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Geometry, Algebra and Applications: From Mechanics to Cryptography

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Geometry, Algebra and Applications: From Mechanics to Cryptography

In Honor of Jaime Muñoz Masqué

Springer

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A las aladas almas de las rosas
del almendro de nata te requiero,
que tenemos que hablar de muchas cosas,
compañero del alma, compañero.

Elegía a Ramón Sijé (1936)
Miguel Hernández

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About the Editors

Prof. Marco Castrillón López read mathematics and physics in Madrid. He is currently Professor Titular at the Universidad Complutense de Madrid, where he also received his Ph.D. in Mathematics. He was a postdoc at École Polytechnique Fédérale de Lausanne (Switzerland), and Faculty Visitor at Caltech (Pasadena, USA), PIMS (Vancouver, Canada), Imperial College (London, UK), TATA Institute (Mumbai, India) and PUC (Rio de Janeiro, Brazil). His research work mainly focuses on geometric variational calculus, gauge theories and Riemannian geometry with applications to relativity, classical field theories and other topics in theoretical physics. His has over 50 publications and books to his name.

Luis Hernández Encinas graduated in mathematics at the University of Salamanca (Spain) in 1980, and received his Ph.D. in Mathematics from the same university in 1992. He is Researcher at the Department of Information Processing and Cryptography (TIC) in the Institute of Physical and Information Technologies (ITEFI), Spanish National Research Council (CSIC) in Madrid. He has participated in more than 30 research projects. He is author of nine books, nine patents, more than 150 papers, and over 100 contributions to workshops and conferences. He has also supervised several doctoral theses. His current research interests include cryptography and cryptanalysis of public key cryptosystems, digital signature schemes, authentication and identification protocols, crypto-biometry, side channel attacks and number theory problems.

Prof. Pedro Martínez Gadea taught at the Universities of Santiago de Compostela and Valladolid in Spain. He is now scientific Researcher at the Instituto de Física Fundamental, CSIC, Madrid, Spain. He has published almost 70 research papers on several topics of differential geometry and algebraic topology. He has also been advisor for four Ph.D. theses. His current interests include differential geometry, more specifically in homogeneous spin Riemannian manifolds and Ricci-flat invariant Kähler structures.

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Introduction

This volume presents a collection of articles to honour Prof. Jaime Muñoz Masqué on the occasion of his 65th birthday. Jaime was born on 20 September 1950 in Sabadell, Barcelona (Spain), to his parents, Manuel and Rosenda. He attended high school in his home village, initially showing an inclination towards literature and poetry, which he subsequently combined with a strong interest in mathematical problems. He devoted more time to the latter; for instance, he spent a summer of his adolescence exploring the intriguing question of the unsolvability of the equations defined by fifth degree polynomials. This clearly indicated that his destiny was to study mathematics, which he did at the University of Barcelona.

During his studies at the university, Jaime made acquaintance of two important persons: First, María Sicilia, his future wife, who was also studying mathematics, and second, Pedro Luis García Pérez, with whom he decided to do his Ph.D. As Prof. García held a post in the University of Salamanca, the newly established family moved to this city after both María and Jaime had completed their studies in 1973. Jaime won his position as High School Professor (Catedrático) in 1975, working first in Zamora and then in Alba de Tormes (Salamanca). Jaime helped some of his colleagues at the High School María de Molina to prepare for their national-level exams in order to obtain permanent positions. They all remember these years with affection.

In the meantime, Jaime had also begun to lecture at the University of Salamanca. In 1983, Jaime defended his doctoral thesis at that university, entitled *Hamilton-Cartan Theory for higher-order variational problems on fibered manifolds* (*Teoría de Hamilton-Cartan para los problemas variacionales de orden superior sobre variedades fibradas*). He also began his fruitful scientific career with the publication of his papers. In his first article, *Higher-order structure forms and infinitesimal contact transformations* (*Formes de structure et transformations infinitésimales de contact d'ordre supérieur*, CR Acad Sci Paris Sér I Math 1984; 298, no. 8:185–8), he formalized the geometry behind the natural lift of vector fields from a bundle to its jet extension for arbitrary degree. This tool was essential for his work on higher-order variational calculus, the topic of his thesis, on which he became a world expert. At the same time, his family grew as María and Jaime had their three children: Ana, Joaquín and Teresa.

From 1984 to 1989, Jaime was first Assistant Professor and then Associate Professor at the University of Salamanca, where he continued his scientific work, mainly in the fields of differential geometry and algebra. In 1989 he was appointed as Researcher at CSIC (Spanish National Research Council) and the family moved to Madrid. While working in his new position he added cryptography to the list of his interests, and joined research projects in this field. He continued collaborating in CSIC and has carried out research on these topics until now.

In parallel with his scientific work, Jaime delivered courses in different universities where he showed his rare talent of explaining complex mathematics in a clean, simple and rigorous language. In association with this academic work, he has been the advisor for nine doctoral theses (Marco Castrillón López, Raúl Durán Díaz, Víctor Fernández Mateos, Roberto Ferreiro, Ángel Martín del Rey, Alberto Peinado Domínguez, Luis Pozo Coronado, Eugenia Rosado María, Antonio Valdés) covering a varied collection of topics in geometry and algebra, from variational calculus, Riemannian geometry and theory of invariants to cryptography. We have borne in mind this versatility for choosing the title of this volume, which offers an indication of Jaime's vast knowledge and wide-ranging scientific works. In this respect, the database of the Mathematical Reviews of the AMS includes as many as 162 contributions from Jaime, including both books and articles, on which he has worked with 39 collaborators.

