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Finite Element Methods for Incompressible Flow Problems

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Finite Element Methods for Incompressible Flow Problems

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For Anja and Josephine

Preface

Incompressible flow problems appear in many models of physical processes and applications. Their numerical simulation requires in particular a spatial discretization. Finite element methods belong to the mathematically best understood discretization techniques.

This monograph is devoted mainly to the mathematical aspects of finite element methods for incompressible flow problems. It addresses researchers, Ph.D. students, and even students aiming for the master's degree. The presentation of the material, in particular of the mathematical arguments, is performed in detail. This style was chosen in the hope to facilitate the understanding of the topic, especially for nonexperienced readers.

Most parts of this monograph were presented in three consecutive master's level courses taught at the Free University of Berlin, and this monograph is based on the corresponding lecture notes. First of all, I like to thank the students who attended these courses. Many of them wrote finally their master's thesis under my supervision. Then, I like to thank two collaborators of mine, Julia Novo (Madrid) and Gabriel R. Barrenechea (Glasgow), who read parts of this monograph and gave valuable suggestions for improvement. Above all, I like to thank my beloved wife Anja and my daughter Josephine for their continual encouragement. Their efforts to manage our daily life and to save me working time were an invaluable contribution for writing this monograph in the past 3 years.

Colbitz, Germany Volker John
July 2016

Contents

Chapter 1
Introduction

The behavior of incompressible fluids is modeled with the system of the incompressible Navier–Stokes equations. These equations describe the conservation of linear momentum and the conservation of mass. In the special case of a steady-state situation and large viscosity of the fluid, the Navier–Stokes equations can be approximated by the Stokes equations. Incompressible flow problems are not only of interest by themselves, but they are part of many complex models for describing phenomena in nature or processes in engineering and industry.

Usually it is not possible to find an analytic solution of the Stokes or Navier–Stokes equations such that numerical methods have to be employed for approximating the solution. To this end, a so-called discretization has to be applied to the equations, in the general case a temporal and a spatial discretization. Concerning the spatial discretization, this monograph considers finite element methods. Finite element methods are very popular and they are understood quite well from the mathematical point of view.

First applications of finite element methods for the simulation of the Stokes and Navier–Stokes equations were performed in the 1970s. Also the finite element analysis for these equations started in this decade, e.g., by introducing in Babuška (1971) and Brezzi (1974) the inf-sup condition which is a basis of the well-posedness of the continuous as well as of the finite element problem. The early works on the finite element analysis cumulated in the monograph (Girault and Raviart 1979). The extended version of this monograph, Girault and Raviart (1986), became the classical reference work. Three decades have been passed since its publication. Of course, in this time, there were many new developments and new results have been obtained. More recent monographs that study finite element methods for incompressible flow problems, or important aspects of this topic, include Layton (2008), Boffi et al. (2008), Elman et al. (2014).

This monograph covers on the one hand a wide scope, from the derivation of the Navier–Stokes equations to the simulation of turbulent flows. On the other hand, there are many topics whose detailed presentation would amount in a monograph

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itself and they are only sketched here. The main emphasis of the current monograph is on mathematical issues. Besides many results for finite element methods, also a few fundamental results concerning the continuous equations are presented in detail, since a basic understanding of the analysis of the continuous problem provides a better understanding of the considered problem in its entirety.

A main feature of this monograph is the detailed presentation of the mathematical tools and of most of the proofs. This feature arose from the experience in sometimes spending (wasting) a lot of time in understanding certain steps in proofs that are written in the short form which is usual in the literature. Often, such steps would have been easy to understand if there would have been just an additional hint or one more line in the estimate. Thus, the presentation is mostly self-contained in the way that no other literature has to be used for understanding the majority of the mathematical results. Altogether, the monograph is directed to a broad audience: experienced researchers on this topic, young researchers, and master students. The latter point was successfully verified. Most parts of this monograph were presented in master courses held at the Free University of Berlin, in particular from 2013–2015. As a result, several master's theses were written on topics related to these courses.

1.1 Contents of this Monograph

Chapter 2 sketches the derivation of the incompressible Navier–Stokes equations on the basis of the conservation of mass and the conservation of linear momentum. Important properties of the stress tensor are derived from physical considerations. The non-dimensionalized equations are introduced and appropriate boundary conditions are discussed.

The following structure of this monograph is based on the inherent difficulties of the incompressible Navier–Stokes equations pointed out in Chap. 2.

- First, the coupling of velocity and pressure is studied:
  - Chapter 3 presents an abstract theory and discusses the choice of appropriate finite element spaces.
  - Chapter 4 applies the abstract theory to the Stokes equations.
- Second, the issue of dominant convection is also taken into account:
  - Chapter 5 studies this topic for the Oseen equations, which are a kind of linearized Navier–Stokes equations.
- Third, the nonlinearity of the Navier–Stokes equations is considered in addition to the other two difficulties:
  - Chapter 6 studies stationary flows that occur only for small Reynolds numbers.
  - Chapter 7 considers laminar flows that arise for medium Reynolds numbers.
  - Chapter 8 studies turbulent flows that occur for large Reynolds numbers.

The coupling of velocity and pressure in incompressible flow problems does not allow the straightforward use of arbitrary pairs of finite element spaces. For obtaining a well-posed problem, the spaces have to satisfy the so-called discrete inf-sup condition. This condition is derived in Chap. 3. The derivation is based on the study of the well-posedness of an abstract linear saddle point problem. The abstract theory is applied first to a continuous linear incompressible flow problem, thereby identifying appropriate function spaces for velocity and pressure. These spaces satisfy the so-called inf-sup condition. Then, it is discussed that the satisfaction of the inf-sup condition does not automatically lead to the satisfaction of the discrete inf-sup condition. Examples of velocity and pressure finite element spaces that do not satisfy this condition are given. Next, some techniques for proving the discrete inf-sup condition are presented and important inf-sup stable pairs of finite element spaces are introduced. For some pairs, the proof of the discrete inf-sup condition is presented. In addition, a way for computing the discrete inf-sup constant is described. The final section of this chapter discusses the Helmholtz decomposition.

