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T. E. Govindan

Yosida Approximations of Stochastic Differential Equations in Infinite Dimensions and Applications

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In fond memory of my maternal great grandmother and my maternal grandmother

To my mother Mrs. G. Suseela and to my father Mr. T. E. Sarangan

In fond memory of my Kutty

Preface

It is well known that the celebrated Hille-Yosida theorem, discovered independently by Hille [1] and Yosida [1], gave the first characterization of the infinitesimal generator of a strongly continuous semigroup of contractions. This was the beginning of a systematic development of the theory of semigroups of bounded linear operators. The bounded linear operator A_λ appearing in the sufficiency part of Yosida's proof of this theorem is called the Yosida approximation of A; see Pazy [1]. The objective of this research monograph is to present a systematic study on Yosida approximations of stochastic differential equations in infinite dimensions and applications.

On the other hand, a study on stochastic differential equations (SDEs) in infinite dimensions was initiated in the mid-1960s; see, for instance, Curtain and Falb [1, 2], Chojnowska-Michalik [1], Ichikawa [1–4], and Metivier and Pistone [1] using the semigroup theoretic approach and Pardoux [1] using the variational approach of Lions [1] from the deterministic case. Note, however, that a strong foundation of SDEs, in infinite dimensions in the semilinear case was first laid by Ichikawa [1–4]. It is also worth mentioning here the earlier works of Haussman [1] and Zabczyk [1]. All these aforementioned attempts in infinite dimensions were generalizations of stochastic ordinary differential equations introduced by K. Itô in the 1940s and independently by Gikhman [1] in a different form, perhaps motivated by applications to stochastic partial differential equations in one dimension, like heat equations. Today, SDEs in the sense of Itô, in infinite dimensions are a well-established area of research; see the excellent monographs by Curtain and Pritchard [1], Itô [1], Rozovskii [1], Ahmed [1], Da Prato and Zabczyk [1], Kallianpur and Xiong [1], and Gawarecki and Mandrekar [1]. Throughout this book, we shall use mainly the semigroup theoretic approach as it is our interest to study mild solutions of SDEs in infinite dimensions. However, we shall also use the variational approach to study stochastic evolution equations with delay and multivalued stochastic partial differential equations.

To the best of our knowledge, Ichikawa [2] was the first to use Yosida approximations to study control problems for SDEs. It is a well-known fact that Itô's formula is not applicable to mild solutions; see Curtain [1]. This motivates the need to look for a way out, and Yosida approximations come in handy as these Yosida approximating SDEs have the so-called strong solutions and Itô's formula is applicable only to strong solutions. Yosida approximations, since then, have been used widely for various classes of SDEs; see Chapters 3 and 4 below, to study many diverse problems considered in Chapters 5 and 6.

The book begins in Chapter 1 with a brief introduction mentioning motivating problems like heat equations, an electric circuit, an interacting particle system, a lumped control system, and the option and stock price dynamics to study the corresponding abstract stochastic equations in infinite dimensions like stochastic evolution equations including such equations with delay, McKean-Vlasov stochastic evolution equations, neutral stochastic partial differential equations, and stochastic evolution equations with Poisson jumps. The book also deals with stochastic integrodifferential equations, multivalued stochastic differential equations, stochastic evolution equations with Markovian switchings driven by Lévy martingales, and time-varying stochastic evolution equations.

In Chapter 2, to make the book as self-contained as possible and reader friendly, some important mathematical machinery, namely, concepts and definitions, lemmas, and theorems, that will be needed later on in the book will be provided. As the book studies SDEs using mainly the semigroup theory, it is first intended to provide this theory starting with the fundamental Hille-Yosida theorem and then define precisely the Yosida approximations as well as such approximations for multivalued monotone maps. There is an interesting connection between the semigroup theory and the probability theory. Using this, we shall also delve into some recent results on asymptotic expansions and optimal convergence rate of Yosida approximations. Next, some basics from probability and analysis in Banach spaces are considered like those of the concepts of probability and random variables, Wiener process, Poisson process, and Lévy process, among others. With this preparation, stochastic calculus in infinite dimensions is dealt with next, namely, the concepts of Itô stochastic integral with respect to Q-Wiener and cylindrical Wiener processes, stochastic integral with respect to a compensated Poisson random measure, and Itô's formulas in various settings. In some parts of the book, the theory of stochastic convolution integrals is needed. So, we then state some results from this theory without proofs. This chapter coupled with Appendices dealing with multivalued maps, maximal monotone operators, duality maps, random multivalued maps, and operators on Hilbert spaces, more precisely, notions of trace class operators, nuclear and Hilbert-Schmidt operators, etc., should give a sound background. Since there are many excellent references on this subject matter like Curtain and Pritchard [1], Ahmed [1], Altman [1], Bharucha-Reid [1], Bichteler [1], Da Prato and Zabczyk [1, 2], Dunford and Schwartz [1], Ichikawa [3], Gawarecki and Mandrekar [1], Joshi and Bose [1], Pazy [1], Barbu [1, 2], Knoche [1], Peszat and Zabczyk [1], Prévôt and Röckner [1], Padgett [1], Padgett and Rao [1], Stephan [1], Tudor [1], Yosida [1], and Vilkiene [1–3], among others, the objective here is to keep this chapter brief.