The general consensus among the people who work with Jaime is that he is not only a hard worker but also possesses a very broad knowledge of mathematics and physics (as well as poetry and philosophy!) and an incredible capability to tackle problems in very different areas in an interdisciplinary atmosphere. We all enjoy his warm personality, the conversations with him over a cup of coffee and especially his generosity, in all senses of the word. Jaime is a person who loves mathematics and with whom one feels that excitement which accompanies the search for a solution or the thrilling experience of finding those hidden mathematical gems accessible only to a select group—a group of which Jaime is undoubtedly a member.

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A Survey on Homogeneous Structures on the Classical Hyperbolic Spaces

Wafaa Batat, P.M. Gadea and José A. Oubiña

Dedicated to our colleague and friend Jaime Muñoz Masqué, a good mathematician, with affection and admiration, on the occasion of his 65th birthday

Abstract This is a survey on homogeneous Riemannian, Kähler or quaternionic Kähler structures on the real, complex or quaternionic hyperbolic spaces $\mathbb{R}\mathrm{H}(n)$, $\mathbb{C}\mathrm{H}(n)$ and $\mathbb{H}\mathrm{H}(n)$, respectively.

Keywords Homogeneous Riemannian structures · Classical hyperbolic spaces

1 Introduction

Real, complex and quaternionic hyperbolic spaces and the Cayley hyperbolic plane are known to be important spaces and have been and are subject of much research. Two general references are Chen and Greenberg [10] and Ratcliffe [22].

On the other hand, homogeneous Riemannian structures were introduced by Ambrose and Singer [3], and further studied in depth by Tricerri and Vanhecke

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(see for instance [25]) and then by other authors. There exist three basic geometric types, $\mathscr{S}_1$, $\mathscr{S}_2$, $\mathscr{S}_3$. Later, homogeneous Kähler structures were defined and studied by Abbena and Garbiero in [1] and then by several authors. This time there are four basic types, $\mathscr{K}_1, \ldots, \mathscr{K}_4$. Further, homogeneous quaternionic Kähler structures were introduced by Fino [11], who moreover gave a Lie-theoretical description of the five basic types, $\mathscr{QK}_1, \ldots, \mathscr{QK}_5$, and then studied by several authors. (In the sequel we shall denote $\mathscr{S}_i \oplus \mathscr{S}_j$ simply by $\mathscr{S}_{ij}$; $\mathscr{K}_i \oplus \mathscr{K}_j$ by $\mathscr{K}_{ij}$; $\mathscr{QK}_i \oplus \mathscr{QK}_j$ by $\mathscr{QK}_{ij}$, and so on.)

Homogeneous Riemannian structures have found some useful applications. Two of them are: The characterization of $\mathbb{R}\mathrm{H}(n)$, $\mathbb{C}\mathrm{H}(n)$ and $\mathbb{H}\mathrm{H}(n)$ by such structures and the characterization of the homogeneous spin Riemannian manifolds whose Dirac operator is like that on a Riemannian symmetric spin space (see [15]). In our opinion, Tricerri's and Vanhecke's classification of geometric types is so natural, that more nice applications are to be expected.

The present survey is on the characterization of each of the classical hyperbolic spaces by linear homogeneous structures and on the geometric types of homogeneous structures on them. Recall that the characterization of $\mathbb{R}\mathrm{H}(n)$ by homogeneous Riemannian structures of type $\mathscr{S}_1$ was given by Tricerri and Vanhecke in [25], that of $\mathbb{C}\mathrm{H}(n)$ in terms of homogeneous Kähler structures of type $\mathscr{K}_{24}$ was obtained in [16], and that of $\mathbb{H}\mathrm{H}(n)$ by homogeneous quaternionic Kähler structures of type $\mathscr{QK}_{123}$ with nonzero projection to $\mathscr{QK}_3$ (actually, of type $\mathscr{QK}_3$) was given in [7].

The vector spaces $\mathscr{S}_1$, $\mathscr{K}_{24}$ and $\mathscr{QK}_{123}$ have dimension growing linearly according to the dimension of the homogeneous manifold admitting some homogeneous structure in each of them, that is, hyperbolic spaces. For this reason, these structures are sometimes called of linear type. However, this is not the unique type that hyperbolic spaces admit.

As for the contents, we recall in Sect. 2 some definitions on homogeneous Riemannian, Kähler and quaternionic Kähler structures, and recall the classification of geometric types for each of the three cases.

In Sect. 3 we give some results on the types of homogeneous structures that $\mathbb{R}\mathrm{H}(n)$, $\mathbb{C}\mathrm{H}(n)$ or $\mathbb{H}\mathrm{H}(n)$ admit.