Chapter 4 applies the theory developed in the previous chapter to the Stokes equations. The Stokes equations, being a system of linear equations, are the simplest model of incompressible flows, modeling only the flow caused by viscous forces. First, the existence, uniqueness, and stability of a weak solution is discussed. The next section presents results from the finite element error analysis. Conforming finite element methods are considered in the first part of this section and a low order non-conforming finite element discretization is studied in the second part. Some remarks concerning the implementation of the finite element methods are given. Next, a basic introduction to a posteriori error estimation is presented and its application for adaptive mesh refinement is sketched. It follows a presentation of methods that allow to circumvent the discrete inf-sup condition. Such methods enable the usage of the same finite element spaces with respect to the piecewise polynomials for velocity and pressure, which is appealing from the practical point of view. A detailed numerical analysis of one of these methods, the PSPG method, is provided and a couple of other methods are discussed briefly. Finite element methods satisfy in general the conservation of mass only approximately. This chapter concludes with a survey of methods that reduce the violation of mass conservation or even guarantee its conservation.

The Oseen equations, i.e., a linear equation with viscous (second order), convective (first order), and reactive (zeroth order) term are the topic of Chap. 5. These equations arise in various numerical methods for solving the Navier–Stokes equations. Usually, the convective or the reactive term dominate the viscous term. A major issue in the analysis consists in tracking the dependency of the stability and error bounds on the coefficients of the problem. After having established the existence and uniqueness of a solution of the equations, the standard Galerkin finite element method is studied. It turns out that the stability and error bounds become large if convection or reaction dominates. Numerical studies support this statement. For improving the numerical solutions, stabilized methods have to be applied. The analysis of a residual-based stabilized method, the SUPG/PSPG/grad-div method, is presented in detail and some further stabilized methods are reviewed briefly.

In Chap. 6, the first nonlinear model of an incompressible flow problem is studied—the steady-state Navier–Stokes equations. At the beginning of this chapter, the nonlinear term is investigated. Different forms of this term are introduced and various properties are derived. Then, the solution of the continuous steady-state Navier–Stokes equations is studied. It turns out that a unique solution can be expected only for sufficiently small external forces, which do not depend on time, and sufficiently large viscosity. For this situation, a finite element error analysis is presented, with the emphasis on bounding the nonlinear term. Next, iterative methods for solving the nonlinear problem are discussed. The final section of this chapter presents the Dual Weighted Residual (DWR) method. This method is an approach for the a posteriori error estimation with respect to quantities of interest.

Chapter 7 starts with the investigation of the time-dependent incompressible Navier–Stokes equations. From the point of view of finite element discretizations, so-called laminar flows are considered, i.e., flows where a standard Galerkin finite element method is applicable. At the beginning of this chapter, a short introduction into the analysis concerning the existence and uniqueness of a weak solution of the time-dependent incompressible Navier–Stokes equations is given. In particular, the mathematical reason is highlighted that prevents to prove the uniqueness in the practically relevant three-dimensional case. Then, the finite element error analysis for the Galerkin finite element method in the so-called continuous-in-time case is presented, i.e., without the consideration of a discretization with respect to time. For practical simulations, a temporal discretization has to be applied. The next part of this chapter introduces a number of time stepping schemes that require the solution of a coupled velocity-pressure problem in each discrete time. In particular, θ-schemes are discussed in detail. It follows the presentation of a finite element error analysis for the fully discretized equations at the example of the backward Euler scheme. The approaches presented so far in this chapter require the solution of saddle point problems, which might be computational expensive. Projection methods, which circumvent the solution of such problems, are presented in the last section of this chapter. In these methods, only scalar equations for each component of the velocity field and for the pressure have to be solved.

The topic of Chap. 8 is the simulation of turbulent flows. There is no mathematical definition of what is a turbulent flow. Thus, this chapter starts with a description of characteristics of flow fields that are considered to be turbulent. In addition, a mathematical approach for describing turbulence is sketched. It turns out that turbulent flows possess scales that are much too small to be representable on grids with affordable fineness. The impact of these scales on the resolvable scales has to be modeled with a so-called turbulence model. The bulk of this chapter presents turbulence models that allow mathematical or numerical analysis or whose derivation is based on mathematical arguments. A very popular approach for turbulence modeling is large eddy simulation (LES). LES aims at simulating only large (resolved) scales that are defined by spatial averaging. In the first section on LES, the derivation of equations for these scales is discussed, in particular the underlying assumption of commuting differentiation and spatial averaging. It is shown that usually commutation errors occur that are not negligible. The next section presents the most popular LES model, the Smagorinsky model. For the Smagorinsky model, a well developed mathematical and numerical analysis is available. Then, LES models are described that are derived on the basis of approximations in wave number space. The final section on LES considers so-called Approximate Deconvolution models (ADM). As next turbulence model, the Leray-α model is presented. This model is based on a regularization of the velocity field. Afterward, the Navier–Stokes-α model is considered. Its derivation is performed by studying a Lagrangian functional and the corresponding trajectory. The last class of turbulence models that is discussed is the class of Variational Multiscale (VMS) methods. VMS methods define the large scales, which should be simulated, in a different way than LES models, namely by projections in appropriate function spaces. Two principal types of VMS methods can be distinguished, those based on a two-scale decomposition and those using a three-scale decomposition of the flow field. Five different realizations of VMS methods are described in detail. The final section of Chap. 8 presents a few numerical studies of turbulent flow simulations using the Smagorinsky model and one representative of the VMS models.

The linearization and discretization of the incompressible Navier–Stokes equations results for many methods in coupled algebraic systems for velocity and pressure. Chapter 9 gives a brief introduction into solvers for such equations. One can distinguish between sparse direct solvers and iterative solvers, where the latter solvers have to be used with appropriate preconditioners. Some emphasis in the presentation is on the preconditioner that was used for simulating most of the numerical examples presented in this monograph, namely a coupled multigrid method.

Appendix A provides some basic notations from functional analysis. A number of inequalities and theorems are given that are used in the analysis and numerical analysis presented in this monograph. Some basics of the finite element method are provided in Appendix B. In particular, those finite element spaces are described in some detail that are used for discretizing incompressible flow problems. The approximation of functions from Sobolev spaces with finite element functions by interpolation or projection is the topic of Appendix C. The corresponding estimates are heavily used in the finite element error analysis. Finally, Appendix D describes a number of examples for numerical simulations, which are divided into three groups:

- examples for steady-state flow problems,
- examples for laminar time-dependent flow problems,
- examples for turbulent flow problems.

The described examples were utilized for performing numerical simulations whose results are presented in this monograph.