Chapter 3 addresses the main results on Yosida approximations of stochastic differential equations in infinite dimensions in the sense of Itô. The chapter begins by motivating this study from linear stochastic evolution equations. After a brief discussion on linear equations, the pioneering work by Ichikawa (1982) on semilinear stochastic evolution equations is considered in detail next. We introduce Yosida approximating system as it has strong solutions so that Itô's formula can be applied. It will be interesting to show that these approximating strong solutions converge to mild solutions of the original system in mean square. This result is then generalized to stochastic evolution equations with delay. We next consider a special form of a stochastic evolution equation that is related to the so-called McKean-Vlasov measure-valued stochastic evolution equation. We introduce Yosida approximations to this class of equations, showing their existence and uniqueness of strong solutions and also the mean-square convergence of these strong solutions to the mild solutions of the original system. We next generalize this theory to McKean-Vlasov-type stochastic evolution equations with a multiplicative diffusion. In the rest of the chapter, we consider Yosida approximation problems of many more general stochastic models including neutral stochastic partial functional differential equations, stochastic integrodifferential equations, multivalued-valued stochastic differential equations, and time-varying stochastic evolution equations. The chapter concludes with some interesting Yosida approximations of controlled stochastic differential equations, notably, stochastic evolution equations driven by stochastic vector measures, McKean-Vlasov measure-valued evolution equations, and also stochastic equations with partially observed relaxed controls.

In Chapter 4, we consider Yosida approximations of stochastic differential equations with Poisson jumps. More precisely, we introduce Yosida approximations to stochastic delay evolution equations with Poisson jumps, stochastic evolution equations with Markovian switching driven by Lévy martingales, multivalued-valued stochastic differential equations driven by Poisson noise, and also such equations with a general drift term with respect to a general measure. As before, we shall also obtain mean-square convergence results of strong solutions of such Yosida approximate systems to mild solutions of the original equations.

In Chapter 5, many consequences and applications of Yosida approximations to stochastic stability theory are given. First, we consider the pioneering work of Ichikawa (1982) on exponential stability of semilinear stochastic evolution equation in detail and also the stability in distribution of mild solutions of such semilinear equations. As an interesting consequence, exponential stabilizability for mild solutions of semilinear stochastic evolution equations is considered next. Since an uncertainty is present in the system, we obtain robustness in stability of such systems with constant and general decays. This study is then generalized to stochastic equations with delay; that is, polynomial stability with a general decay is established for such delay systems. Consequently, robust exponential stabilization of such delay equations is obtained. Subsequently, stability in distribution is considered for stochastic evolution equations with delays driven by Poisson jumps. Moreover, moment exponential stability and also almost sure exponential stability of sample paths of mild solutions of stochastic evolution equations with Markovian switching with Poisson jumps are dealt with. We also study the weak convergence of induced probability measures of mild solutions of McKean-Vlasov stochastic evolution equations, neutral stochastic partial functional differential equations, and stochastic integrodifferential equations. Furthermore, the exponential stability of mild solutions of McKean-Vlasov-type stochastic evolution equations with a multiplicative diffusion, stochastic integrodifferential evolution equations, and time-varying stochastic evolution equations are considered.

Finally, in Chapter 6, it will be interesting to consider some applications of Yosida approximations to stochastic optimal control problems like optimal control over finite time horizon, a periodic control problem of stochastic evolution equations, and an optimal control problem of McKean-Vlasov measure-valued evolution equations. Moreover, we also consider some necessary conditions of optimality of relaxed controls of stochastic evolution equations. The chapter as well as the book concludes with optimal feedback control problems of stochastic evolution equations driven by stochastic vector measures.

I have tried to keep the work of various authors drawn from all over the literature as original as possible. I thank sincerely all of them whose work have been included in the book with due citations they deserve in the bibliographical notes and remarks and elsewhere. I believe to the best of my knowledge that I have covered in this monograph all the work that I have known. There may be more interesting materials, but it is impossible to include all in one book. I apologize to those authors in case I have missed out their work. This is certainly not deliberate.