2 Homogeneous Riemannian, Kähler or Quaternionic Kähler Structures

2.1 Homogeneous Riemannian Structures

A homogeneous structure on a Riemannian manifold (M, g) is a tensor field S of type $(1, 2)$ satisfying

$$\widetilde{\nabla} g = 0, \quad \widetilde{\nabla} R = 0, \quad \widetilde{\nabla} S = 0, \tag{1}$$

where $\widetilde{\nabla}$ is (see [25]) the connection determined by $\widetilde{\nabla} = \nabla - S$, ∇ being the Levi–Civita connection of g. The condition $\widetilde{\nabla} g = 0$ is equivalent to $S_{XYZ} = -S_{XZY}$, where $S_{XYZ} = g(S_X Y, Z)$.

Ambrose and Singer [3] gave the following characterization of homogeneous Riemannian manifolds: *A connected, simply connected and complete Riemannian manifold* (M, g) *is homogeneous if and only if it admits a homogeneous structure* S.

Let V be a real vector space endowed with an inner product $\langle\cdot,\cdot\rangle$, which is the model for each tangent space T_pM, $p \in M$, of a (homogeneous) Riemannian manifold. Consider the vector space $\mathscr{S}(V)$ of tensors of type (0, 3) on $(V, \langle\cdot,\cdot\rangle)$ satisfying the same algebraic symmetry that a homogeneous Riemannian structure S, that is, $\mathscr{S}(V) = \{S \in \otimes^3 V^* : S_{XYZ} = -S_{XZY}, X, Y, Z \in V\}$.

Tricerri and Vanhecke studied the decomposition of $\mathscr{S}(V)$ into invariant and irreducible subspaces $\mathscr{S}_i(V)$, $i = 1, 2, 3$, under the action of the orthogonal group $O(n)$ given by $(aS)_{XYZ} = S_{a^{-1}X\, a^{-1}Y\, a^{-1}Z}$, $a \in O(n)$. The inner product on V induces in a natural way an inner product on $\mathscr{S}(V)$, given by $\langle S, S'\rangle = \sum_{i,j,k=1}^{n} S_{e_ie_je_k} S'_{e_ie_je_k}$, where $\{e_i\}$ is an orthonormal basis of V. Let $c_{12}(S)(Z) = \sum_{i=1}^{n} S_{e_ie_iZ}$, $Z \in V$.

From the theory of representations of the orthogonal group (cf. [26, pp. 153–159]) it follows that $\mathscr{S}(V)$ decomposes into the orthogonal direct sum of three invariant and irreducible subspaces under the action of $O(n)$. Specifically, the subspace of c_{12}-traceless tensors of the subspace $\mathscr{Y}$ of $\otimes^3 V^*$ corresponding to the nonstandard Young symmetrizer id + (12) − (23) − (132), the n -dimensional subspace of tensors corresponding to the above c_{12}-trace, and the subspace $\wedge^3 V^*$. Then, one has

Theorem 1 ([25]) *If* $\dim V \geqslant 3$, *then* $\mathscr{S}(V)$ *decomposes into the orthogonal direct sum of subspaces which are invariant and irreducible under the action of* $O(n)$, $\mathscr{S}(V) = \mathscr{S}_1(V) \oplus \mathscr{S}_2(V) \oplus \mathscr{S}_3(V)$, *where*

$$
\begin{aligned}
\mathscr{S}_1(V) &= \{S \in \mathscr{S}(V) : S_{XYZ} = \langle X, Y\rangle\theta(Z) - \langle X, Z\rangle\theta(Y),\ \theta \in V^*\},\\
\mathscr{S}_2(V) &= \{S \in \mathscr{S}(V) : \mathfrak{S}_{XYZ} S_{XYZ} = 0,\ c_{12}(S) = 0\},\\
\mathscr{S}_3(V) &= \{S \in \mathscr{S}(V) : S_{XYZ} + S_{YXZ} = 0\},
\end{aligned}
$$

with dimensions n, $\frac{1}{3}n(n^2-4)$, $\frac{1}{6}n(n-1)(n-2)$, *respectively. If* $\dim V = 2$ *then* $\mathscr{S}(V) = \mathscr{S}_1(V)$.

We say that the homogeneous Riemannian structure S on (M, g) is of *type* $\{0\}$, $\mathscr{S}_i$ $(i = 1, 2, 3)$, $\mathscr{S}_{ij}$ $(1 \leqslant i < j \leqslant 3)$, or $\mathscr{S}_{123}$ if, for each point $p \in M$, $S(p) \in \mathscr{S}(T_pM)$ belongs to $\{0\}$, $\mathscr{S}_i(T_pM)$, $\mathscr{S}_{ij}(T_pM)$ or $\mathscr{S}_{123}(T_pM)$, respectively.

The similar terminology and notation will be used for the homogeneous Kähler (Sect. 2.2) and homogeneous quaternionic Kähler (Sect. 2.3) geometric types, that is, for the different types obtained from the basic types $\mathscr{K}_i$ $(i = 1, \ldots, 4)$ and $\mathscr{QK}_i$ $(i = 1, \ldots, 5)$, respectively.