The master courses held at the Free University of Berlin covered the following topics:

- Course 1: Chaps. 2, and 3, Sect. 4.1–4.3,
- Course 2: Sects. 4.4–4.6, Chaps. 5–7, and 9,
- Course 3: Chap. 8.

Of course, the presentation in these courses concentrated on the most important issues of each topic.

Chapter 2
The Navier–Stokes Equations as Model for Incompressible Flows

Remark 2.1 (Basic Principles and Variables) The basic equations of fluid dynamics are called Navier–Stokes equations. In the case of an isothermal flow, i.e., a flow at constant temperature, they represent two physical conservation laws: the conservation of mass and the conservation of linear momentum. There are various ways for deriving these equations. Here, the classical one of continuum mechanics will be outlined. This approach contains some heuristic steps.

The flow will be described with the variables

- $\rho(t, \boldsymbol{x})$: density [$^{\mathrm{kg}}/_{\mathrm{m}^3}$],
- $\boldsymbol{v}(t, \boldsymbol{x})$: velocity [$^{\mathrm{m}}/_{\mathrm{s}}$],
- $P(t, \boldsymbol{x})$: pressure [$\mathrm{Pa} = {}^{\mathrm{N}}/_{\mathrm{m}^2}$],

which are assumed to be sufficiently smooth functions in the time interval $[0, T]$ and the domain $\Omega \subset \mathbb{R}^3$. □

2.1 The Conservation of Mass

Remark 2.2 (General Conservation Law) Let ω be an arbitrary open volume in Ω with sufficiently smooth surface $\partial\omega$, which is constant in time, and with mass

$$m(t) = \int_\omega \rho(t, \boldsymbol{x})\, d\boldsymbol{x} \text{ [kg]}.$$

If mass in ω is conserved, the rate of change of mass in ω must be equal to the flux of mass $\rho\boldsymbol{v}(t, \boldsymbol{x})$ [kg/(m^{2}s)] across the boundary $\partial\omega$ of ω

$$\frac{d}{dt} m(t) = \frac{d}{dt} \int_\omega \rho(t, \boldsymbol{x})\, d\boldsymbol{x} = -\int_{\partial\omega} (\rho\boldsymbol{v})(t, \boldsymbol{s}) \cdot \boldsymbol{n}(\boldsymbol{s})\, ds, \tag{2.1}$$

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where $\boldsymbol{n}(\boldsymbol{s})$ is the outward pointing unit normal on $\boldsymbol{s} \in \partial\omega$. Since all functions and $\partial\omega$ are assumed to be sufficiently smooth, the divergence theorem can be applied (integration by parts), which gives

$$\int_\omega \nabla \cdot (\rho \boldsymbol{v})(t, \boldsymbol{x})\, d\boldsymbol{x} = \int_{\partial\omega} (\rho \boldsymbol{v})(t, \boldsymbol{s}) \cdot \boldsymbol{n}(\boldsymbol{s})\, d\boldsymbol{s}.$$

Inserting this identity in (2.1) and changing differentiation with respect to time and integration with respect to space leads to

$$\int_\omega \left(\partial_t \rho(t, \boldsymbol{x}) + \nabla \cdot (\rho \boldsymbol{v})(t, \boldsymbol{x})\right)\, d\boldsymbol{x} = 0.$$

Since ω is an arbitrary volume, it follows that

$$(\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}))(t, \boldsymbol{x}) = 0\, forallst \in (0, T],\ \boldsymbol{x} \in \Omega. \tag{2.2}$$

This relation is the first equation of fluid dynamics, which is called continuity equation. □

Remark 2.3 (Time-Dependent Domain) It is also possible to consider a time-dependent domain $\omega(t)$. In this case, the Reynolds transport theorem can be applied. Let $\phi(t, \boldsymbol{x})$ be a sufficiently smooth function defined on an arbitrary volume $\omega(t)$ with sufficiently smooth boundary $\partial\omega(t)$, then the Reynolds transport theorem has the form

$$\frac{d}{dt} \int_{\omega(t)} \phi(t, \boldsymbol{x})\, d\boldsymbol{x} = \int_{\omega(t)} \partial_t \phi(t, \boldsymbol{x})\, d\boldsymbol{x} + \int_{\partial\omega(t)} (\phi \boldsymbol{v} \cdot \boldsymbol{n})(t, \boldsymbol{s})\, ds. \tag{2.3}$$

In the special case that $\phi(t, \boldsymbol{x})$ is the density, one gets for the change of mass

$$\frac{d}{dt} \int_{\omega(t)} \rho(t, \boldsymbol{x})\, d\boldsymbol{x} = \int_{\omega(t)} \partial_t \rho(t, \boldsymbol{x})\, d\boldsymbol{x} + \int_{\partial\omega(t)} (\rho \boldsymbol{v} \cdot \boldsymbol{n})(t, \boldsymbol{s})\, ds.$$

Conservation of mass and the divergence theorem yields

$$0 = \int_{\omega(t)} (\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}))(t, \boldsymbol{x})\, d\boldsymbol{x}.$$

Since $\omega(t)$ is assumed to be arbitrary, Eq. (2.2) follows. □

Remark 2.4 (Incompressible, Homogeneous Fluids) If the fluid is incompressible and homogeneous, i.e., composed of one fluid only, then $\rho(t, \boldsymbol{x}) = \rho > 0$ and (2.2) reduces to

$$\left(\partial_x v_1 + \partial_y v_2 + \partial_z v_3\right)(t, \boldsymbol{x}) = \nabla \cdot \boldsymbol{v}(t, \boldsymbol{x}) = 0\, forallst \in (0, T],\ \boldsymbol{x} \in \Omega, \tag{2.4}$$

where

$$\boldsymbol{v}(t,\boldsymbol{x}) = \begin{pmatrix} v_1(t,\boldsymbol{x}) \\ v_2(t,\boldsymbol{x}) \\ v_3(t,\boldsymbol{x}) \end{pmatrix}.$$

Thus, the conservation of mass for an incompressible, homogeneous fluid imposes a constraint on the velocity only. □

2.2 The Conservation of Linear Momentum

Remark 2.5 (Newton's Second Law of Motion) The conservation of linear momentum is the formulation of Newton's second law of motion

$$\text{net force} = \text{mass} \times \text{acceleration} \tag{2.5}$$

for flows. It states that the rate of change of the linear momentum must be equal to the net force acting on a collection of fluid particles. □