Mexico City, Mexico T. E. Govindan
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Notations and Abbreviations

Abbreviations

a.e.	Almost everywhere
P-a.s.	Probability almost surely or with probability 1
i.i.d.	Independently and identically distributed
w.l.g.	Without loss of generality
l.s.c.	Lower semicontinuous
u.s.c.	Upper semicontinuous
SDE	Stochastic differential equation
SEE	Stochastic evolution equation
SPDE	Stochastic partial differential equation
RNP	Radon-Nikodym property
HJB	Hamilton-Jacobi-Bellman
fBm	Fractional Brownian motion

Notations

$\square$	Signals end of proof
$:=$	Equality by definition
$I_B(x)$	Indicator function of a set B
$\mathbb{N}$	Set of natural numbers
R^n	n-dimensional Euclidean space with the usual norm, $n \in \mathbb{N}$
R	Real line, i.e., $R = (-\infty, \infty)$
R^+	Nonnegative real line, i.e., $R^+ = [0, \infty)$
$\text{Re}\lambda$	Real part of λ
$\text{Im}\lambda$	Imaginary part of λ
$(X, \|\cdot\|_X)$	Banach space with its norm

$(X^*, \|\|\cdot\|\|_{X^*})$	Dual of a Banach space with its norm
$_{X^*}\langle x^*, x\rangle_X$	Duality pairing between X^* and X
$\mathcal{B}(X)$	Borel σ-algebra of subsets of X
$M(X)$	Space of probability measures on $\mathcal{B}(X)$ carrying the usual topology of weak convergence
$BC(Z)$	Space of bounded continuous functions on Z with the topology of sup norm where Z is a normal topological space
$\mathcal{L}(x)$	Probability law of x
$D(A)$	Domain of an operator A
$\rho(A)$	Resolvent set of an operator A
$R(\lambda, A)$	Resolvent of an operator A
A_λ	Yosida approximation of an operator A
$\mathrm{tr}Q$	Trace of an operator Q
$L(Y,X)$	Space of all bounded linear operators from Y into X
$L(X)$	$L(X,X)$
$L_1(Y,X)$	Space of all nuclear operators from Y into X
$L_2(Y,X)$	Space of all Hilbert-Schmidt operators from Y into X
$\|\|\cdot\|\|_{L_2}$	Hilbert-Schmidt norm
$L^p(\Omega, \mathcal{F}, P; X)$	Banach space of all functions from Ω to X which are p-integrable with respect to (w.r.t.) P, $1 \le p < \infty$
$L^p(\Omega, \mathcal{F}, P)$	$L^p(\Omega, \mathcal{F}, P; R)$, $1 \le p < \infty$
$L^p([0,T], X)$	Banach space of all X-valued Borel measurable functions on $[0,T]$ which are p-integrable, $1 \le p < \infty$
$L^p[0,T]$	$L^p([0,T], R)$, $1 \le p < \infty$
$co\{\cdot\}$	Closed convex hull of $\{\cdot\}$
$\mathcal{G}(A)$	Graph of an operator A
$\{S(t) : t \ge 0\}$	C_0-semigroup
$\{U(t,s) : s < t\}$	Evolution operator
$\{\beta(t), t \ge 0\}$	Real-valued Brownian motion or Wiener process
$\{w(t), t \ge 0\}$	Q-Wiener process or cylindrical Wiener process
$E(x)$	Expectation of x
$E(x\|\mathcal{A})$	Conditional expectation of x given $\mathcal{A}$
$Q^{1/2}$	Square root of $Q \in L(X)$
T^*	Adjoint operator of $T \in L(Y,X)$
T^{-1}	(Pseudo) Inverse of $T \in L(Y,X)$
w^*	Weak star
(V, H, V^*)	Gelfand triple
$\int_0^t \Phi(s)dw(s)$	Itô stochastic integral w.r.t. $w(t)$
$\tilde{N}(t,A)$	Compensated Poisson random measure
$\int_0^t \int_Z \Phi(s,z)\tilde{N}(ds,dz)$	Stochastic integral w.r.t. a compensated Poisson measure $\tilde{N}(dt,du)$

$C([0,T],X)$	Banach space of X-valued continuous functions on $[0,T]$ with the usual sup norm
$M_T^2(X)$	Space of all X-valued continuous, square integrable martingales
$\langle\langle x(\cdot)\rangle\rangle$	The process of quadratic variation of x
$N(m,Q)$	Gaussian law with mean m and covariance operator Q
$T(\omega)$	Random operator
$N_w^2(0,T;L_2^0)$	Simply $N_w^2(0,T)$ is a Hilbert space of all L_2^0-predictable processes Φ such that $\|\|\Phi\|\|_T < \infty$

Chapter 1
Introduction and Motivating Examples

Stochastic differential equations are well known to model stochastic processes observed in the study of dynamic systems arising from many areas of science, engineering, and finance. Existence and uniqueness of mild, strong, relaxed, and weak solutions; stability, stabilizability, and control problems; regularity and continuous dependence on initial values; approximation problems notably of Yosida; among others, of solutions of stochastic differential equations in infinite dimensions have been investigated by several authors, see, for instance, Ahmed [1, 6, 8] Bharucha-Reid [1], Curtain and Pritchard [1], Da Prato [2], Da Prato and Zabczyk [1, 3, 4], Gawarecki and Mandrekar [1], Kotelenez [1], Liu [2], Mandrekar and Rüdiger [1], McKibben [2], and Prévôt and Röckner [1] and the references therein. Yosida approximations play a key role in many of these problems.

