TIONS 11

Specialization of Quadratic and Symmetric Bilinear Forms

Translated by Thomas Unger

Springer

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Manfred Knebusch

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Dedicated to the memory of my teachers

Emil Artin 1898–1962

Hel Braun 1914–1986

Ernst Witt 1911–1991

Preface

A Mathematician Said Who
Can Quote Me a Theorem that's True?
For the ones that I Know
Are Simply not So,
When the Characteristic is Two!

This pretty limerick first came to my ears in May 1998 during a talk by T.Y. Lam on field invariants from the theory of quadratic forms.[1] It is—poetic exaggeration allowed—a suitable motto for this monograph.

What is it about? At the beginning of the seventies I drew up a specialization theory of quadratic and symmetric bilinear forms over fields [32]. Let $\lambda : K \to L \cup \infty$ be a place. Then one can assign a form $\lambda_*(\varphi)$ to a form φ over K in a meaningful way if φ has "good reduction" with respect to λ (see §1.1). The basic idea is to simply apply the place λ to the coefficients of φ, which must therefore be in the valuation ring of λ.

The specialization theory of that time was satisfactory as long as the field L, and therefore also K, had characteristic $\neq 2$. It served me in the first place as the foundation for a theory of generic splitting of quadratic forms [33], [34]. After a very modest beginning, this theory is now in full bloom. It became important for the understanding of quadratic forms over fields, as can be seen from the book [26] of Izhboldin–Kahn–Karpenko–Vishik for instance. One should note that there exists a theory of (partial) generic splitting of central simple algebras and reductive algebraic groups, parallel to the theory of generic splitting of quadratic forms (see [29] and the literature cited there).

In this book I would like to present a specialization theory of quadratic and symmetric bilinear forms with respect to a place $\lambda : K \to L \cup \infty$, without the assumption that $\operatorname{char} L \neq 2$. This is where complications arise. We have to make a distinction

[1] "Some reflections on quadratic invariants of fields", 3 May 1998 in Notre Dame (Indiana) on the occasion of O.T. O'Meara's 70th birthday.

between bilinear and quadratic forms and study them both over fields and valuation rings. From the viewpoint of reductive algebraic groups, the so-called regular quadratic forms (see below) are the natural objects. But, even if we are interested only in such forms, we have to know a bit about specialization of nondegenerate symmetric bilinear forms, since they occur as "multipliers" of quadratic forms: if φ is such a bilinear form and ψ is a regular quadratic form, then we can form a tensor product $\varphi \otimes \psi$, see §1.5. This is a quadratic form, which is again regular when ψ has even dimension ($\dim \psi$ = number of variables occurring in ψ). However—and here already we run into trouble—when $\dim \psi$ is odd, $\varphi \otimes \psi$ is not necessarily regular.

Even if we only want to understand quadratic forms over a field K of characteristic zero, it might be necessary to look at specializations with respect to places from K to fields of characteristic 2, especially in arithmetic investigations. When K itself has characteristic 2, an often more complicated situation may occur, for which we are not prepared by the available literature. Certainly fields of characteristic 2 were already allowed in my work on specializations in 1973 [32], but from today's point of view satisfactory results were only obtained for symmetric bilinear forms. For quadratic forms there are gaping holes. We have to study quadratic forms over a valuation ring in which 2 is not a unit. Even the beautiful and extensive book of Ricardo Baeza [6] doesn't give us enough for the theory of specializations, although Baeza even allows semilocal rings instead of valuation rings. He studies only quadratic forms whose associated bilinear forms are nondegenerate. This forces those forms to have even dimension.

Let me now discuss the contents of this book. After an introduction to the problem in §1.1, which can be understood without any previous knowledge of quadratic and bilinear forms, the specialization theory of symmetric bilinear forms is presented in §1.2–§1.3. There are good, generally accessible sources available for the foundations of the algebraic theory of symmetric bilinear forms. Therefore many results are presented without a proof, but with a reference to the literature instead. As an important application, the outlines of the theory of generic splitting in characteristic $\neq 2$ are sketched in §1.4, nearly without proofs.

From §1.5 onwards we address the theory of quadratic forms. In characteristic 2 fewer results can be found in the literature for such forms than for bilinear forms, even at the basic level. Therefore we present most of the proofs. We also concern ourselves with the so-called "weak specialization" (see §1.1) and get into areas which may seem strange even to specialists in the theory of quadratic forms. In particular we have to require a quadratic form over K to be "obedient" in order to weakly specialize it with respect to a place $\lambda : K \to L \cup \infty$ (see §1.7). I have never encountered such a thing anywhere in the literature.

At the end of Chapter 1 we reach a level in the specialization theory of quadratic forms that facilitates a generic splitting theory, useful for many applications. In the first two sections (§2.1, §2.2) of Chapter 2 we produce such a generic splitting theory in two versions, both of which deserve interest in their own right.

We call a quadratic form φ over a field k *nondegenerate* when its quasilinear part (cf. Arf [3]), which we denote by $QL(\varphi)$, is anisotropic. We further call—deviating from Arf [3]—φ *regular* when $QL(\varphi)$ is at most one-dimensional and *strictly regular* when $QL(\varphi) = 0$ (cf. §1.6, Definition 1.59). When k has characteristic $\neq 2$, every nondegenerate form is strictly regular, but in characteristic 2 the quasilinear part causes complications. For in this case φ can become degenerate under a field extension $L \supset k$. Only in the regular case is this impossible.