Remark 2.6 (Conservation of Linear Momentum) The linear momentum in an arbitrary volume ω is given by

$$\int_\omega \rho \boldsymbol{v}(t,\boldsymbol{x})\, d\boldsymbol{x} \quad [\mathrm{Ns}].$$

Then, the conservation of linear momentum in ω can be formulated analogously to the conservation of mass in (2.1)

$$\frac{d}{dt}\int_\omega \rho \boldsymbol{v}(t,\boldsymbol{x})\, d\boldsymbol{x} = -\int_{\partial\omega} (\rho\boldsymbol{v})\,(\boldsymbol{v}\cdot\boldsymbol{n})\,(t,\boldsymbol{s})\, d\boldsymbol{s} + \int_\omega \boldsymbol{f}_{\text{net}}(t,\boldsymbol{x})\, d\boldsymbol{x}\ [\mathrm{N}],$$

where the term on the left-hand side describes the change of the momentum in ω, the first term on the right-hand side models the flux of momentum across the boundary of ω and $\boldsymbol{f}_{\text{net}}$ $[\mathrm{N/m^3}]$ represents the force density in ω. It is

$$\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{n}) = \begin{pmatrix} v_1v_1n_1 + v_1v_2n_2 + v_1v_3n_3 \\ v_2v_1n_1 + v_2v_2n_2 + v_2v_3n_3 \\ v_3v_1n_1 + v_3v_2n_2 + v_3v_3n_3 \end{pmatrix} = \boldsymbol{v}\boldsymbol{v}^T\boldsymbol{n}.$$

Applying integration by parts and changing differentiation with respect to time and integration on ω gives

$$\int_\omega \left(\partial_t(\rho\boldsymbol{v}) + \nabla\cdot\left(\rho\boldsymbol{v}\boldsymbol{v}^T\right)\right)(t,\boldsymbol{x})\, d\boldsymbol{x} = \int_\omega \boldsymbol{f}_{\text{net}}(t,\boldsymbol{x})\, d\boldsymbol{x}.$$

The product rule yields

$$\int_\omega \left(\partial_t \rho \boldsymbol{v} + \rho \partial_t \boldsymbol{v} \;+\; \boldsymbol{v}\boldsymbol{v}^T \nabla \rho + \rho(\nabla \cdot \boldsymbol{v})\boldsymbol{v} + \rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}\right)(t, \boldsymbol{x})\, d\boldsymbol{x} = \int_\omega \boldsymbol{f}_{\text{net}}(t, \boldsymbol{x})\, d\boldsymbol{x}. \tag{2.6}$$

In the usual notation $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$, one can think of $\boldsymbol{v} \cdot \nabla = v_1 \partial_x + v_2 \partial_y + v_3 \partial_z$ acting on each component of $\boldsymbol{v}$. In the literature, one often finds the notation $\boldsymbol{v} \cdot \nabla \boldsymbol{v}$.

In the case of incompressible flows, i.e., ρ is constant, it is known that $\nabla \cdot \boldsymbol{v} = 0$, see (2.4), such that (2.6) simplifies to

$$\int_\omega \rho \left(\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}\right)(t, \boldsymbol{x})\, d\boldsymbol{x} = \int_\omega \boldsymbol{f}_{\text{net}}(t, \boldsymbol{x})\, d\boldsymbol{x}.$$

Since ω was chosen to be arbitrary, one gets the conservation law

$$\rho \left(\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}\right) = \boldsymbol{f}_{\text{net}} \qquad \forall\, t \in (0, T],\ \boldsymbol{x} \in \Omega.$$

The same conservation law can be derived for a time-dependent volume $\omega(t)$ using the Reynolds transport theorem (2.3). □

Remark 2.7 (External Forces) The forces acting on ω are composed of external (body) forces and internal forces. External forces include, e.g., gravity, buoyancy, and electromagnetic forces (in liquid metals). These forces are collected in a body force term

$$\int_\omega \boldsymbol{f}_{\text{ext}}(t, \boldsymbol{x})\, d\boldsymbol{x}.$$

□

Remark 2.8 (Internal Forces, Cauchy's Principle, and the Stress Tensor) Internal forces are forces which a fluid exerts on itself. These forces include the pressure and the viscous drag that a 'fluid element' exerts on the adjacent element. The internal forces of a fluid are contact forces, i.e., they act on the surface of the fluid element ω. Let $\boldsymbol{t}$ [N/m^2] denote this internal force vector, which is called Cauchy stress vector or torsion vector, then the contribution of the internal forces on ω is

$$\int_{\partial\omega} \boldsymbol{t}(t, \boldsymbol{s})\, d\boldsymbol{s}.$$

Adding the external and internal forces, the equation for the conservation of linear momentum is, for an arbitrary constant-in-time volume ω,

$$\int_\omega \rho(t, \boldsymbol{x}) \left(\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}\right)(t, \boldsymbol{x})\, d\boldsymbol{x} = \int_\omega \boldsymbol{f}_{\text{ext}}(t, \boldsymbol{x})\, d\boldsymbol{x} + \int_{\partial\omega} \boldsymbol{t}(t, \boldsymbol{s})\, d\boldsymbol{s}. \tag{2.7}$$

The right-hand side of (2.7) describes the net force acting on and inside ω. Now, a detailed description of the internal forces represented by $\boldsymbol{t}(t, \boldsymbol{s})$ is necessary.

The foundation of continuum mechanics is the stress principle of Cauchy. The idea of Cauchy on internal contact forces was that on any (imaginary) plane on $\partial\omega$ there is a force that depends (geometrically) only on the orientation of the plane. Thus, it is $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{n})$, where $\boldsymbol{n}$ is the outward pointing unit normal vector of the imaginary plane.