In this chapter, we motivate the study of some of the abstract stochastic differential equations considered in this book by modeling real-life problems such as a heat equation, an electric circuit, an interacting particle system, and the stock and option price dynamics in a loose language. Rigorous formulations of many concrete problems and theoretical examples are taken up later on in the subsequent chapters.

1.1 A Heat Equation

Let us consider the following heat equation with a stochastic perturbation

$$dx(z,t) = \frac{\partial^2}{\partial z^2}x(z,t)dt + \sigma x(z,t)d\beta(t), \quad t > 0, \tag{1.1}$$
$$x(0,t) = x(1,t) = 0, \quad x(z,0) = x_0(z),$$

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where σ is a real number and $\beta(t)$ is a real standard Wiener process. Consider also the semilinear stochastic heat equation of the form

$$dx(z,t) = \left[\frac{\partial^2}{\partial z^2}x(z,t) - \frac{x(z,t)}{1+|x(z,t)|}\right]dt + \frac{\sigma x(z,t)}{1+|x(z,t)|}d\beta(t), \quad t>0, \tag{1.2}$$
$$x_z(0,t) = x_z(1,t) = 0, \quad x(z,0) = x_0(z),$$

where $|\cdot|$ is the absolute value on $R = (-\infty,\infty)$. For details, we refer to Ichikawa [2, 3].

1.1.1 Stochastic Evolution Equations

The equations (1.1) and (1.2) can be formulated in the abstract setting as follows:

Take $X = L^2(0,1)$ and $Y = R$. Define $A = d^2/dz^2$ with $D(A) = \{x \in X | x, x'$ are absolutely continuous with x', $x'' \in X$, $x(0) = x(1) = 0\}$. Equation (1.1) can be expressed in a real Hilbert space X by

$$dx(t) = Ax(t)dt + g(x(t))dw(t), \quad t>0, \tag{1.3}$$
$$x(0) = x_0,$$

where $g(x) = \sigma x$ and $w(t)$ is a Y-valued Q-Wiener process. From Ichikawa [3], the explicit solution of equation (1.3) takes the form

$$x(t) = e^{-\sigma^2 t/2 + \sigma\beta(t)} S(t)x_0,$$

where $\{S(t) : t \geq 0\}$ is the C_0-semigroup generated by A given by

$$S(t)x_0 = \sum_{n=1}^{\infty} e^{-2n^2\pi^2 t} \sin n\pi z \int_0^1 x_0(r) \sin n\pi r dr.$$

To model the second equation (1.2), take X and Y as defined earlier. Define $A = d^2/dz^2$ with $D(A) = \{x \in X | x, x'$ absolutely continuous, x', $x'' \in X$, $x'(0) = x'(1) = 0\}$. Equation (1.2) can be expressed as a semilinear stochastic evolution equation in the Hilbert space X as

$$dx(t) = [Ax(t) + f(x(t))]dt + g(x(t))dw(t), \quad t>0, \tag{1.4}$$
$$x(0) = x_0,$$

where $w(t)$ is a Y-valued Q-Wiener process and

$$f(x) = -\frac{g(x)}{\sigma} = -\frac{x}{1+||x||_X}, \quad x \in X.$$

The concept of a Q-Wiener process will be defined precisely later on in Chapter 2.

Linear stochastic evolution equations of the form (1.3) will be considered in Sections 3.1 and 6.1 in connection with optimal control problems. The semilinear stochastic equations of the form (1.4) will be discussed in detail in Section 3.2 and later on in Sections 5.1 and 5.2. More general time-varying semilinear stochastic equations will be studied in Sections 3.8 and 5.9. See also Section 6.2.

1.2 An Electric Circuit

An electric circuit is considered in which two resistances, a capacitance and an inductance, are connected in series. Assume that the current is flowing through the loop, and its value at time t is $x(t)$ amperes. Let us use the following units: volts for the voltage, ohms for the resistance R, henry for the inductance L, farads for the capacitance c, coloumbs for the charge on the capacitance, and seconds for the time. It is well known that with this system of units, the voltage drop across the inductance is $\mathsf{L}dx(t)/dt$, and that across the resistances R and R_1 is $(\mathsf{R}+\mathsf{R}_1)x(t)$. The voltage drop across the capacitance is q/c, where q is the charge on the capacitance. It is also known that $x(t) = dq/dt$. A fundamental Kirchhoff's law states that the sum of the voltage drops around the loop must be equal to the applied voltage:

$$\mathsf{L}\frac{dx(t)}{dt} + (\mathsf{R}+\mathsf{R}_1)x(t) + \frac{q}{c} = 0. \tag{1.5}$$

On differentiating equation (1.5) with respect to t, we deduce

$$\mathsf{L}\frac{d^2x(t)}{dt^2} + (\mathsf{R}+\mathsf{R}_1)\frac{dx(t)}{dt} + \frac{1}{c}x(t) = 0. \tag{1.6}$$