2.2 Homogeneous Kähler Structures

An almost Hermitian manifold (M, g, J) is called a homogeneous almost Hermitian manifold if there exists a Lie group of almost complex isometries acting transitively and effectively on M. In [24], Sekigawa proved that a simply connected and complete almost Hermitian manifold (M, g, J) is homogeneous if and only if it admits a tensor field S of type $(1, 2)$ satisfying the Ambrose–Singer equations (1) and $\widetilde{\nabla} J = 0$. Such a tensor field S is called a homogeneous almost Hermitian structure (or a homogeneous Kähler structure if (M, g, J) is Kähler). Moreover, a homogeneous Riemannian structure on a Kähler manifold (M, g, J) is a homogeneous Kähler structure if and only if $S_{ZXY} = S_{ZJXJY}$ for all vector fields X, Y, Z on M.

The classification of homogeneous Kähler structures was obtained by Abbena and Garbiero. We recall here their result: Let V be a $2n$-dimensional real vector space (which is the model for the tangent space at any point of a manifold equipped with a Kähler homogeneous structure) endowed with a complex structure J and a Hermitian inner product $\langle\ ,\ \rangle$, that is, $J^2 = -I$, $\langle JX, JY\rangle = \langle X, Y\rangle, X, Y \in V$, where I denotes the identity isomorphism of V.

Denoting complexifications by a superscript c, we now consider the decompositions in $(\pm \mathrm{i})$-eigenspaces $V^c = V^{1,0} \oplus V^{0,1}$ and $V^{*c} = \lambda^{1,0} \oplus \lambda^{0,1}$, with respect to the complexified J^c of the complex structure J. In Salamon's notation [23], let $\lambda^{p,q}$ denote the space of forms of type (p, q), which is isomorphic to $\Lambda^p\lambda^{1,0} \otimes \Lambda^q\lambda^{0,1}$. We can decompose the space $\mathscr{S}(V)^c = \{S \in \otimes^3 V : S_{XYZ} = -S_{XZY}\}, X, Y, Z \in V^c$, into two subspaces invariant under the action of $U(n)$. One summand (that is, $\mathscr{S}(V)^c_- = V^{*c} \otimes (\lambda^{2,0} \oplus \lambda^{0,2})$) is related to homogeneous almost Hermitian structures. The other summand is

$$\mathscr{S}(V)^c_+ = V^{*c} \otimes \lambda^{1,1} \cong \{S \in \otimes^3 V : S_{XYZ} = -S_{XZY} = S_{XJ^cYJ^cZ}\},$$

$X, Y, Z \in V^c$, which is the complexified of Abbena–Garbiero's space $\mathscr{S}(V)_+$ (see [1]). The space $S(V)_+$ decomposes ([12, (2.1)]) into four subspaces invariant and irreducible under the action of $U(n)$. The sum of the first and second subspaces corresponds with the irreducible complex representation of $U(n)$ of the highest weight $(1, 1, 0, \ldots, 0, -1)$. The related real tensors of trace zero and those corresponding to that trace give rise to the first and second types in Theorem 2 below. Similarly, the sum of the third and four subspaces in that theorem, corresponds to the irreducible complex representation of $U(n)$ of the highest weight $(2, 0, \ldots, 0, -1)$. Taking traceless real tensors one gets the third subspace and the fourth one comes from that trace. We recall that Abbena and Garbiero [1, Theorem 4.4] proved the invariance and irreducibility by using quadratic invariants. In [12], Young diagrams and symmetrizers are used instead.

The standard representation of $U(n)$ on V induces a representation of $U(n)$ on $\mathscr{S}(V)_+$ given by $(A(S))_{XYZ} = S_{A^{-1}XA^{-1}YA^{-1}Z}, A \in U(n)$. Moreover, the scalar product in V induces in a natural way the scalar product in $\mathscr{S}(V)$ given by $\langle S, S'\rangle = \sum_{i,j,k=1}^{2n} S_{e_ie_je_k} S'_{e_ie_je_k}$, for any orthonormal basis $\{e_1, \ldots, e_{2n}\}$ of V. The expression of the tensors in each basic geometric type was given by Abbena and Garbiero and is as follows.

Theorem 2 ([1]) *If* $\dim V \geqslant 6$, *$\mathscr{S}(V)_+$ decomposes into the orthogonal direct sum of the following subspaces invariant and irreducible under the action of the group $U(n)$:*

$$\begin{aligned}
\mathscr{K}_1 =\ & \{S \in \mathscr{S}(V) : S_{XYZ} = \tfrac{1}{2}(S_{YZX} + S_{ZXY} + S_{JYJZX} + S_{JZXJY}),\ c_{12}(S) = 0\},\\
\mathscr{K}_2 =\ & \{S \in \mathscr{S}(V) : S_{XYZ} = \langle X, Y\rangle\theta_1(Z) - \langle X, Z\rangle\theta_1(Y) + \langle X, JY\rangle\theta_1(JZ)\\
& \qquad - \langle X, JZ\rangle\theta_1(JY) - 2\langle JY, Z\rangle\theta_1(JX),\ \theta_1 \in V^*\},\\
\mathscr{K}_3 =\ & \{S \in \mathscr{S}(V) : S_{XYZ} = -\tfrac{1}{2}(S_{YZX} + S_{ZXY} + S_{JYJZX} + S_{JZXJY}), c_{12}(S) = 0\},\\
\mathscr{K}_4 =\ & \{S \in \mathscr{S}(V) : S_{XYZ} = \langle X, Y\rangle\theta_2(Z) - \langle X, Z\rangle\theta_2(Y) + \langle X, JY\rangle\theta_2(JZ)\\
& \qquad - \langle X, JZ\rangle\theta_2(JY) + 2\langle JY, Z\rangle\theta_2(JX),\ \theta_2 \in V^*\},
\end{aligned}$$

