\infty} u_{k+j} + \sum_{j=1}^{\infty} u_j u_{j+k} - \frac{1}{2}\sum_{j=1}^{k-1} u_j u_{k-j}.$$

It is easily checked that

$$u_j u_{k-j} = u_k(1 + u_j + u_{k-j})$$

and

$$u_{k+j} + u_j u_{k+j} = u_k(u_j - u_{k+j})$$

so that

$$c_k = u_k\left\{\frac{1}{2} + \sum_{j=1}^{\infty}(u_j - u_{k+j}) - \frac{1}{2}\sum_{j=1}^{k-1}(1 + u_j + u_{k-j})\right\}.$$

The first sum telescopes and we obtain

$$\begin{aligned} c_k &= u_k\left\{\frac{1}{2} + (u_1 + u_2 + \cdots + u_k) - \frac{1}{2}(k-1) - (u_1 + u_2 + \cdots + u_{k-1})\right\} \\ &= u_k(1 + u_k - \frac{1}{2}k). \end{aligned}$$

Hence

$$\begin{aligned} L^2 &= \left(\frac{1}{4}\cot\frac{\theta}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} n u_n + \sum_{k=1}^{\infty} u_k(1 + u_k - \frac{1}{2}k)\cos k\theta \\ &= \left(\frac{1}{4}\cot\frac{\theta}{2}\right)^2 + \sum_{k=1}^{\infty} u_k(1 + u_k)\cos k\theta + \frac{1}{2}\sum_{k=1}^{\infty} k u_k(1 - \cos k\theta), \end{aligned}$$

which is the desired result. □

The methods utilised above to determine formulas for $r_2(n)$ and $r_4(n)$ can be extended for other values $r_{2k}(n)$ for certain values of k. We refer the reader to consult [11] for more examples. In [11], the student will also find a discussion for finding formulas for $r_k(n)$ when k is odd and this is intimately connected with the theory of modular forms of half-integral weight which is beyond the scope of this book.

1.5 Supplementary problems

Exercise 1.5.1. *Define* $\tau(n)$ *by*

$$q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n.$$

Show that $\tau(n)$ *is odd if and only if* $n=(2m+1)^2$ *for some* m.

Exercise 1.5.2. *Let* $r_k(n)$ *be the number of ways of writing* n *as a sum of* k *squares. Show that*

$$r_k(n) = \sum_{a=0}^{n} r_i(a)r_{k-i}(n-a)$$

for any i *satisfying* $1 \le i \le k$.

Exercise 1.5.3. *Define the q-logarithm function* $\mathcal{L}_q(x)$ *as*

$$\mathcal{L}_q(x) = \sum_{n=1}^{\infty}\frac{x^n}{q^n-1}.$$

Suppose $|x|<|q|$ *and* $|q|>1$. *Show that*

$$\mathcal{L}_q(x) - \mathcal{L}_q\left(\frac{x}{q}\right) = \frac{x}{q-x}.$$

Deduce that

$$\mathcal{L}_q(x) = \sum_{n=1}^{\infty}\frac{x}{q^n-x}.$$

Exercise 1.5.4. *Show that if* $|x|<|q|$ *and* $|q|>1$ *then*

$$\mathcal{L}_q(x) = \frac{x\mathcal{E}_q'(-x)}{\mathcal{E}_q(-x)},$$

where the derivative is with respect to x.

Chapter 2
The Modular Group

2.1 The full modular group

Exercise 2.1.1. *Let R be a commutative ring with identity. Show that the set*

$$\mathrm{SL}_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{matrix} a, b, c, d \in R \\ ad - bc = 1 \end{matrix} \right\}$$

forms a group under matrix multiplication.

The *(full) modular group* $\mathrm{SL}_2(\mathbb{Z})$ plays a pivotal role in the theory of modular forms. One also considers $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$. The relationship of $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{R})$ is similar to the relationship of $\mathbb{Z}$ to $\mathbb{R}$ in the sense that $\mathbb{Z}$ is a discrete subgroup of $\mathbb{R}$ and $\mathrm{SL}_2(\mathbb{Z})$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. We will show below that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the elements

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Theorem 2.1.2. *The matrices S and T generate* $\mathrm{SL}_2(\mathbb{Z})$.

Proof. Observe that

$$T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

so that

$$T^n \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + nc & b + nd \\ c & d \end{pmatrix}. \tag{2.1}$$

Also $S^2 = -I$ and

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}.$$

M.R. Murty et al., *Problems in the Theory of Modular Forms*, HBA Lecture Notes in Mathematics, DOI 10.1007/978-981-10-2651-5_2

Now let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be any element of $\mathrm{SL}_2(\mathbb{Z})$. If $c = 0$, then $ad = 1$ implies that $a = d = \pm 1$. In this case

$$g = \pm \begin{pmatrix} 1 & b' \\ 0 & 1 \end{pmatrix} = \pm T^{b'}.$$

where $b' = \pm b$. Since $S^2 = -I$, either $g = T^{b'}$ or $S^2 T^{b'}$. If $c \neq 0$, we proceed as follows. Without loss of generality we may suppose $|a| \geq |c|$, for otherwise we can apply S to arrange this. By the Division Algorithm, we can write $a = cq + r$ with $0 \leq r < |c|$. Then $T^{-q} g$ has upper left entry $r = a - qc$ which is smaller than $|c|$. Applying S switches the rows (with a sign change) and so we can iterate the process if $r \neq 0$. After a finite number of steps, we are reduced to the case $c = 0$ and we are done. □

Exercise 2.1.3. *Show that S has order 4, ST has order 6, and T has infinite order.*

Exercise 2.1.4. *Show that $\mathrm{SL}_2(\mathbb{Z})$ is generated by two elements of finite order, namely S and ST of order 4 and 6 respectively.*

Exercise 2.1.5. *Show that any homomorphism*

$$\phi : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}^\times$$

has image contained in the finite subgroup of $\mathbb{C}^\times$ consisting of 12^{th} roots of unity.

Exercise 2.1.6. *Show that $\mathrm{SL}_2(\mathbb{Z})$ is generated by*

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Exercise 2.1.7. *Suppose that $(c, d, N) = 1$. Show that there are elements $c' = c + tN$ and $d' = d + sN$ for some integers s, t such that $(c', d') = 1$.*

2.2 Subgroups of the modular group

For each natural number N, the *principal congruence subgroup* of level N, denoted $\Gamma(N)$, is the group

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

In particular, $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$.

Exercise 2.2.1. *Show that the natural map*