The voltage drop across R_1 is applied to a nonlinear amplifier A_1. The output is provided with a special phase-shifting network P. This introduces a constant time lag between the input and the output P. The voltage drop across R in series with the output P is

$$e(t) = qg(\dot{x}(t-r));$$

where q is the gain of the amplifier to R measured through the network. The equation (1.6) takes the form

$$\mathsf{L}\frac{d^2x(t)}{dt^2} + \mathsf{R}\dot{x}(t) + qg(\dot{x}(t-r)) + \frac{1}{c}x(t) = 0.$$

Finally, a second device is introduced to help stabilize the fluctuations in the current. If $\dot{x}(t) = y(t)$, the controlled system may be described by

$$\begin{aligned} \dot{x}(t) &= y(t) + u_1(t) \\ \dot{y}(t) &= -\frac{\mathsf{R}}{\mathsf{L}} y(t) - \frac{q}{\mathsf{L}} g(y(t-r)) - \frac{1}{c\mathsf{L}} x(t) + u_2(t). \end{aligned} \tag{1.7}$$

The controlled system (1.7) can be expressed in the matrix form

$$\dot{\mathbf{X}}(t) = A\mathbf{X}(t) + G(\mathbf{X}(t-r)) + B\mathbf{U}, \tag{1.8}$$

where

$$\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -1/c\mathsf{L} & -\mathsf{R}/\mathsf{L} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$G(\mathbf{X}(t-r)) = \begin{pmatrix} 0 \\ -qg(y(t-r))/\mathsf{L} \end{pmatrix}.$$

The controlled vector $\mathbf{U}$ is created and introduced by the stabilizer.

1.2.1 Stochastic Evolution Equations with Delay

Motivated by this electric circuit and stochastic partial differential equations with delay, consider the following stochastic evolution equation with delay in a real Hilbert space X:

$$\begin{aligned} dx(t) &= [Ax(t) + f(x(t-r))]dt + g(x(t-r))dw(t), \quad t > 0, \\ x(t) &= \varphi(t), \quad t \in [-r, 0], \quad 0 \le r < \infty, \end{aligned} \tag{1.9}$$

where $A : D(A) \to X$ (possibly unbounded) is the infinitesimal generator of a C_0-semigroup $\{S(t) : t \ge 0\}$, $f : X \to X$ and $g : X \to L(Y, X)$ (space of all bounded linear operators from Y into X), where Y is another real Hilbert space and $w(t)$ is a Y-valued Q-Wiener process. We assume that the past process $\{\varphi(t), -r \le t \le 0\}$ is known.

We shall be considering such stochastic evolution equations with a constant delay in Section 3.3.1 and stochastic equations with a variable delay in Sections 3.3.2 and 3.3.3. See also Sections 5.3.1 and 5.4.

1.3 An Interacting Particle System

Consider a biological, chemical, or physical interacting particle system in which each particle moves in some space according to the dynamics described by the following system of N coupled semilinear stochastic evolutions equations:

$$\begin{aligned} dx_k(t) &= [Ax_k(t) + f(x_k(t), \mu_N(t))]dt + \sqrt{\mathbb{Q}}dw_k(t), \quad t > 0, \\ x_k(0) &= x_0, \quad k = 1, 2, \ldots, N, \end{aligned} \tag{1.10}$$

where $\mu_N(t)$ is the empirical measure given by

$$\mu_N(t) = \frac{1}{N} \sum_{k=1}^{N} \delta x_k(t)$$

of the N particles $x_1(t)$, $x_2(t)$, ..., $x_N(t)$ at time t. According to McKean-Vlasov theory, see, for example, McKean [1], Dawson and Gärtner [1], and Gärtner [1], under proper conditions, the empirical measure-valued process μ_N converges in probability to a deterministic measure-valued function μ as N goes to infinity. It is interesting to observe that the limit μ corresponds to the probability distribution of a stochastic process determined by the equation (1.11) given next. We also refer to Kurtz and Xiong [1].

1.3.1 McKean-Vlasov Stochastic Evolution Equations

Consider the following stochastic process described by a semilinear Itô equation in a real separable Hilbert space X:

$$\begin{aligned} dx(t) &= [Ax(t) + f(x(t), \mu(t))]dt + \sqrt{\mathbb{Q}}dw(t), \quad t > 0, \\ \mu(t) &= \text{ probability distribution of x(t)}, \\ x(0) &= x_0, \end{aligned} \tag{1.11}$$

where $w(t)$ is a given X-valued cylindrical Wiener process; $A : D(A) \subset X \to X$ (possibly unbounded) is the infinitesimal generator of a strongly continuous semigroup $\{S(t) : t \geq 0\}$ of bounded linear operators on X; f is an appropriate X-valued function defined on $X \times M_{\gamma^2}(X)$, where $M_{\gamma^2}(X)$ denotes a proper subset of probability measures on X; $\mathbb{Q}$ is a positive, symmetric, bounded operator on X; and x_0 is a given X-valued random variable. For details, see Section 3.4.1.

We shall also consider more general Mc-Kean-Vlasov type stochastic systems in Section 3.4.2 and subsequently in Sections 3.11.1, 5.5, and 6.3.