Next, it will be discussed that $\boldsymbol{t}$ depends linearly on $\boldsymbol{n}$. To this end, consider a tetrahedron ω with the vertices $\boldsymbol{p}_0 = (0,0,0)^T$, $\boldsymbol{p}_1 = (x_1,0,0)^T$, $\boldsymbol{p}_2 = (0,y_2,0)^T$, $\boldsymbol{p}_3 = (0,0,z_3)^T$, and with $x_1, y_2, z_3 > 0$, see Fig. 2.1 for an illustration. Denote the plane containing $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ by $\partial\omega^{(\boldsymbol{n})}$. The unit outward pointing normal of $\partial\omega^{(\boldsymbol{n})}$ is given by

$$\begin{aligned} \boldsymbol{n} &= \frac{(\boldsymbol{p}_2 - \boldsymbol{p}_1) \times (\boldsymbol{p}_3 - \boldsymbol{p}_1)}{\|(\boldsymbol{p}_2 - \boldsymbol{p}_1) \times (\boldsymbol{p}_3 - \boldsymbol{p}_1)\|_2} \\ &= \frac{1}{\|(\boldsymbol{p}_2 - \boldsymbol{p}_1) \times (\boldsymbol{p}_3 - \boldsymbol{p}_1)\|_2} \begin{pmatrix} y_2 z_3 \\ z_3 x_1 \\ x_1 y_2 \end{pmatrix} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}. \end{aligned} \tag{2.8}$$

The face of the tetrahedron with the normal $-\boldsymbol{e}_i$ will be denoted by $\partial\omega^{(\boldsymbol{e}_i)}$, $i = 1,2,3$. Let $\boldsymbol{t}^{(\boldsymbol{n})}$ be the Cauchy stress vector at $\partial\omega^{(\boldsymbol{n})}$. Assuming that the Cauchy stress vectors depend only on the normal of the respective face, they are constant on

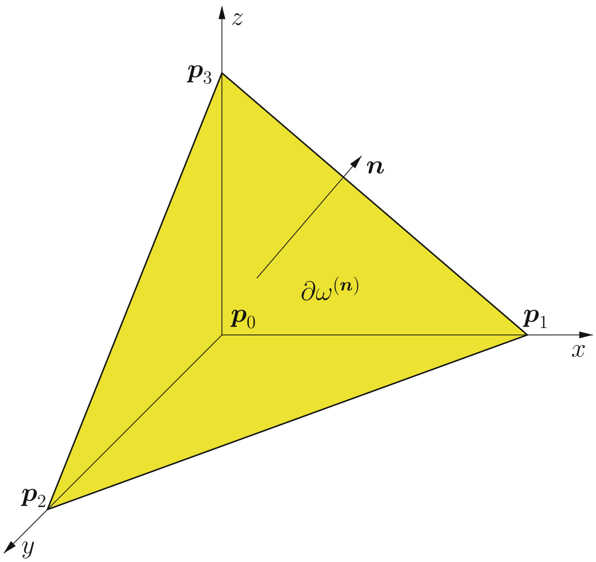

Fig. 2.1 Illustration of the tetrahedron used for discussing the linear dependency of the Cauchy stress vector on the normal

each face of the tetrahedron and the integrals on the faces can be computed easily. Applying in addition Newton's second law of motion (2.5) and the formula for the volume of a tetrahedron leads to

$$\underbrace{\boldsymbol{t}^{(\boldsymbol{n})}\left|\partial\omega^{(\boldsymbol{n})}\right| - \sum_{i=1}^{3}\boldsymbol{t}^{(\boldsymbol{e}_i)}\left|\partial\omega^{(\boldsymbol{e}_i)}\right|}_{\text{internal force}} = \underbrace{\rho\frac{h^{(\boldsymbol{n})}}{3}\left|\partial\omega^{(\boldsymbol{n})}\right|}_{\text{mass}}\boldsymbol{a}, \quad [\mathrm{N}] \tag{2.9}$$

where $|\cdot|$ is the area of the faces, $\boldsymbol{t}^{(\boldsymbol{e}_i)}$ the constant stress vector at face $\partial\omega^{(\boldsymbol{e}_i)}$, $\boldsymbol{a}$ $[\mathrm{m}/\mathrm{s}^2]$ is an acceleration, and $h^{(\boldsymbol{n})}$ is the distance of the face $\partial\omega^{(\boldsymbol{n})}$ to the origin. The area of $\partial\omega^{(\boldsymbol{n})}$ can be calculated with the cross product, giving

$$\left|\partial\omega^{(\boldsymbol{n})}\right| = \frac{1}{2}\left|(\boldsymbol{p}_2 - \boldsymbol{p}_1)\times(\boldsymbol{p}_3 - \boldsymbol{p}_1)\right| = \frac{1}{2}\left\|\begin{pmatrix} y_2 z_3 \\ z_3 x_1 \\ x_1 y_2 \end{pmatrix}\right\|_2 .$$

Using the representation (2.8) of the normal leads to

$$\left|\partial\omega^{(\boldsymbol{e}_1)}\right| = \frac{1}{2}y_2 z_3 = \frac{1}{2}n_1\left\|(\boldsymbol{p}_2 - \boldsymbol{p}_1)\times(\boldsymbol{p}_3 - \boldsymbol{p}_1)\right\|_2 = \left|\partial\omega^{(\boldsymbol{n})}\right| n_1 .$$

Analogous formulas are derived for $\left|\partial\omega^{(\boldsymbol{e}_2)}\right|$ and $\left|\partial\omega^{(\boldsymbol{e}_3)}\right|$. Inserting these formulas into (2.9) gives

$$\boldsymbol{t}^{(\boldsymbol{n})} - \sum_{i=1}^{3}\boldsymbol{t}^{(\boldsymbol{e}_i)}n_i = \rho\frac{h^{(\boldsymbol{n})}}{3}\boldsymbol{a}. \tag{2.10}$$

Shrinking now the tetrahedron to the origin, where $\partial\omega^{(\boldsymbol{n})}$ moves in the direction $\boldsymbol{n}$, the left-hand side of (2.10) stays constant whereas the right-hand side vanishes since $h^{(\boldsymbol{n})} \to 0$. Hence, one obtains in the limit that

$$\boldsymbol{t}^{(\boldsymbol{n})} = \sum_{i=1}^{3}\boldsymbol{t}^{(\boldsymbol{e}_i)}n_i = \left(\boldsymbol{t}^{(\boldsymbol{e}_1)}\ \boldsymbol{t}^{(\boldsymbol{e}_2)}\ \boldsymbol{t}^{(\boldsymbol{e}_3)}\right)\boldsymbol{n},$$

where $(\cdot,\cdot,\cdot)$ denotes a tensor with the respective columns. This relation means that $\boldsymbol{t}^{(\boldsymbol{n})}$ depends linearly on $\boldsymbol{n}$.