In §2.1 we study the splitting behaviour of a regular quadratic form φ over k under field extensions, while in §2.2 any nondegenerate form φ, but only separable extensions of k are allowed. The theory of §2.1 incorporates the theory of §1.4, so the missing proofs of §1.4 are subsequently filled in.

Until the end of §2.2 our specialization theory is based on an obvious "canonical" concept of *good reduction* of a form φ over a field K (quadratic or symmetric bilinear) to a valuation ring $\mathfrak{o}$ of K, similar to what is known under this name in other areas of mathematics (e.g. abelian varieties). There is nothing wrong with this theory; however, for many applications it is too limited.

This is particularly clear when studying specializations with respect to a place $\lambda : K \to L \cup \infty$ with $\operatorname{char} K = 0$, $\operatorname{char} L = 2$. If φ is a nondegenerate quadratic form over K with good reduction with respect to λ, then the specialization $\lambda_*(\varphi)$ is automatically strictly regular. However, we would like to have a more general specialization concept, in which forms with quasilinear part $\neq 0$ can arise over L. Conversely, if the place λ is surjective, i.e. $\lambda(K) = L \cup \infty$, we would like to "lift" *every* nondegenerate quadratic form ψ over L with respect to λ to a form φ over K, i.e. to find a form φ over K which specializes to ψ with respect to λ. Then we could use the theory of forms over K to make statements about ψ.

We present such a general specialization theory in §2.3. It is based on the concept of "*fair reduction*", which is less orthodox than good reduction, but which nevertheless possesses quite satisfying properties.

Next, in §2.4, we present a theory of generic splitting, which unites the theories of §1.4, §2.1 and §2.2 under one roof and which incorporates fair reduction. This theory is deepened in §2.5 and §2.6 through the study of generic splitting towers, and thus we reach the end of Chapter 2.

Chapter 3 (§3.1–§3.13) is a long chapter in which we present a panorama of results about quadratic forms over fields for which specialization and generic splitting of forms play an important role. This only scratches the surface of applications of the specialization theory of Chapters 1 and 2. Certainly many more results can be unearthed.

We return to the foundations of specialization theory in the final short Chapter 4 (§4.1–§4.5). Quadratic and bilinear forms over a field can be specialized with respect to a more general "quadratic place" $\Lambda : K \to L \cup \infty$ (defined in §4.1) instead of a usual place $\lambda : K \to L \cup \infty$. This represents a considerable broadening of the specialization theory of Chapters 1 and 2. Of course we require again "obedience" from a quadratic form q over K in order for its specialization $\Lambda_*(q)$ to reasonably exist. It then turns out that the generic splitting behaviour of $\Lambda_*(q)$ is governed by the splitting behaviour of q and Λ, in so far as good or fair reduction is present in a weak sense, as elucidated for ordinary places in Chapter 2.

Why are quadratic places of interest, compared with ordinary places? To answer this question we observe the following. If a form q over K has bad reduction with respect to a place $\lambda : K \to L \cup \infty$, it often happens that λ can be "enlarged" to a quadratic place $\Lambda : K \to L \cup \infty$ such that q has good or fair reduction with respect to Λ in a weak sense, and the splitting properties of q are handed down to $\Lambda_*(q)$ while there is no form $\lambda_*(q)$ available for which this would be the case. The details of such a notion of reduction are much more tricky compared with what happens in Chapters 1 and 2. The central term which renders possible a unified theory of generic splitting of quadratic forms is called "stably conservative reduction", see §4.4.

One must get used to the fact that for bilinear forms there is in general no Witt cancellation rule, in contrast to quadratic forms. Nevertheless the specialization theory is in many respects easier for bilinear forms than for quadratic forms.

On the other hand we do not have any theory of generic splitting for symmetric bilinear forms over fields of characteristic 2. Such a theory might not even be possible in a meaningful way. This may well be connected to the fact that the automorphism groups of such forms can be very far from being reductive groups (which may also account for the absence of a good cancellation rule).

This book is intended for audiences with different interests. For a mathematician with perhaps only a little knowledge of quadratic or symmetric bilinear forms, who just wants to get an impression of specialization theory, it suffices to read §1.1–§1.4. The theory of generic splitting in characteristic $\neq 2$ will acquaint the reader with an important application area.

From §1.5 onwards the book is intended for scholars working in the algebraic theory of quadratic forms, and also for specialists in the area of algebraic groups. They have always been given something to look at by the theory of quadratic forms.

On reaching §2.2 of the book, readers can lean back in their chair and take a well-deserved break. They will have learned about the specialization theory, which is based on the concept of good reduction, and will have gained a certain perspective on specific phenomena in characteristic 2. Furthermore, they will have been introduced to the foundations of generic splitting and so will have seen the specialization theory in action. Admittedly, readers will not yet have seen independent applications of the weak specialization theory (§1.3, §1.7), for this theory has only appeared up to then as an auxiliary one.