1.4 A Lumped Control System

A method to stabilize lumped control systems is to use a hereditary proportional-integral-differential (PID) feedback control. Consider a linear distributed hereditary system with a finite delay of the form

$$\frac{dx(t)}{dt} = Ax(t) + f(x_t) + Bu(t), \quad t > 0, \tag{1.12}$$

where $x(t) \in X$ represents the state, $u(t) \in R^m$ (m-dimensional Euclidean space) denotes the control, $x_t(s) = x(t+s)$, $-r \le s \le 0$, $A : D(A) \subset X \to X$ is the infinitesimal generator of an analytic semigroup $\{S(t) : t \ge 0\}$, and $B : R^m \to X$.

The feedback control $u(t)$ will be a PID-hereditary control defined by

$$u(t) = K_0 x(t) - \frac{d}{dt}\int_{-r}^{t} K_1(t-s)x(s)ds, \tag{1.13}$$

where $K_0 : X \to R^m$ is a bounded linear operator and $K_1 : [0,\infty) \to L(X,R^m)$ is a strongly continuous operator-valued map. The closed system corresponding to the PID-hereditary control (1.13) takes the form

$$\frac{d}{dt}\left[x(t) + B\int_{-r}^{t} K_1(t-s)x(s)ds\right] = (A + BK_0)x(t) + f(x_t), \quad t > 0.$$

It is known that $A + BK_0$ is the infinitesimal generator of an analytic semigroup.

1.4.1 Neutral Stochastic Partial Differential Equations

Consider a neutral stochastic partial differential equation in a real separable Hilbert space X of the form:

$$\begin{aligned} d[x(t) + f(t,x_t)] &= [Ax(t) + a(t,x_t)]dt + b(t,x_t)dw(t), \quad t > 0, \\ x(t) &= \varphi(t), \quad t \in [-r,0]\ (0 \le r < \infty); \end{aligned} \tag{1.14}$$

where $x_t(s) := x(t+s)$, $-r \le s \le 0$, $-A : D(-A) \subset X \to X$ (possibly unbounded) is the infinitesimal generator of a C_0-semigroup $\{S(t) : t \ge 0\}$ on X, $w(t)$ is a Y-valued Q-Wiener process, $a : R^+ \times X \to X$, where $R^+ = [0,\infty)$, $b : R^+ \times X \to L(Y,X)$ and $f : R^+ \times X \to D((-A)^\alpha)$, $0 < \alpha \le 1$, and $\varphi(t)$ is the past stochastic process assumed to be known. For details, see Section 3.5 below.

Such equations will be considered again in Section 5.6.

1.5 A Hyperbolic Equation

Consider the hyperbolic type deterministic integral equation

$$\begin{aligned} u_{tt}(t,z) &= \Delta u(t,z) + \int_0^t b(t-s)\Delta u(s,z)ds + f(t,z), \quad t>0, \\ u(t,0) &= u(t,\pi) = 0, \end{aligned} \tag{1.15}$$

where $\Delta = \partial^2/\partial z^2$, or the equivalent system

$$u_t = v, \quad v_t = \Delta u + \int_0^t b(t-s)\Delta u(s,\cdot)ds + f(t,\cdot).$$

The equation (1.15) may be written in the form

$$x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + F(t), \quad t>0, \tag{1.16}$$

where

$$x = \begin{pmatrix} u \\ v \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

and

$$A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & 0 \\ b(t)\Delta & 0 \end{pmatrix}.$$

1.5.1 Stochastic Integrodifferential Equations

Integrodifferential equations arise, for example, in mechanics, electromagnetic theory, heat flow, nuclear reactor dynamics, and population dynamics, see Kannan and Bharucha-Reid [1] and the references therein for details. Note that a dynamic system with memory may lead to integrodifferential equations.

Consider a stochastic version of the Volterra integrodifferential equation (1.16) of the form

$$\begin{aligned} x'(t) &= Ax(t) + \int_0^t B(t-s)x(s)d\beta(s) + f(t), \quad t>0, \\ x(0) &= x_0, \end{aligned} \tag{1.17}$$

where A (possibly unbounded) is the infinitesimal generator of a C_0-semigroup $\{S(t) : t \geq 0\}$ on a real separable Hilbert space X with domain $D(A)$, f belongs to a function space $\mathscr{A}$ on X-valued functions, $B(t)$ is a (not necessarily bounded) convolution kernel type linear operator on the domain $D(A)$ (for each $t \geq 0$) such that $B(\cdot)x \in \mathscr{A}$ for each $x \in D(A)$, x_0 is an X-valued random variable, and $\beta(\cdot)$ is a Hilbert-Schmidt operator-valued Brownian motion. For details, see Section 3.6.1 below and Kannan and Bharucha-Reid [1].