Thus, the model for the Cauchy stress vector is

$$\boldsymbol{t} = \mathbb{S}\boldsymbol{n}, \tag{2.11}$$

where $\mathbb{S}(t,\boldsymbol{x})$ $[\mathrm{N}/\mathrm{m}^2]$ is a 3×3-tensor that is called stress tensor. The stress tensor represents all internal forces of the flow. Inserting (2.11) in the term representing the internal forces in (2.7) and applying the divergence theorem gives

$$\int_{\partial\omega} \boldsymbol{t}(t, \boldsymbol{s})\, d\boldsymbol{s} = \int_{\omega} \nabla \cdot \mathbb{S}(t, \boldsymbol{x})\, d\boldsymbol{x},$$

where the divergence of a tensor is defined row-wise

$$\nabla \cdot \mathbb{A} = \begin{pmatrix} \partial_x a_{11} + \partial_y a_{12} + \partial_z a_{13} \\ \partial_x a_{21} + \partial_y a_{22} + \partial_z a_{23} \\ \partial_x a_{31} + \partial_y a_{32} + \partial_z a_{33} \end{pmatrix}.$$

Since (2.7) holds for every volume ω, it follows that

$$\rho\left(\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}\right) = \nabla \cdot \mathbb{S} + \boldsymbol{f}_{\text{ext}} \quad \forall\, t \in (0, T],\ \boldsymbol{x} \in \Omega. \tag{2.12}$$

This relation is the momentum equation. □

Remark 2.9 (Symmetry of the Stress Tensor) Let ω be an arbitrary volume with sufficiently smooth boundary $\partial\omega$ and let the net force be given by the right-hand side of (2.7). The torque in ω with respect to the origin $\mathbf{0}$ of the coordinate system is then defined by

$$\boldsymbol{M}_{\mathbf{0}} = \int_{\omega} \boldsymbol{r} \times \boldsymbol{f}_{\text{ext}}\, d\boldsymbol{x} + \int_{\partial\omega} \boldsymbol{r} \times (\mathbb{S}\boldsymbol{n})\ d\boldsymbol{s} \quad [\text{Nm}], \tag{2.13}$$

where (2.11) was used. In (2.13), $\boldsymbol{r} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + z\boldsymbol{e}_3$ is the vector pointing from $\mathbf{0}$ to a point $\boldsymbol{x} \in \overline{\omega}$. A straightforward calculation shows that

$$\boldsymbol{r} \times (\mathbb{S}\boldsymbol{n}) = (\boldsymbol{r} \times \mathbb{S}_{*1}\ \boldsymbol{r} \times \mathbb{S}_{*2}\ \boldsymbol{r} \times \mathbb{S}_{*3})\, \boldsymbol{n},$$

where $\mathbb{S}_{*i}$ is the i-th column of $\mathbb{S}$ and $(\cdot)$ denotes here the tensor with the respective columns. Inserting this expression in (2.13), applying integration by parts, and using the product rule leads to

$$\begin{aligned} \boldsymbol{M}_{\mathbf{0}} &= \int_{\omega} \boldsymbol{r} \times \boldsymbol{f}_{\text{ext}}\, d\boldsymbol{x} + \int_{\omega} \nabla \cdot ((\boldsymbol{r} \times \mathbb{S}_{*1}\ \boldsymbol{r} \times \mathbb{S}_{*2}\ \boldsymbol{r} \times \mathbb{S}_{*3}))\, d\boldsymbol{x} \\ &= \int_{\omega} \boldsymbol{r} \times (\boldsymbol{f}_{\text{ext}} + \nabla \cdot \mathbb{S})\, d\boldsymbol{x} \\ &\quad + \int_{\omega} \partial_x \boldsymbol{r} \times \mathbb{S}_{*1} + \partial_y \boldsymbol{r} \times \mathbb{S}_{*2} + \partial_z \boldsymbol{r} \times \mathbb{S}_{*3}\, d\boldsymbol{x}. \end{aligned} \tag{2.14}$$

Consider now a fluid in equilibrium state, i.e., the net forces acting on this fluid are zero. Hence, the right-hand side of (2.12) vanishes and so the first integral of (2.14). In addition, equilibrium requires in particular that $\boldsymbol{M}_{\mathbf{0}} = \mathbf{0}$. Thus, from (2.14) it follows that

$$\mathbf{0} = \int_{\omega} \partial_x \boldsymbol{r} \times \mathbb{S}_{*1} + \partial_y \boldsymbol{r} \times \mathbb{S}_{*2} + \partial_z \boldsymbol{r} \times \mathbb{S}_{*3} \, d\boldsymbol{x}. \tag{2.15}$$

Using now

$$\partial_x \boldsymbol{r} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)\boldsymbol{e}_1 - x\boldsymbol{e}_1}{\Delta x} = \boldsymbol{e}_1,$$

$\partial_y \boldsymbol{r} = \boldsymbol{e}_2$, $\partial_z \boldsymbol{r} = \boldsymbol{e}_3$, and inserting these equations in (2.15) leads finally to

$$\mathbf{0} = \int_{\omega} \begin{pmatrix} \mathbb{S}_{32} - \mathbb{S}_{23} \\ \mathbb{S}_{13} - \mathbb{S}_{31} \\ \mathbb{S}_{21} - \mathbb{S}_{12} \end{pmatrix} (t, \boldsymbol{x}) \, d\boldsymbol{x}$$

for an arbitrary volume ω. From this relation, one deduces that $\mathbb{S}$ has to be symmetric, $\mathbb{S} = \mathbb{S}^T$, and $\mathbb{S}$ possesses six unknown components. □

Remark 2.10 (Decomposition of the Stress Tensor) To model the stress tensor in the basic variables introduced in Remark 2.1, this tensor is decomposed into

$$\mathbb{S} = \mathbb{V} - P\mathbb{I}. \tag{2.16}$$

Here, $\mathbb{V}$ [N/m²] is the so-called viscous stress tensor, representing the forces coming from the friction of the particles, and P [Pa] is the pressure, describing the forces acting on the surface of each fluid volume ω, where $\mathbb{I}$ is the identity tensor. The viscous stress tensor will be modeled in terms of the velocity, see Remark 2.12. □

Remark 2.11 (The Pressure) The pressure P acts on a surface of a fluid volume ω only normal to that surface and it is directed into ω. This property is reflected by the negative sign in the ansatz (2.16) since

$$-\int_{\partial\omega} P\boldsymbol{n} \, ds = -\int_{\omega} \nabla P \, d\boldsymbol{x} = -\int_{\omega} \nabla \cdot (P\mathbb{I}) \, d\boldsymbol{x}.$$