The remaining sections §2.3–§2.6 of Chapter 2 develop the specialization theory sufficiently far to allow an understanding of the classical algebraic theory of quadratic forms (as presented in the books of Lam [43], [44] and Scharlau [55]) without the usual restriction that the characteristic should be different from 2. Precisely this happens in Chapter 3 where readers will also obtain sufficient illustrations to enable them to relieve other classical theorems from the characteristic $\neq 2$ restriction, although this is often a nontrivial task.

The final Chapter 4 is ultimately intended for mathematicians who want to embark on a more daring expedition in the realm of quadratic forms over fields. It cannot be mere coincidence that the specialization theory for quadratic places works just as well as the specialization theory for ordinary places. It is therefore a safe prediction that quadratic places will turn out to be generally useful and important in a future theory of quadratic forms over fields.

Regensburg, *Manfred Knebusch*
June 2007

Postscript (October 2009)

I had the very good luck to find a translator of the German text into English, who, besides having two languages, could also understand the mathematical content of this book in depth. I met Professor Thomas Unger within the framework of the European network "Linear Algebraic Groups, Algebraic K-theory, and Related Topics", and most of the translation and our collaboration has been done under the auspices of this network, which we acknowledge gratefully.

I further owe deep thanks to my former secretary Rosi Bonn, who typed the whole German text in various versions and large parts of the English text.

The German text, *Spezialisierung von quadratischen und symmetrischen bilinearen Formen*, can be found on my homepage.[2]

[2] http://www-nw.uni-regensburg.de/~.knm22087.mathematik.uni-regensburg.de

Contents

Chapter 1
Fundamentals of Specialization Theory

1.1 Introduction: on the Problem of Specialization of Quadratic and Bilinear Forms

Let φ be a nondegenerate symmetric bilinear form over a field K, in other words

$$\varphi(x, y) = \sum_{i,j=1}^{n} a_{ij} x_i y_j,$$

where $x = (x_1, \dots, x_n) \in K^n$ and $y = (y_1, \dots, y_n) \in K^n$ are vectors, (a_{ij}) is a symmetric $(n \times n)$-matrix with coefficients $a_{ij} = a_{ji} \in K$ and $\det(a_{ij}) \neq 0$. We would like to write $\varphi = (a_{ij})$. The number of variables n is called the *dimension of* φ, $n = \dim \varphi$.

Let also $\lambda : K \to L \cup \infty$ be a place, $\mathfrak{o} = \mathfrak{o}_\lambda$ the valuation ring associated to K and $\mathfrak{m}$ the maximal ideal of $\mathfrak{o}$. We denote the group of units of $\mathfrak{o}$ by $\mathfrak{o}^*$, $\mathfrak{o}^* = \mathfrak{o} \setminus \mathfrak{m}$.

We would like λ to "specialize" φ to a bilinear form $\lambda_*(\varphi)$ over L. When is this possible in a reasonable way? If all $a_{ij} \in \mathfrak{o}$ and if $\det(a_{ij}) \in \mathfrak{o}^*$, then one can associate the nondegenerate form $(\lambda(a_{ij}))$ over L to φ. This naive idea leads us to the following:

Definition 1.1. We say that φ has *good reduction with respect to* λ when φ is isometric to a form (c_{ij}) over K with $c_{ij} \in \mathfrak{o}$, $\det(c_{ij}) \in \mathfrak{o}^*$. We then call the form $(\lambda(c_{ij}))$ "the" *specialization of* φ with respect to λ. We denote this specialization by $\lambda_*(\varphi)$.

Note. $\varphi = (a_{ij})$ is isometric to (c_{ij}) if and only if there exists a matrix $S \in \mathrm{GL}(n, K)$ with $(c_{ij}) = {}^tS\,(a_{ij})S$. In this case we write $\varphi \cong (c_{ij})$.

We also allow the case $\dim \varphi = 0$, standing for the unique bilinear form on the zero vector space, the form $\varphi = 0$. We agree that the form $\varphi = 0$ has good reduction and set $\lambda_*(\varphi) = 0$.

Problem 1.2. Is this definition meaningful? Up to isometry $\lambda_*(\varphi)$ should be independent of the choice of the matrix (c_{ij}).

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We shall later see that this is indeed the case, provided $2 \notin \mathfrak{m}$, so that L has characteristic $\neq 2$. If L has characteristic 2, then $\lambda_*(\varphi)$ is well-defined up to "stable isometry" (see §1.3).

Problem 1.3. Is there a meaningful way in which one can associate a symmetric bilinear form over L to φ, when φ has bad reduction?

With regard to this problem we would like to recall a classical result of T.A. Springer, which leads us to suspect that finding a solution to the problem is not completely beyond hope. Let $v : K \to \mathbb{Z} \cup \infty$ be a discrete valuation of a field K with associated valuation ring $\mathfrak{o}$. Let π be a generator of the maximal ideal $\mathfrak{m}$ of $\mathfrak{o}$, so that $\mathfrak{m} = \pi\mathfrak{o}$. Finally, let $k = \mathfrak{o}/\mathfrak{m}$ be the residue class field of $\mathfrak{o}$ and $\lambda : K \to k \cup \infty$ the canonical place with valuation ring $\mathfrak{o}$. We suppose that $2 \notin \mathfrak{m}$, so that $\operatorname{char} k \neq 2$ is.