$X, Y, Z \in V$, where c_{12} is defined by $c_{12}(S)(X) = \sum_{i=1}^{2n} S_{e_i e_i X}$, for any $X \in V$ and $\{e_1, \ldots, e_{2n}\}$ being an orthonormal basis of V; $\theta_1(X) = (1/(2(n-1)))c_{12}(S)(X)$ and $\theta_2(X) = (1/(2(n+1)))c_{12}(S)(X)$, $X \in V$. The dimensions are $n(n+1)(n-2)$, $2n$, $n(n-1)(n+2)$ and $2n$, respectively. If $\dim V = 4$, *then $\mathscr{S}(V)_+ = \mathscr{K}_2 \oplus \mathscr{K}_3 \oplus \mathscr{K}_4$. If* $\dim V = 2$, *then $\mathscr{S}(V)_+ = \mathscr{K}_4$.*

2.3 Homogeneous Quaternionic Kähler Structures

Let (M, g, υ^3) be an almost quaternion-Hermitian $4n$-manifold, υ^3 being the structure subbundle of the bundle of $(1, 1)$ tensors on M and let ∇ denote the Levi–Civita connection. The manifold is said to be quaternion-Kähler if one has locally (cf. Ishihara [19]) that

$$\nabla_X J_1 = \tau^3(X)J_2 - \tau^2(X)J_3, \quad \text{etc.}, \tag{2}$$

for certain differential 1-forms τ^1, τ^2, τ^3. Here and in the sequel we write "etc." to indicate the similar formulas obtained by cyclic permutation of (123). The holonomy group is contained in $Sp(n)Sp(1)$. A quaternion-Kähler manifold (M, g, υ^3) is said to be a *homogeneous quaternion-Kähler manifold* if it admits a transitive group of isometries (cf. Alekseevsky and Cortés [2, p.218] and [7, Remark 2.2]). A connected, simply connected and complete quaternion-Kähler manifold (M, g, υ^3) is homogeneous if and only if it admits a *homogeneous quaternionic Kähler structure*, that is, a $(1, 2)$ tensor field S satisfying the Ambrose–Singer equations (1) and equations

$$\widetilde{\nabla}_X J_1 = \tilde{\tau}^3(X)J_2 - \tilde{\tau}^2(X)J_3, \quad \text{etc.}, \tag{3}$$

for three differential 1-forms $\tilde{\tau}^1, \tilde{\tau}^2, \tilde{\tau}^3$. Let $\theta^a = \tau^a - \tilde{\tau}^a$, $a = 1, 2, 3$. Then, from formulas (2) and (3) we have that

$$S_{XJ_1YJ_1Z} - S_{XYZ} = \theta^3(X)g(J_2Y, J_1Z) - \theta^2(X)g(J_3Y, J_1Z), \quad \text{etc.},$$

which, together with the condition $S_{XYZ} = -S_{XZY}$, are the algebraic symmetries satisfied by a homogeneous quaternionic Kähler structure S.

Denote by E the standard representation of $Sp(n)$ on $\mathbb{C}^{2n}$, by $S^r E$ the rth-symmetric power of E (so that $S^2E \cong \mathfrak{sp}(n) \otimes \mathbb{C}$), by K the irreducible $Sp(n)$-module of the highest weight $(2, 1, 0, \ldots, 0)$ in $E \otimes S^2E = S^3E \oplus K \oplus E$, and by H the standard representation of $Sp(1) \cong SU(2)$ on $\mathbb{C}^2$, so that $S^2H \cong \mathfrak{sp}(1) \otimes \mathbb{C}$ and S^3H is the 4-dimensional irreducible representation of $Sp(1)$.

Let $\mathscr{S}(V)_+$ denote the set of homogeneous quaternionic Kähler structures. The geometric types were classified from a representation-theoretic point of view as follows.

Theorem 3 (Fino [11, Lemma 5.1])

$$\mathscr{S}(V)_+ = [EH] \otimes (\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)) \cong [EH] \oplus [ES^3H] \oplus [EH] \oplus [S^3EH] \oplus [KH].$$

Here, $[V]$ denotes the real representation whose complexification is V and the tensor products signs are omitted, that is, one writes EH instead of $E \otimes H$, and so on.

The standard representation $[EH]$ of $Sp(n)Sp(1)$ on V induces a representation of $Sp(n)Sp(1)$ on $\mathscr{S}(V)_+$ given by $(A(S))_{XYZ} = S_{A^{-1}XA^{-1}YA^{-1}Z}$, $A \in Sp(n)Sp(1)$. Moreover, the scalar product in V induces in a natural way the scalar product in $\mathscr{S}(V)_+$ given by $\langle S, S' \rangle = \sum_{i,j,k=1}^{4n} S_{e_ie_je_k} S'_{e_ie_je_k}$, for any orthonormal basis $\{e_1, \ldots, e_{4n}\}$ of V. The classification of homogeneous quaternionic Kähler structures in terms of real tensors was given in [7], as we now recall (except for the explanation of a few notations).