We shall also be interested in considering a semilinear stochastic integrodifferential equation of the form

$$\begin{aligned} x'(t) &= Ax(t) + \int_0^t B(t,s)f(s,x(s))ds \\ &\quad + \int_0^t C(t,s)g(s,x(s))dw(s) + F(t,x(t)), \quad t > 0, \qquad (1.18) \\ x(0) &= x_0, \end{aligned}$$

where A is a linear operator (possibly unbounded) is the infinitesimal generator of a C_0-semigroup $\{S(t) : t \geq 0\}$ on a real separable Hilbert space X with domain $D(A)$; $B(t,s)_{0 \leq s \leq t \leq T}$ and $C(t,s)_{0 \leq s \leq t \leq T}$ $(0 < T < \infty)$ are linear operators mapping X into X, $F : [0,\infty) \times X \to X$, $f : [0,\infty) \times X \to X$ and $g : [0,\infty) \times X \to L(Y,X)$, $w(t)$ is a Y-valued Q-Wiener process and x_0 is a known random variable. For details, see Section 3.6.2 below.

See also Sections 3.6.2 and 5.7 for another class of such equations.

1.6 The Stock Price and Option Price Dynamics

This problem was proposed by R. Merton (1976). The total change in the stock price is posited to be the composition of two types of changes: First, the normal vibrations in price, for example, due to temporary imbalance between supply and demand, changes in capitalization rates, changes in the economic outlook, or other new information that causes marginal changes in the stock's value. In essence, the impact of such information per unit time on the stock price is to produce a marginal change in the price P-$a.s.$. This component is modeled by a standard geometric Brownian motion with a constant variance per unit time and it has a continuous sample path. The abnormal vibrations in price are due to the arrival of important new information about the stock that has more than a marginal effect on price. Usually such information will be specific to the firm. It is reasonable to expect that there will be active times in the stock when such information arrives and quiet times when it does not although the active and quiet times are random. By its very nature, important information arrives only at discrete points in time. This component is modeled by a jump process reflecting the non-marginal impact of the information.

To be consistent with the general efficient market hypothesis of Fama [1] and Samuelson [1], the dynamics of the unanticipated part of the stock price motions should be a martingale. Just as once the dynamics are posited to be continuous-time process, the natural prototype process for the continuous component of the stock price change is a Wiener process, so the prototype for the jump component is a Poisson driven process.

Given that the Poisson event occurs (i.e., some important information on the stock arrives), then there is a drawing from a distribution to determine the impact of this information on the stock price, i.e., if $S(t)$ is the stock price at time t and $\mathbf{Y}$ is the random variable description of this drawing, neglecting the continuous part, the stock price at time $t+h$, $S(t+h)$, will be the random variable $S(t+h) = S(t)\mathbf{Y}$, given that one such arrival occurs between t and $t+h$. It is assumed throughout that $\mathbf{Y}$ has a probability measure with compact support and $\mathbf{Y} \geq 0$. Moreover, the $\{\mathbf{Y}\}$ from successive drawings are *i.i.d.*.

As discussed in Merton [2], the posited stock price returns are a mixture of both types and can be formally written as a stochastic differential equation

$$\frac{dS(t)}{S(t)} = (\alpha - \gamma k)dt + \sigma d\beta(t) + dN(t), \quad t > 0, \tag{1.19}$$

where α is the instantaneous expected return on the stock, σ^2 is the instantaneous variance of the return, conditional on no arrivals of important new information (i.e., the Poisson event does not occur); $\beta(t)$ is a standard Wiener process; $N(t)$ is the Poisson process; $N(t)$ and $\beta(t)$ are assumed to be independent; γ is the mean number of arrivals per unit time; $k = E(\mathbf{Y}-1)$ where $\mathbf{Y}-1$ is the random variable percentage change in the stock price if the Poisson event occurs.

The $\sigma d\beta(t)$ part describes the instantaneous part of the unanticipated return due to the normal price vibrations, and the $dN(t)$ part describes the abnormal price vibrations. If $\gamma = 0$ (and thereafter, $dN(t) \equiv 0$), then the return dynamics would be identical to those posited in Black and Scholes [1] and Merton [3]. Equation (1.19) can be rewritten in a somewhat more cumbersome form as

$$\frac{dS(t)}{S(t)} = (\alpha - \gamma k)dt + \sigma d\beta(t),$$

if the Poisson event does not occur, and

$$\frac{dS(t)}{S(t)} = (\alpha - \gamma k)dt + \sigma d\beta(t) + (\mathbf{Y}-1),$$

if the Poisson event occurs, where with *P-a.s.*, no more than one Poisson event occurs in an instant, and if the event does not occur, then $\mathbf{Y}-1$ is an impulse function producing a finite jump in S to $S\mathbf{Y}$.