Let φ be a nondegenerate symmetric bilinear form over K. Then there exists a decomposition $\varphi \cong \varphi_0 \perp \pi\varphi_1$, where φ_0 and φ_1 have good reduction with respect to λ. Indeed, we can choose a diagonalization $\varphi \cong \langle a_1, \dots, a_n \rangle$. {As usual $\langle a_1, \dots, a_n \rangle$ denotes the diagonal matrix $\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$.} Then we can arrange that $v(a_i) = 0$ or 1 for each i, by multiplying the a_i by squares and renumbering indices to get $a_i \in \mathfrak{o}^*$ for $1 \leq i \leq t$ and $a_i = \pi\varepsilon_i$, where $\varepsilon_i \in \mathfrak{o}^*$ for $t < i \leq n$. {Possibly $t = 0$, so that $\varphi_0 = 0$, or $t = n$, so that $\varphi_1 = 0$.}

Theorem 1.4 (Springer 1955 [56]). *Let K be complete with respect to the discrete valuation v. If φ is anisotropic (i.e. there is no vector $x \neq 0$ in K^n with $\varphi(x, x) = 0$), then the forms $\lambda_*(\varphi_0)$ and $\lambda_*(\varphi_1)$ are anisotropic and up to isometry independent of the choice of decomposition $\varphi \cong \varphi_0 \perp \pi\varphi_1$.*

Conversely, if ψ_0 and ψ_1 are anisotropic forms over k, then there exists up to isometry a unique anisotropic form φ over K with $\lambda_*(\varphi_0) \cong \psi_0$ and $\lambda_*(\varphi_1) \cong \psi_1$.

Given any place $\lambda : K \to L \cup \infty$ and any form φ over K, Springer's theorem suggests to look for a "weak specialization" $\lambda_W(\varphi)$ by orthogonally decomposing φ in a form φ_0 with good reduction and a form φ_1 with "extremely bad" reduction, subsequently forgetting φ_1 and setting $\lambda_W(\varphi) = \lambda_*(\varphi_0)$.

Given an arbitrary valuation ring $\mathfrak{o}$, this sounds like a daring idea. Nonetheless we shall see in §1.3 that a weak specialization can be defined in a meaningful way. Admittedly $\lambda_W(\varphi)$ is not uniquely determined by φ and λ up to isometry, but up to so-called Witt equivalence. In the situation of Springer's theorem, $\lambda_W(\varphi)$ is then the Witt class of φ_0 and $\lambda_W(\pi\varphi)$ the Witt class of φ_1.

A *quadratic form q* of dimension n over K is a function $q : K^n \to K$, defined by a homogeneous polynomial of degree 2,

$$q(x) = \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j$$

($x = (x_1, \dots, x_n) \in K^n$). We can associate (a possibly degenerate) symmetric bilinear form

$$B_q(x,y) = q(x+y) - q(x) - q(y) = \sum_{i=1}^{n} 2a_{ii}x_iy_i + \sum_{i<j} a_{ij}(x_iy_j + x_jy_i)$$

to q. It is clear that $B_q(x,x) = 2q(x)$ for all $x \in K^n$.

If $\operatorname{char} K \neq 2$, then any symmetric bilinear form φ over K corresponds to just one quadratic form q over K with $B_q = \varphi$, namely $q(x) = \frac{1}{2}\varphi(x,x)$. In this way we can interpret a quadratic form as a symmetric bilinear form and vice versa. In characteristic 2, however, quadratic forms and symmetric bilinear forms are very different objects.

Problem 1.5. Let $\lambda : K \to L \cup \infty$ be a place.

(a) To which quadratic forms q over K can we associate "specialized" quadratic forms $\lambda_*(q)$ over L in a meaningful way?
(b) Let $\operatorname{char} L = 2$ and $\operatorname{char} K \neq 2$, hence $\operatorname{char} K = 0$. Should one specialize a quadratic form q over K with respect to λ as a quadratic form, or rather as a symmetric bilinear form?

In what follows we will present a specialization theory for arbitrary nondegenerate symmetric bilinear forms (§1.3), but only for a rather small class of quadratic forms, the so-called "obedient" quadratic forms (§1.7). Problem 1.5(b) will be answered unequivocally. If q is obedient, B_q will determine a really boring bilinear form $\lambda_*(B_q)$ (namely a hyperbolic form) which gives almost no information about q. However, $\lambda_*(q)$ can give important information about q. If possible, a specialization in the quadratic sense is thus to be preferred over a specialization in the bilinear sense.

1.2 An Elementary Treatise on Symmetric Bilinear Forms

In this section a "form" will always be understood to be a *nondegenerate* symmetric bilinear form over a field. So let K be a field.

Theorem 1.6 ("Witt decomposition").
(*a*) *Any form φ over K has a decomposition*

$$\varphi \cong \varphi_0 \perp \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} a_r & 1 \\ 1 & 0 \end{pmatrix}$$

with φ_0 anisotropic and $a_1, \ldots, a_r \in K$ $\quad (r \geq 0)$.
(*b*) *The isometry class of φ_0 is uniquely determined by φ.* (*Therefore* $\dim \varphi_0$ *and the number r are uniquely determined.*)

To clarify these statements, let us recall the following:

(1) A form φ_0 over K is called *anisotropic* if $\varphi_0(x,x) \neq 0$ for all vectors $x \neq 0$.