Theorem 4 ([7, Theorem 1.1]) *If* $n \geqslant 2$, *then* $\mathscr{V}$ *decomposes into the orthogonal direct sum of the following subspaces invariant and irreducible under the action of* $Sp(n)Sp(1)$:

$$\mathscr{QK}_1 = \left\{\Theta \in \widetilde{\mathscr{V}} \,:\, \Theta_{XYZ} = \textstyle\sum_{a=1}^{3} \theta(J_aX)\langle J_aY, Z\rangle,\ \theta \in V^*\right\},$$

$$\mathscr{QK}_2 = \left\{\Theta \in \widetilde{\mathscr{V}} \,:\, \Theta_{XYZ} = \textstyle\sum_{a=1}^{3} \theta^a(X)\langle J_aY, Z\rangle,\ = \sum_{a=1}^{3} \theta^a \circ J_a = 0,\ \theta^1, \theta^2, \theta^3 \in V^*\right\},$$

$$\mathscr{QK}_3 = \Big\{S \in \widehat{\mathscr{V}} \,:\, S_{XYZ} = \langle X, Y\rangle\theta(Z) - \langle X, Z\rangle\theta(Y) + \textstyle\sum_{a=1}^{3} \big(\langle X, J_aY\rangle\theta(J_aZ) - \langle X, J_aZ\rangle\theta(J_aY)\big), \theta \in V^*\Big\},$$

$$\mathscr{QK}_4 = \left\{S \in \widehat{\mathscr{V}} \,:\, S_{XYZ} = \tfrac{1}{6}\big(\mathfrak{S}_{XYZ} S_{XYZ} + \textstyle\sum_{a=1}^{3} \mathfrak{S}_{XJ_aYJ_aZ} S_{XJ_aYJ_aZ}\big), c_{12}(S) = 0\right\},$$

$$\mathscr{QK}_5 = \left\{S \in \widehat{\mathscr{V}} \,:\, \mathfrak{S}_{XYZ} S_{XYZ} = 0\right\},$$

with dimensions $4n$, $8n$, $4n$, $\frac{4}{3}n(n+1)(2n+1)$, $\frac{16}{3}n(n^2-1)$, *respectively.*

3 Types of Homogeneous Structures on $\mathbb{R}\mathrm{H}(n)$, $\mathbb{C}\mathrm{H}(n)$ or $\mathbb{H}\mathrm{H}(n)$

The usual homogeneous description of each hyperbolic space is as a rank-one noncompact Riemannian symmetric space, that is, as $\mathbb{R}\mathrm{H}(n) = SO(n,1)/O(n)$, $\mathbb{C}\mathrm{H}(n) = SU(n,1)/S(U(n)\times U(1))$ and $\mathbb{H}\mathrm{H}(n) = Sp(n,1)/(Sp(n)\times Sp(1))$, respectively. Then the corresponding homogeneous tensor S vanish.

We have the next result.

Proposition 1 **(i)** ([25, Theorem 5.2]) *A connected, simply connected and complete Riemannian manifold of dimension $n \geqslant 2$ admits a nontrivial homogeneous structure $S \in \mathscr{S}_1$ if and only if it is isometric to $\mathbb{R}\mathrm{H}(n)$.*

(ii) ([16, Theorem 1.1]) *A connected, simply connected and complete irreducible Kähler manifold of dimension $2n \geqslant 4$ admits a nontrivial homogeneous Kähler structure $S \in \mathscr{K}_{24}$ if and only if it is holomorphically isometric to $\mathbb{C}\mathrm{H}(n)$.*

(iii) ([7, Theorem 1.1]) *A connected, simply connected and complete quaternionic Kähler manifold of dimension $4n \geqslant 8$ admits a nontrivial homogeneous quaternionic Kähler structure $S \in \mathscr{QK}_{123}$ if and only if it is isometric to $\mathbb{H}\mathrm{H}(n)$. In this case, the homogeneous structure is necessarily of type $\mathscr{QK}_3$.*

Recall (Heintze [18, Theorem 4]), that a connected homogeneous Kähler $2n$-manifold of negative curvature is holomorphically isometric to $\mathbb{C}\mathrm{H}(n)$. Hence from Proposition 1, (ii), it follows the next

Corollary 1 *Any connected homogeneous Kähler manifold of real dimension $2n \geqslant 4$ and negative curvature admits a Kähler homogeneous structure $S \in \mathscr{K}_{24}$.*

However, hyperbolic spaces admit more types of homogeneous structures. We first recall

Proposition 2 **(i)** ([8, Theorem 3.1]) *The connected groups acting transitively on $\mathbb{R}\mathrm{H}(n)$ are the full isometry group $SO(n,1)$ and the groups $G = F_r N$, where N is the nilpotent factor in the Iwasawa decomposition of $SO(n,1)$ and F_r is a connected closed subgroup of $SO(n-1)\mathbb{R}$ with nontrivial projection to $\mathbb{R}$.*