□

Remark 2.12 (The Viscous Stress Tensor) Friction between fluid particles can only occur if the particles move with different velocities. For this reason, the viscous stress tensor is modeled to depend on the gradient of the velocity. For the reason of symmetry, Remark 2.9, it is modeled to depend on the symmetric part of the gradient, the so-called velocity rate-of-deformation tensor or shortly velocity deformation tensor

$$\mathbb{D}(\boldsymbol{v}) = \frac{\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T}{2} \quad [1/\mathrm{s}].$$

The gradient of the velocity is a tensor with the components

$$(\nabla \boldsymbol{v})_{ij} = \partial_j v_i = \frac{\partial v_i}{\partial x_j}, \quad i,j = 1,2,3.$$

If the velocity gradients are not too large, one can assume that first the dependency is linear and second that higher order derivatives can be neglected. Since there is no friction for a flow with constant velocity, such that $\mathbb{V}$ vanishes in this case, lower order terms than first order derivatives of the velocity should not appear in the model. The most general form of a tensor that satisfies all conditions is

$$\mathbb{V} = a\mathbb{D}(\boldsymbol{v}) + b(\nabla \cdot \boldsymbol{v})\mathbb{I},$$

where a and b do not depend on the velocity. Introducing the first order viscosity μ [$^{\text{kg}}/_{(\text{m s})}$] and the second order viscosity ζ [$^{\text{kg}}/_{(\text{m s})}$], one writes this tensor in fluid dynamics in the form

$$\mathbb{V} = 2\mu\mathbb{D}(\boldsymbol{v}) + \left(\zeta - \frac{2\mu}{3}\right)(\nabla \cdot \boldsymbol{v})\mathbb{I} \quad [^{\text{N}}/_{\text{m}^2}]. \tag{2.17}$$

The viscosity μ is also called dynamic or shear viscosity. The law (2.17) is for fluids the analog of Hooke's law for solids. □

Example 2.13 (Steady Rotation) There is no viscous stress, i.e., $\mathbb{V} = \mathbb{0}$, if the fluid is rotating steadily. In this situation, the velocity is given by

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{x} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \omega_3 y - \omega_2 z \\ \omega_1 z - \omega_3 x \\ \omega_2 x - \omega_1 y \end{pmatrix},$$

where $\boldsymbol{\omega}$ [$^1/_\text{s}$] is a constant angular velocity. One has obviously $\nabla \cdot \boldsymbol{v} = 0$ and

$$\nabla \boldsymbol{v} = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \implies \mathbb{D}(\boldsymbol{v}) = \mathbb{0}.$$

Hence, (2.17) is an appropriate model in this case. □

Remark 2.14 (Newtonian Fluids) The linear relation (2.17) is only an approximation for a real fluid. In general, the relation will be nonlinear. Only for small stresses, a linear approximation of the general stress-deformation relation can be used. A linear stress-deformation relation was postulate by Newton. For this reason, a fluid satisfying assumption (2.17) is called Newtonian fluid. More general relations than (2.17) exist, however they are less well understood from the mathematical point of view. □

Remark 2.15 (Normal and Shear Stresses, Trace of the Stress Tensor) The diagonal components $\mathbb{S}_{11}, \mathbb{S}_{22}, \mathbb{S}_{33}$ of the stress tensor are called normal stresses and the off-diagonal components shear stresses.

For incompressible flows one gets with (2.4), (2.16), and (2.17)

$$\mathbb{S} = 2\mu\mathbb{D}(\boldsymbol{v}) - P\mathbb{I}. \tag{2.18}$$

The trace of the stress tensor is the sum of the normal stresses

$$\begin{aligned} \operatorname{tr}(\mathbb{S}) &= \mathbb{S}_{11} + \mathbb{S}_{22} + \mathbb{S}_{33} \\ &= 2\mu(\partial_x v_1 + \partial_y v_2 + \partial_z v_3) + 3\left(\zeta - \frac{2\mu}{3}\right)(\nabla \cdot \boldsymbol{v}) - 3P \\ &= 3\zeta(\nabla \cdot \boldsymbol{v}) - 3P. \end{aligned}$$

For incompressible fluids, it follows that

$$\operatorname{tr}(\mathbb{D}(\boldsymbol{v})) = \frac{1}{2\mu}(\operatorname{tr}(\mathbb{S}) + \operatorname{tr}(P\mathbb{I})) = \frac{1}{2\mu}(-3P + 3P) = 0$$

and

$$P(t, \boldsymbol{x}) = -\frac{1}{3}(\mathbb{S}_{11} + \mathbb{S}_{22} + \mathbb{S}_{33})(t, \boldsymbol{x}). \tag{2.19}$$

□

Remark 2.16 (The Navier–Stokes Equations) Now, the pressure part of the stress tensor and the model (2.17) of the viscous stress tensor can be inserted in (2.12) giving the general Navier–Stokes equations (including the conservation of mass)

$$\begin{aligned} &\rho(\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}) - 2\nabla \cdot (\mu\mathbb{D}(\boldsymbol{v})) \\ &-\nabla \cdot \left(\left(\zeta - \frac{2\mu}{3}\right)(\nabla \cdot \boldsymbol{v})\mathbb{I}\right) + \nabla P = \boldsymbol{f}_{\text{ext}} \text{ in } (0, T] \times \Omega, \\ &\qquad \partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0 \quad \text{in } (0, T] \times \Omega. \end{aligned} \tag{2.20}$$

If the fluid is incompressible and homogeneous, such that μ and ρ are positive constants, the Navier–Stokes equations simplify to

$$\begin{aligned} \partial_t \boldsymbol{v} - 2\nu\nabla \cdot \mathbb{D}(\boldsymbol{v}) + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \nabla\frac{P}{\rho} &= \frac{\boldsymbol{f}_{\text{ext}}}{\rho} \text{ in } (0, T] \times \Omega, \\ \nabla \cdot \boldsymbol{v} &= 0 \quad \text{in } (0, T] \times \Omega. \end{aligned} \tag{2.21}$$