(2) If char $K \neq 2$, then we have for every $a \in K^*$ that

$$\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong \langle 1, -1 \rangle \cong \langle a, -a \rangle.$$

If char $K = 2$, however, and $a \neq 0$, then $\left(\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right) \not\cong \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Indeed if $\varphi = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ we have $\varphi(x, x) = 0$ for every vector $x \in K^2$, while this is not the case for $\varphi = \left(\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right)$. In characteristic 2 we still have $\left(\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right) \cong \langle a, -a \rangle \quad (a \in K^*)$, but $\left(\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *need not* be isometric to $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \cong \langle 1, -1 \rangle$.

(3) The form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is given the name "hyperbolic plane" (even in characteristic 2), and every form φ, isometric to an orthogonal sum $r \times \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ of r copies of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, is called "*hyperbolic*" ($r \geq 0$).

(4) Forms which are isometric to an orthogonal sum $\left(\begin{smallmatrix} a_1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \perp \cdots \perp \left(\begin{smallmatrix} a_r & 1 \\ 1 & 0 \end{smallmatrix}\right)$ are called *metabolic* ($r \geq 0$). If char $K \neq 2$, then every metabolic form is hyperbolic. This is not the case if char $K = 2$.

(5) If char $K = 2$, then φ is hyperbolic exactly when *every* vector x of the underlying vector space K^n is isotropic, i.e. $\varphi(x, x) = 0$. If φ is not hyperbolic, we can always find an orthogonal basis such that $\varphi \cong \langle a_1, \ldots, a_n \rangle$ for suitable $a_i \in K^*$.

One can find a proof of Theorem 1.6 in any book about quadratic forms when char $K \neq 2$ (see in particular [10], [43], [55]). Part (b) of the theorem is then an immediate consequence of Witt's Cancellation Theorem. There is no general cancellation theorem in characteristic 2, as the following example shows:

$$\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \perp \langle -a \rangle \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \langle -a \rangle \tag{1.1}$$

for all $a \in K^*$. If e, f, g is a basis of K^3 which has the left-hand side of (1.1) as value matrix, then $e + g, f, g$ will be a basis which has the right-hand side of (1.1) as value matrix. For characteristic 2 one can find proofs of Theorem 1.6 and the other statements we made in [50, Chap. I and Chap. III, §1], [31, §8], [49, §4]. The following is clear from formula (1.1):

Lemma 1.7. *If a form φ with* $\dim \varphi = 2r$ *is metabolic, then there exists a form ψ such that $\varphi \perp \psi \cong r \times \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \perp \psi$.*

Definition 1.8.

(a) In the situation of Theorem 1.6, we call the form φ_0 the *kernel form* of φ and r the (*Witt*) *index* of φ. We write $\varphi_0 = \ker(\varphi)$, $r = \operatorname{ind}(\varphi)$. {In the literature one frequently sees the notation $\varphi_0 = \varphi_{\text{an}}$ ("anisotropic part" of φ).}

(b) Two forms φ, ψ over K are called *Witt equivalent*, denoted by $\varphi \sim \psi$, if $\ker \varphi \cong \ker \psi$. We write $\varphi \approx \psi$ when $\ker \varphi \cong \ker \psi$ and $\dim \varphi = \dim \psi$. On the basis of the next theorem, we then call φ and ψ *stably isometric*.

Theorem 1.9. *$\varphi \approx \psi$ exactly when there exists a form χ such that $\varphi \perp \chi \cong \psi \perp \chi$.*

We omit the proof. It is easy when one uses Theorem 1.6, Lemma 1.7 and the following lemma.

Lemma 1.10. *The form* $\chi \perp (-\chi)$ *is metabolic for every form* χ.

Proof. From Theorem 1.6(a) we may suppose that χ is anisotropic. If χ is different from the zero form, then $\chi \cong \langle a_1, \ldots, a_n \rangle$ with elements $a_i \in K^*$ ($n \geq 1$). Finally, $\langle a_i \rangle \perp \langle -a_i \rangle \cong \left(\begin{smallmatrix} a_i & 1 \\ 1 & 0 \end{smallmatrix}\right)$. □

As is well-known, Witt's Cancellation Theorem (already mentioned above) is valid if char $K \neq 2$. It says that two stably isometric forms are already isometric: $\varphi \approx \psi \Rightarrow \varphi \cong \psi$.