(ii) ([8, Theorem 4.1]) *The connected groups acting transitively on $\mathbb{C}\mathrm{H}(n)$ are the full isometry group $SU(n,1)$ and the groups $G = F_r N$, where N is the nilpotent factor in the Iwasawa decomposition KAN of $SU(n,1)$ and F_r is a connected closed subgroup of $S(U(n-1)U(1))\mathbb{R}$ with nontrivial projection to $\mathbb{R}$.*

(iii) ([7, Theorem 5.2]) *The connected groups acting transitively on $\mathbb{H}\mathrm{H}(n)$ are the full isometry group $Sp(n,1)$ and the groups $G = F_r N$, where N is the nilpotent factor in the Iwasawa decomposition KAN of $Sp(n,1)$ and F_r is a connected closed subgroup of $Sp(n-1)Sp(1)\mathbb{R}$ with nontrivial projection to $\mathbb{R}$.*

The simplest choice is $F_r = A$, giving the description of $\mathbb{R}\mathrm{H}(n)$, $\mathbb{C}\mathrm{H}(n)$ or $\mathbb{H}\mathrm{H}(n)$ as the solvable group AN, and one has

Proposition 3 **(i)** ([8, Subsection 3.1]) *Any homogeneous Riemannian structure on* $\mathbb{R}\mathrm{H}(n) \equiv AN$ *with trivial holonomy lies in the class* $\mathscr{S}_1$.

(ii) ([8, Proposition 4.2]) *Any homogeneous Kähler structure on* $\mathbb{C}\mathrm{H}(n) \equiv AN$ *with trivial holonomy lies in the class* $\mathscr{K}_{234}$.

(iii) ([7, Proposition 5.3]) *Any homogeneous quaternionic Kähler structure on* $\mathbb{H}\mathrm{H}(n) \equiv AN$ *with trivial holonomy lies in the class* $\mathscr{QK}_{134}$.

For structures of linear type one has

Proposition 4 ([25, p. 55], [8, Subsection 3.1]) **(i)** *The homogeneous Riemannian structures of linear type on* $\mathbb{R}\mathrm{H}(n)$ *can be realized by the homogeneous model AN, where AN stands for the solvable part of the Iwasawa decomposition of the full isometry group* $SO(n, 1)$.

(ii) ([8, Theorem 4.4]) *The homogeneous Kähler structures of linear type on* $\mathbb{C}\mathrm{H}(n)$ *can be realized by the homogeneous model* $U(1)AN/U(1)$*, where AN stands for the solvable part of the Iwasawa decomposition of the full isometry group* $SU(n, 1)$.

(iii) ([7, Theorem 5.4]) *The homogeneous quaternionic Kähler structures of linear type on* $\mathbb{H}\mathrm{H}(n)$ *can be realized by the homogeneous model* $Sp(1)AN/Sp(1)$*, where AN stands for the solvable part of the Iwasawa decomposition of the full isometry group* $Sp(n, 1)$.

In the case of $\mathbb{R}\mathrm{H}(n)$, even all the holonomy algebras of canonical connections and the types of the corresponding homogeneous structures are known, see Proposition 5 below. We first recall some definitions and notations.

Assume that $G = F_r N$ acts transitively on $\mathbb{R}\mathrm{H}(n)$ as in Proposition 2. This implies that $\mathbb{R}\mathrm{H}(n) = G/H$, with $H = F_r \cap SO(n-1)$. Then H is reductive, and thus $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_{ss}$, where $\mathfrak{h}_0$ is abelian and $\mathfrak{h}_{ss}$ is semisimple. Let $\mathfrak{f}_r = \mathfrak{h} \oplus \mathfrak{a}_r$, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{a}_r \oplus \mathfrak{n}$, with $\mathfrak{a}_r$ projecting nontrivially to $\mathfrak{a} = \mathbb{R}_{>0}$. Also $\mathfrak{f}_r$ is reductive, with $\mathfrak{f}_r = (\mathfrak{h}_0 \oplus \mathfrak{a}_r) \oplus \mathfrak{h}_{ss}$. Let $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ and $\mathfrak{s}_r = \mathfrak{a}_r \oplus \mathfrak{n}$, where $\mathfrak{a}_r$ is any one-dimensional complement to $\mathfrak{h}_0 \oplus \mathfrak{n}$ in $\mathfrak{s}_f = (\mathfrak{f}_r)_0 \oplus \mathfrak{n}$. A homogeneous Riemannian structure on G/H depends on a choice of ad_H-invariant complement $\mathfrak{m}$ to $\mathfrak{h}$ in $\mathfrak{g}$, which is the graph of an $\mathfrak{h}$ -equivariant map $\varphi_r\colon \mathfrak{s}_r \to \mathfrak{h}$. For any $\mathfrak{h}$-equivariant map $\chi_r\colon \mathfrak{s} \to \mathfrak{s}_r$ extending the identity on $\mathfrak{n}$, one defines $\varphi\colon \mathfrak{s} \to \mathfrak{h}$ as $\varphi = \varphi_r \circ \chi_r$. Then we have

Proposition 5 ([9, Theorems 1.1,5.2]) *The holonomy algebras of canonical connections on* $\mathbb{R}\mathrm{H}(n)$ *are* $\mathfrak{so}(n)$ *and all the reductive algebras* $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_{ss}$ *of compact type with* $\mathfrak{k}_0 \cong \mathbb{R}^r$ *abelian and* $\mathfrak{k}_{ss}$ *semisimple such that* $3r + \dim \mathfrak{k}_{ss} \leqslant n - 1$.