Here, $\nu = \mu/\rho$ [m^2/s] is the kinematic viscosity of the fluid. □

2.3 The Dimensionless Navier–Stokes Equations

Remark 2.17 (Characteristic Scales) Mathematical analysis and numerical simulations are based on dimensionless equations. To derive dimensionless equations from system (2.21), the quantities

- L [m]—a characteristic length scale of the flow problem,
- U [m/s]—a characteristic velocity scale of the flow problem,
- T^* [s]—a characteristic time scale of the flow problem,

are introduced. □

Remark 2.18 (The Navier–Stokes Equations in Dimensionless Form) Denote by $(t', \boldsymbol{x}')$ [s, m] the old variables. Applying the transform of variables

$$\boldsymbol{x} = \frac{\boldsymbol{x}'}{L}, \quad \boldsymbol{u} = \frac{\boldsymbol{v}}{U}, \quad t = \frac{t'}{T^*}, \tag{2.22}$$

one obtains from (2.21) and a rescaling

$$\begin{aligned} \frac{L}{UT^*}\partial_t \boldsymbol{u} - \frac{2\nu}{UL}\nabla\cdot\mathbb{D}(\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\frac{P}{\rho U^2} &= \frac{L}{\rho U^2}\boldsymbol{f}_{\text{ext}} \quad \text{in } (0,T]\times\Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \qquad\qquad \text{in } (0,T]\times\Omega, \end{aligned}$$

where all derivatives are with respect to the new variables. Without having emphasized this issue in the notation, also the domain and the time interval are now dimensionless. Defining

$$p = \frac{P}{\rho U^2}, \quad \text{Re} = \frac{UL}{\nu}, \quad \text{St} = \frac{L}{UT^*}, \quad \boldsymbol{f} = \frac{L}{\rho U^2}\boldsymbol{f}_{\text{ext}}, \tag{2.23}$$

the incompressible Navier–Stokes equations in dimensionless form

$$\begin{aligned} \text{St}\partial_t \boldsymbol{u} - \frac{2}{\text{Re}}\nabla\cdot\mathbb{D}(\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } (0,T]\times\Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } (0,T]\times\Omega, \end{aligned} \tag{2.24}$$

are obtained. The constant Re is called Reynolds number and the constant St Strouhal number. These numbers allow the classification and comparison of different flows. □

Remark 2.19 (Inherent Difficulties of the Dimensionless Navier–Stokes Equations) To simplify the notations, one uses the characteristic time scale $T^* = L/U$ such that (2.24) simplifies to

$$\begin{aligned} \partial_t \boldsymbol{u} - 2\nu\nabla\cdot\mathbb{D}(\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \text{ in } (0,T]\times\Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \text{ in } (0,T]\times\Omega, \end{aligned} \tag{2.25}$$

with the dimensionless viscosity $\nu = \mathrm{Re}^{-1}$. Here, with an abuse of notation, the same symbol is used as for the kinematic viscosity.

This transform and the resulting Eq. (2.25) are the basic equations for the mathematical analysis of the incompressible Navier–Stokes equations and the numerical simulation of incompressible flows. System (2.25) comprises two important difficulties:

- the coupling of velocity and pressure,
- the nonlinearity of the convective term.

Additionally, difficulties for the numerical simulation occur if

- the convective term dominates the viscous term, i.e., if ν is small.

□

Remark 2.20 (Different Forms of Terms in (2.25)) With the help of the divergence constraint, i.e., the second equation in (2.25), the viscous and the convective term of the Navier–Stokes equations can be reformulated equivalently.

Assume that $\boldsymbol{u}$ is sufficiently smooth with $\nabla \cdot \boldsymbol{u} = 0$. Then, straightforward calculations, using the Theorem of Schwarz and the second equation of (2.25), give

$$\nabla \cdot (\nabla \boldsymbol{u}) = \Delta \boldsymbol{u}, \quad \nabla \cdot \left(\nabla \boldsymbol{u}^T\right) = \nabla \left(\nabla \cdot \boldsymbol{u}\right) = \begin{pmatrix} \partial_x \left(\nabla \cdot \boldsymbol{u}\right) \\ \partial_y \left(\nabla \cdot \boldsymbol{u}\right) \\ \partial_z \left(\nabla \cdot \boldsymbol{u}\right) \end{pmatrix} = \boldsymbol{0}. \tag{2.26}$$

Thus, the viscous term becomes

$$-2\nu \nabla \cdot \mathbb{D}\left(\boldsymbol{u}\right) = -\nu \Delta \boldsymbol{u}. \tag{2.27}$$

For the convective term, ones uses the identity (product rule)

$$\begin{aligned} \nabla \cdot \left(\boldsymbol{u}\boldsymbol{v}^T\right) &= \begin{pmatrix} \partial_x \left(u_1 v_1\right) + \partial_y \left(u_1 v_2\right) + \partial_z \left(u_1 v_3\right) \\ \partial_x \left(u_2 v_1\right) + \partial_y \left(u_2 v_2\right) + \partial_z \left(u_2 v_3\right) \\ \partial_x \left(u_3 v_1\right) + \partial_y \left(u_3 v_2\right) + \partial_z \left(u_3 v_3\right) \end{pmatrix} \\ &= \begin{pmatrix} u_1 \left(\partial_x v_1 + \partial_y v_2 + \partial_z v_3\right) \\ u_2 \left(\partial_x v_1 + \partial_y v_2 + \partial_z v_3\right) \\ u_3 \left(\partial_x v_1 + \partial_y v_2 + \partial_z v_3\right) \end{pmatrix} + \begin{pmatrix} v_1 \partial_x u_1 + v_2 \partial_y u_1 + v_3 \partial_z u_1 \\ v_1 \partial_x u_2 + v_2 \partial_y u_2 + v_3 \partial_z u_2 \\ v_1 \partial_x u_3 + v_2 \partial_y u_3 + v_3 \partial_z u_3 \end{pmatrix} \\ &= (\nabla \cdot \boldsymbol{v})\boldsymbol{u} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{u}. \end{aligned} \tag{2.28}$$

In the case $\boldsymbol{v} = \boldsymbol{u}$ with $\nabla \cdot \boldsymbol{u} = 0$, it follows that

$$\left(\boldsymbol{u} \cdot \nabla\right) \boldsymbol{u} = \nabla \cdot \left(\boldsymbol{u}\boldsymbol{u}^T\right). \tag{2.29}$$

A detailed presentation and discussion of different forms of the convective term is given in Sect. 6.1.2. □