Let φ be a form over K. We call the equivalence class of φ with respect to the relation $\sim$, introduced above, the *Witt class* of φ and denote it by $\{\varphi\}$. We can add Witt classes together as follows:

$$\{\varphi\} + \{\psi\} := \{\varphi \perp \psi\}.$$

The class $\{0\}$ of the zero form, whose members are exactly the metabolic forms, is the neutral element of this addition. From Lemma 1.10 it follows that $\{\varphi\} + \{-\varphi\} = 0$. In this way, the Witt classes of forms over K form an abelian group, which we denote by $W(K)$. We can also multiply Witt classes together:

$$\{\varphi\} \cdot \{\psi\} := \{\varphi \otimes \psi\}.$$

Remark. The definition of the tensor product $\varphi \otimes \psi$ of two forms φ, ψ belongs to the domain of linear algebra [10, §1, No. 9]. For diagonalizable forms we have

$$\langle a_1, \ldots, a_n \rangle \otimes \langle b_1, \ldots, b_m \rangle \cong \langle a_1 b_1, \ldots, a_1 b_m, a_2 b_1, \ldots, a_n b_m \rangle.$$

We also have $\langle a_1, \ldots, a_n \rangle \otimes \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \cong n \times \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Finally, for a form $\left(\begin{smallmatrix} b & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with $b \neq 0$ we have

$$\langle a \rangle \otimes \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix} \cong \langle a \rangle \otimes \langle b, -b \rangle \cong \langle ab, -ab \rangle \cong \begin{pmatrix} ab & 1 \\ 1 & 0 \end{pmatrix}.$$

Now it is clear that the tensor product of any given form and a metabolic form is again metabolic. {For a conceptual proof of this see [31, §3], [50, Chap. I].} Therefore the Witt class $\{\varphi \otimes \psi\}$ is completely determined by the classes $\{\varphi\}$, $\{\psi\}$, independent of the choice of representatives φ, ψ.

With this multiplication, $W(K)$ becomes a commutative ring. The identity element is $\{\langle 1 \rangle\}$. We call $W(K)$ the *Witt ring* of K. For char $K \neq 2$ this ring was already introduced by Ernst Witt in 1937 [58].

We would like to describe the ring $W(K)$ by generators and relations. In characteristic $\neq 2$ this was already known by Witt [oral communication] and is implicitly contained in his work [58, Satz 7].

First we must recall the notion of determinant of a form. For $a \in K^*$, the isometry class of a one-dimensional form $\langle a \rangle$ will again be denoted by $\langle a \rangle$. The tensor product

$\langle a\rangle \otimes \langle b\rangle$ will be abbreviated by $\langle a\rangle\langle b\rangle$. We have $\langle a\rangle\langle b\rangle = \langle ab\rangle$ and $\langle a\rangle\langle a\rangle = \langle 1\rangle$. In this way the isometry classes form an abelian group of exponent 2, which we denote by $Q(K)$. Given $a, b \in K^*$, it is clear that $\langle a\rangle = \langle b\rangle$ exactly when $b = ac^2$ for a $c \in K^*$. So $Q(K)$ is just the *group of square classes* K^*/K^{*2} in disguise. We identify $Q(K) = K^*/K^{*2}$.

It is well-known that for a given form $\varphi = (a_{ij})$ the square class of the determinant of the symmetric matrix (a_{ij}) depends only on the isometry class of φ. We denote this square class by $\det(\varphi)$, so $\det(\varphi) = \langle \det(a_{ij})\rangle$, and call it the *determinant of* φ. A slight complication arises from the fact that the determinant is not compatible with Witt equivalence. To remedy this, we introduce the *signed determinant*

$$d(\varphi) := \langle -1\rangle^{\frac{n(n-1)}{2}} \cdot \det(\varphi)$$

($n := \dim\varphi$). One can easily check that $d(\varphi \perp \left(\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right)) = d(\varphi)$, for any $a \in K$. Hence $d(\varphi)$ depends only on the Witt class $\{\varphi\}$. The signed determinant $d(\varphi)$ also has a disadvantage though. In contrast with $\det(\varphi)$, $d(\varphi)$ does not behave completely well with respect to the orthogonal sum. Let $\nu(\varphi)$ denote the *dimension index* of φ, $\nu(\varphi) = \dim\varphi + 2\mathbb{Z} \in \mathbb{Z}/2\mathbb{Z}$. Then we have (cf. [55, I §2])

$$d(\varphi \perp \psi) = \langle -1\rangle^{\nu(\varphi)\nu(\psi)} d(\varphi)d(\psi).$$

Let us now describe $W(K)$ by means of generators and relations. Every one-dimensional form $\langle a\rangle$ satisfies $d(\langle a\rangle) = \langle a\rangle$. This innocent remark shows that the map from $Q(K)$ to $W(K)$, which sends every isometry class $\langle a\rangle$ to its Witt class $\{\langle a\rangle\}$, is injective. We can thus interpret $Q(K)$ as a subgroup of the group of units of the ring $W(K)$, $Q(K) \subset W(K)^*$.

$W(K)$ is additively generated by the subset $Q(K)$, since every non-hyperbolic form can be written as $\langle a_1, \ldots, a_n\rangle = \langle a_1\rangle \perp \cdots \perp \langle a_n\rangle$. Hence $Q(K)$ is a system of generators of $W(K)$. There is an obviously surjective ring homomorphism

$$\Phi : \ \mathbb{Z}[Q(K)] \twoheadrightarrow W(K)$$

from the group ring $\mathbb{Z}[Q(K)]$ to $W(K)$. Recall that $\mathbb{Z}[Q(K)]$ is the ring of formal sums $\sum_g n_g g$ with $g \in Q(K)$, $n_g \in \mathbb{Z}$, and almost all $n_g = 0$. Φ associates to such a sum the in $W(K)$ constructed sum $\sum_g n_g g$.