Let S be a nonzero homogeneous tensor for $\mathbb{R}\mathrm{H}(n)$ *with holonomy algebra* $\mathfrak{hol}$. *Then S always has a nontrivial component in* $\mathscr{S}_1$ *and S is of type* $\mathscr{S}_1$ *if and only if* $\mathfrak{hol}$ *is* 0. *The structure is of strict type* $\mathscr{S}_{13}$ *if and only if* $\mathfrak{a} \subset \ker \varphi$ *and* $\mathfrak{hol}$ *is a nonzero semisimple algebra acting trivially on* $\ker \varphi$. *Otherwise S is of general type.*

All the homogeneous Kähler structures on the solvable description $\mathbb{C}\mathrm{H}(n) \equiv AN$ of the complex hyperbolic space have been given in a rather explicit way in [17, Theorem 3.1]. As expected, the expression simplifies a great deal for $n = 1$ and $n = 2$, which are of course interesting cases on their own.

On the other hand, the use of the parabolic subgroups of the respective full isometry groups permits us to make explicit more homogeneous descriptions and give the corresponding types of structures. In the case of $\mathbb{H}\mathrm{H}(n)$, $n = 2, 3$, one has the next result (for detailed expressions and more details see [5, Theorem 5]) and [6, Theorem 3.4]).

Proposition 6 *Let $G = KAN$ be the Iwasawa decomposition of $Sp(2,1)$ (resp. $Sp(3,1)$). The homogeneous descriptions of $\mathbb{H}\mathrm{H}(2)$ (resp. $\mathbb{H}\mathrm{H}(3)$) are as in the Table 1, where E is simply connected and abelian. In this case the corresponding types of homogeneous quaternionic Kähler structures are also given. The figure on the third column, if any, stands for the number of parameters of the corresponding n-parametric family of homogeneous quaternionic Kähler structures.*

Consider now the Poincaré half-space model

$$(H^n, g) = \left(\left\{ \left(u^1, \ldots, u^n\right) \in \mathbb{R}^n \,:\, u^1 > 0 \right\},\ -\frac{1}{c(u^1)^2} \sum_{i=1}^{n} \mathrm{d}u^i \otimes \mathrm{d}u^i \right)$$

of $\mathbb{R}\mathrm{H}(n)$, equipped with the metric g of constant curvature $c < 0$, and the Siegel domains

$$D_{\mathbb{C}^n} = \left\{ \left(u^1 = x + \mathrm{i}y, u^2, \ldots, u^n\right) \in \mathbb{C}^n \,:\, x - \sum_{k=2}^{n} |u^k|^2 > 0 \right\},$$

$$D_{\mathbb{H}^n} = \left\{ \left(u^1 = x + \mathrm{i}y + \mathrm{j}z + \mathrm{k}t, u^2, \ldots, u^n\right) \in \mathbb{H}^n \,:\, x - \sum_{k=2}^{n} |u^k|^2 > 0 \right\}.$$

Consider also the next vector fields on the relevant manifolds: ξ, metrically dual to the form θ in the expression of the elements of $\mathscr{S}_1$; ξ and η, metrically dual to the forms $\theta_1 + \theta_2$ and $\theta_1 - \theta_2$ in the expressions of the elements of $\mathscr{K}_2$ and $\mathscr{K}_4$; and

Table 1 Homogeneous descriptions of $\mathbb{H}H(2)$ and $\mathbb{H}H(3)$ and the corresponding types of structures

		dim E	n	Type
$Sp(2,1)/(Sp(2) \times Sp(1))$		0		$\{0\}$
$E_{\lambda,\mu}N$	$(\lambda, \mu \in \mathbb{R}^3 \setminus \{0\})$	1	6	$\mathscr{QK}_{12345}$
$E_{0,\mu}N$	$(\mu \in \mathbb{R}^3 \setminus \{0\})$	1	3	$\mathscr{QK}_{1345}$
$AN = E_{0,0}N$		1		$\mathscr{QK}_{134}$
$Sp(3,1)/(Sp(3) \times Sp(1))$		0		$\{0\}$
$E_{\lambda,\mu,\nu,\gamma}N$	$(\lambda, \mu, \nu \in \mathbb{R}^3 \setminus \{0\},\ \gamma \in \mathbb{R}^4 \setminus \{0\})$	1	13	$\mathscr{QK}_{12345}$
$E_{0,\mu,\nu,\gamma}N$	$(\mu, \nu \in \mathbb{R}^3 \setminus \{0\},\ \gamma \in \mathbb{R}^4 \setminus \{0\})$	1	10	$\mathscr{QK}_{1345}$
$AN = E_{0,0,0,0}N$		1		$\mathscr{QK}_{134}$