Having established the stock price dynamics, let us now consider the dynamics of the option price. Suppose that the option price, W, can be written as a twice-continuously differentiable function of the stock price and time; namely, $W(t) = F(S,t)$. If the stock price follows the dynamics described in equation (1.19), then the option return dynamics can be written in a similar form as

$$\frac{dW(t)}{W(t)} = (\alpha_W - \gamma k_W)dt + \sigma_W d\beta(t) + dN_W(t), \tag{1.20}$$

where α_W is the instantaneous expected return on the option; σ_W^2 is the instantaneous variance of the return, conditional on the Poisson event not occurring, $N_W(t)$ is a Poisson process with parameter γ, where $N_W(t)$ and $\beta(t)$ are assumed to be independent; $k_W \equiv E(\mathbf{Y}_W - 1)$, where $\mathbf{Y}_W - 1$ is the random variable percentage change in the option price if the Poisson event occurs.

1.6.1 Stochastic Evolution Equations with Poisson jumps

Consider the following class of stochastic differential equations with Poisson jumps in a Hilbert space X of the form

$$\begin{aligned} dx(t) &= [Ax(t) + f(x(t))]dt + g(x(t))dw(t) \\ &\quad + \int_Z L(x(t),u)\tilde{N}(dt,du), \quad t > 0, \\ x(0) &= x_0, \end{aligned} \tag{1.21}$$

where $\tilde{N}$ is a compensated Poisson random measure associated with a counting Poisson random measure N; A, generally unbounded, is the infinitesimal generator of a C_0-semigroup $\{S(t) : t \geq 0\}$, the mappings $f : X \to X$, $g : X \to L(Y,X)$ and $L : X \times Y \to X$ are some measurable functions. Let $\tilde{N}(dt,du) = N(dt,du) - dt\nu(du)$ be independent of $w(t)$, a Y-valued Q-Wiener process. Here ν is the characteristic measure associated with a stationary $\mathcal{F}_t$-Poisson point process $\{p(t), t \in D_p\}$ (see Definition 2.20), and x_0 is a known random variable.

We shall consider stochastic equations of the type (1.21) with delay in Sections 4.1 and 5.3.2 and with Markovian switchings in Sections 4.2, 4.3, and 5.8.

Chapter 2
Mathematical Machinery

The purpose of this chapter is to introduce the necessary background from the semigroup theory, particularly, the Yosida approximations and their properties, analysis and probability in Banach spaces, including Itô stochastic calculus, stochastic convolution integrals, among others. As pointed out before, no attempt has been made to make the presentation self-contained as there are many excellent books available in the literature.

2.1 Semigroup Theory

Let $(X, ||\cdot||_X)$ be a Banach space.

Definition 2.1 A one parameter family $\{S(t) : 0 \leq t < \infty\}$ of bounded linear operators mapping X into X is a semigroup of bounded linear operators on X if

(i) $S(0) = I$, (I is the identity operator on X),
(ii) $S(t+s) = S(t)S(s)$ for every $t, s \geq 0$ (the semigroup property).

A semigroup of bounded linear operators, $\{S(t) : t \geq 0\}$, is uniformly continuous if

$$\lim_{t \downarrow 0} ||S(t) - I|| = 0.$$

The linear operator A defined by

$$D(A) = \{x \in X : \lim_{t \downarrow 0} \frac{S(t)x - x}{t} \text{ exists}\} \tag{2.1}$$

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and

$$Ax = \lim_{t \downarrow 0} \frac{S(t)x - x}{t} = \frac{d^+ S(t)x}{dt}|_{t=0} \quad \text{for } x \in D(A), \tag{2.2}$$

is the infinitesimal generator of the semigroup $\{S(t) : t \geq 0\}$, where $D(A)$ is the domain of A.

Theorem 2.1 A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.

Proof See Pazy [1, Theorem 1.2]. □

Definition 2.2 A semigroup $\{S(t) : t \geq 0\}$ of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators if

$$\lim_{t \downarrow 0} S(t)x = x \quad \text{for every} \quad x \in X. \tag{2.3}$$

A strongly continuous semigroup of bounded linear operators on X will be called a C_0-semigroup. A C_0-semigroup $\{S(t) : t > 0\}$ is called compact if it is a compact operator.

Theorem 2.2 Let $\{S(t) : t \geq 0\}$ be a C_0-semigroup. There exist constants $\alpha \geq 0$ and $M \geq 1$ such that

$$||S(t)|| \leq Me^{\alpha t} \quad \text{for} \quad 0 \leq t < \infty. \tag{2.4}$$

Proof See Ahmed [1, Theorem 1.3.1]. □

Corollary 2.1 If $\{S(t) : t \geq 0\}$ is a C_0-semigroup then for every $x \in X$, $t \to S(t)x$ is a continuous function from R^+ into X.

Proof See Ahmed [1, Corollary 1.3.2]. □

Theorem 2.3 Let $\{S(t) : t \geq 0\}$ be a C_0-semigroup and let A be its infinitesimal generator. Then

(a) For $x \in X$,

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} S(s)xds = S(t)x.$$

(b) For $x \in X$,

$$\int_0^t S(t)xdx \in D(A) \quad \text{and} \quad A\left(\int_0^t S(t)xdx\right) = S(t)x - x.$$

(c) For $x \in D(A), S(t)x \in D(A)$ and

$$\frac{d}{dt} S(t)x = AS(t)x = S(t)Ax.$$