The elements of the kernel of Φ are the relations on $Q(K)$ we are looking for. We can write down some of those relations immediately: for every $a \in K^*$, $\langle a\rangle + \langle -a\rangle$ is clearly a relation. For $a, b \in K^*$ and given $\lambda, \mu \in K^*$, the form $\langle a, b\rangle$ represents the element $c := \lambda^2 a + \mu^2 b$. If $c \neq 0$, then we can find another element $d \in K^*$ with $\langle a, b\rangle \cong \langle c, d\rangle$. Comparing determinants shows that $\langle d\rangle = \langle abc\rangle$. Hence

$$\langle a\rangle + \langle b\rangle - \langle c\rangle - \langle abc\rangle = (\langle a\rangle + \langle b\rangle)(\langle 1\rangle - \langle c\rangle)$$

is also a relation. We have the technically important:

Theorem 1.11. *The ideal* Ker Φ *of the ring* $\mathbb{Z}[Q(K)]$ *is additively generated (i.e. as abelian group) by the elements* $\langle a\rangle + \langle -a\rangle$, $a \in K^*$ *and the elements* $\langle a\rangle + \langle b\rangle - \langle c\rangle - \langle abc\rangle$ *with* $a, b \in K^*$, $\langle b\rangle \neq \langle -a\rangle$, $c = \lambda^2 a + \mu^2 b$ *with* $\lambda, \mu \in K^*$.

Remark. Ker Φ is therefore generated as an ideal by the element $\langle 1\rangle + \langle -1\rangle$ and the elements $(\langle 1\rangle + \langle a\rangle)(1 - \langle c\rangle)$ with $\langle a\rangle \neq \langle -1\rangle$, $c = 1 + \lambda^2 a$ with $\lambda \in K^*$. For application in the next section, the additive description of Ker Φ above is more favourable though.

A proof of Theorem 1.11, which also works in characteristic 2, can be found in [31, §5], [38, §1], [35, II, §4] (even over semi-local rings instead of over fields[1]), [50, p. 85]. For characteristic $\neq 2$ the proof is a bit simpler, since every form has an orthogonal basis in this case, see [55, I § 9].

1.3 Specialization of Symmetric Bilinear Forms

In this section, a "form" will again be understood to be a nondegenerate symmetric bilinear form. Let $\lambda : K \to L \cup \infty$ be a place from the field K to a field L. Let $\mathfrak{o} = \mathfrak{o}_\lambda$ be the valuation ring associated to λ and $\mathfrak{m}$ its maximal ideal. As usual for rings, $\mathfrak{o}^*$ stands for the group of units of $\mathfrak{o}$, so that $\mathfrak{o}^* = \mathfrak{o} \setminus \mathfrak{m}$. This is the set of all $x \in K$ with $\lambda(x) \neq 0, \infty$.

We will now denote the Witt class of a one-dimensional form $\langle a\rangle$ over K (or L) by $\{a\}$. The group of square classes $Q(\mathfrak{o}) = \mathfrak{o}^*/\mathfrak{o}^{*2}$ can be embedded in $Q(K) = K^*/K^{*2}$ in a natural way via $a\mathfrak{o}^{*2} \mapsto aK^{*2}$. We interpret $Q(\mathfrak{o})$ as a subgroup of $Q(K)$, so $Q(\mathfrak{o}) = \{\langle a\rangle \mid a \in \mathfrak{o}^*\} \subset Q(K)$. Our specialization theory is based on the following:

Theorem 1.12. *There exists a well-defined additive map* $\lambda_W : W(K) \to W(L)$, *given by* $\lambda_W(\{a\}) = \{\lambda(a)\}$ *if* $a \in \mathfrak{o}^*$, *and* $\lambda_W(\{a\}) = 0$ *if* $\langle a\rangle \notin Q(\mathfrak{o})$ *(i.e.* $(aK^{*2}) \cap \mathfrak{o}^* = \emptyset$*).*[2]

Proof. (Copied from [32, §3].) Our place λ is a combination of the canonical place $K \to (\mathfrak{o}/\mathfrak{m}) \cup \infty$ with respect to $\mathfrak{o}$, and a field extension $\overline{\lambda} : \mathfrak{o}/\mathfrak{m} \hookrightarrow L$. Thus it suffices to prove the theorem for the canonical place. So let $L = \mathfrak{o}/\mathfrak{m}$ and $\lambda(a) = \overline{a} := a + \mathfrak{m}$ for $a \in \mathfrak{o}$.

We have a well-defined additive map $\Lambda : \mathbb{Z}[Q(K)] \to W(L)$ such that $\Lambda(\langle a\rangle) = \{\overline{a}\}$ if $a \in \mathfrak{o}^*$, and $\Lambda(\langle a\rangle) = 0$ if $\langle a\rangle \notin Q(\mathfrak{o})$. Clearly Λ vanishes on all elements $\langle a\rangle + \langle -a\rangle$ with $a \in K^*$. According to Theorem 1.11 we will be finished if we can show that Λ also disappears on every element

$$Z = \langle a_1\rangle + \langle a_2\rangle - \langle a_3\rangle - \langle a_4\rangle$$

with $a_i \in K^*$ and $\langle a_1, a_2\rangle \cong \langle a_3, a_4\rangle$.

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[1] The case where K has only two elements, $K = \mathbb{F}_2$, is not covered by the more general theorems there. The statement of Theorem 1.11 for $K = \mathbb{F}_2$ is trivial however, since K has only one square class $\langle 1\rangle$ and $\langle 1, 1\rangle \sim 0$.

[2] The letter W in the notation λ_W refers to "Witt" or "weak", see §1.1 and §1.7.

This will be the case when the four square classes $\langle a_i \rangle$ are not all in $Q(\mathfrak{o})$. Suppose from now on, without loss of generality, that $a_1 \in \mathfrak{o}^*$. Then we have $Z = \langle a_1 \rangle y$, where

$$y = 1 + \langle c \rangle - \langle b \rangle - \langle bc \rangle$$

is an element such that $\langle 1, c \rangle \cong \langle b, bc \rangle$. So $b = u^2 + w^2 c$ for elements $u, w \in K$. Clearly the equation $\Lambda(\langle a \rangle x) = \{\overline{a}\}\Lambda(x)$ is satisfied for any $a \in \mathfrak{o}^*$, $x \in \mathbb{Z}[Q(K)]$. Therefore it is enough to verify that $\Lambda(y) = 0$. We suppose without loss of generality that u and w are not both zero, otherwise we already have that $y = 0$.

Let us first treat the case $\langle c \rangle \in Q(\mathfrak{o})$, so without loss of generality $c \in \mathfrak{o}^*$. Then we have

$$\Lambda(y) = (1 + \{\overline{c}\})\Lambda(1 - \langle b \rangle).$$

If $\{\overline{c}\} = \{-1\}$, we are done. So suppose from now on that $\{\overline{c}\} \neq \{-1\}$. Then the form $\langle 1, \overline{c} \rangle$ is anisotropic over L. Since we are allowed to replace u and v by gu and gv for some $g \in K^*$, we may additionally assume that u and v are both in $\mathfrak{o}$, but not both in $\mathfrak{m}$. Since $\langle 1, \overline{c} \rangle$ is anisotropic, we have $\overline{b} = \overline{u}^2 + \overline{cw}^2 \neq 0$ and

$$\Lambda(y) = (1 + \{\overline{c}\})(1 - \{\overline{u}^2 + \overline{cw}^2\}) = 0.$$

The case which remains to be tackled is when the square class cK^{*2} doesn't contain a unit from $\mathfrak{o}$. Then $u^{-2}w^2c$ is definitely not a unit and either $b = u^2(1 + d)$ or $b = w^2c(1 + d)$ with $d \in \mathfrak{m}$. Hence $\Lambda(1 - \langle \overline{b} \rangle)$ is 0 or $1 - \{\overline{c}\}$, and both times $\Lambda(y) = 0$. □

Scholium 1.13. *The map $\lambda_W : W(K) \to W(L)$ can be described very conveniently as follows: Let φ be a form over K. If φ is hyperbolic (or, more generally metabolic), then $\lambda_W(\{\varphi\}) = 0$. If φ is not hyperbolic, then consider a diagonalization $\varphi \cong \langle a_1, a_2, \ldots, a_n \rangle$. Multiply each coefficient a_i for which it is possible by a square so that it becomes a unit in $\mathfrak{o}$, and leave the other coefficients as they are. Let for example $a_i \in \mathfrak{o}^*$ for $1 \leq i \leq r$ and $\langle a_i \rangle \notin Q(\mathfrak{o})$ for $r < i \leq n$ (possibly $r = 0$ or $r = n$). Then $\lambda_W(\{\varphi\}) = \{\langle \lambda(a_1), \ldots, \lambda(a_r) \rangle\}$.*

Let us now recall a definition from the Introduction §1.1.

Definition 1.14. We say that a form φ over K has *good reduction with respect to λ*, or that φ is *λ-unimodular* if φ is isometric to a form (a_{ij}) with $a_{ij} \in \mathfrak{o}$ and $\det(a_{ij}) \in \mathfrak{o}^*$. We call such a representation $\varphi \cong (a_{ij})$ a *λ-unimodular represention of φ* (or a unimodular representation with respect to the valuation ring $\mathfrak{o}$).

This definition can be interpreted geometrically as follows. We associate to φ a couple (E, B), consisting of an n-dimensional K-vector space E ($n = \dim \varphi$) and a symmetric bilinear form $B : E \times E \to K$ such that B represents the form φ after a choice of basis of E. We denote this by $\varphi \,\widehat{=}\, (E, B)$. Since φ has good reduction with respect to λ, E contains a free $\mathfrak{o}$-submodule M of rank n with $E = KM$, i.e. $E = K \otimes_{\mathfrak{o}} M$, and with $B(M \times M) \subset \mathfrak{o}$, such that the restriction $B|M \times M : M \times M \to \mathfrak{o}$ is a *nondegenerate bilinear form over* $\mathfrak{o}$, i.e. gives rise to an isomorphism $x \mapsto B(x, -)$ from the $\mathfrak{o}$-module M to the dual $\mathfrak{o}$-module $\check{M} = \mathrm{Hom}_{\mathfrak{o}}(M, \mathfrak{o})$.