P. P. Teodorescu, W. W. Kecs, A. Toma

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*Petre P. Teodorescu, Wilhelm W. Kecs,
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Preface

The solution to many theoretical and practical problems is closely connected to the methods applied, and to the mathematical tools which are used. In the mathematical description of mechanical and physical phenomena, and in the solution of the corresponding boundary value and limit problems, difficulties may appear owing to additional conditions. Sometimes, these conditions result from the limited range of applicability of the mathematical tool which is involved; in general, such conditions may be neither necessary nor connected to the mechanical or physical phenomenon considered.

The methods of classical mathematical analysis are usually employed, but their applicability is often limited. Thus, the fact that not all continuous functions have derivatives is a severe restriction imposed on the mathematical tool; it affects the unity and the generality of the results. For example, it may lead to the conclusion of the nonexistence of the velocity of a particle at any moment during the motion, a conclusion which obviously is not true.

On the other hand, the development of mechanics, of theoretical physics and particularly of modern quantum mechanics, the study of various phenomena of electromagnetism, optics, wave propagation and the solution of certain boundary value problems have all brought about the introduction of new concepts and computations, which cannot be justified within the frame of classical mathematical analysis.

In this way, in 1926 Dirac introduced the delta function (denoted by δ), which from a physical point of view, represents the density of a load equal to unity located at one point. A formalism has been worked out for the function, and its use justifies and simplifies various results. Except for a small number of incipient investigations, it was only during the 1960s that the theory of distributions was included as a new chapter of functional analysis. This theory represents a mathematical tool applicable to a large class of problems, which cannot be solved with the aid of classical analysis. The theory of distributions thus eliminates the restrictions which are not imposed by the physical phenomenon and justifies procedure and results, e.g., those corresponding to the continuous and discontinuous phenomena, which can thus be stated in a unitary and general form.

This monograph presents elements of the theory of distributions, as well as theorems with possibility of application. While respecting the mathematical rigor, a large number of applications of the theory of distributions to problems of general Newtonian mechanics, as well as to problems pertaining to the mechanics of deformable solids, are presented in a systematic manner; special stress is laid upon the introduction of corresponding mathematical models.

Some notions and theorems of Newtonian mechanics are stated in a generalized form; the effect of discontinuities on the motion of a particle and its mechanical interpretation is thus emphasized.

Particular stress is laid upon the mathematical representation of concentrated and distributed loads; in this way, the solution of the problems encountered in the mechanics of deformable solids may be obtained in a unitary form.

Newton's fundamental equation, the equations of equilibrium and of motion of the theory of elasticity are presented in a modified form, which includes the boundary and the initial conditions. In this case, the Fourier and the Laplace transforms may be easily applied to obtain the fundamental solutions of the corresponding differential equations; the use of the convolution product allows the expression of the boundary-value problem solutions for an arbitrary load.

Concerning the mechanics of deformable solids, not only have classical elastic bodies been taken into consideration, but also viscoelastic ones, that is, stress is put into dynamical problems: vibrations and propagation of waves.

Applications in physics have been described (acoustics, optics and electrostatics), as well as in electrotechnics.

The aim of the book is to draw attention to the possibility of applying modern mathematical methods to the study of mechanical and physical phenomena and to be useful to mathematicians, physicists, engineers and researchers, which use mathematical methods in their field of interest.

P.P. Teodorescu, W.W. Kecs, A. Toma *Bucharest, 22 August 2012*

1
Introduction to the Distribution Theory

1.1
Short History

The theory of distributions, or of generalized functions, constitutes a chapter of functional analysis that arose from the need to substantiate, in terms of mathematical concepts, formulae and rules of calculation used in physics, quantum mechanics and operational calculus that could not be justified by classical analysis. Thus, for example, in 1926 the English physicist P.A.M. Dirac [1] introduced in quantum mechanics the symbol $\delta(x)$, called the Dirac delta function, by the formulae

$$\delta(x) = \begin{cases} 0\,, & x \neq 0 \\ \infty\,, & x = 0 \end{cases}\,, \quad \int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1\,. \tag{1.1}$$

By this symbol, Dirac mathematically described a material point of mass density equal to the unit, placed at the origin of the coordinate axis.

We notice immediately that $\delta(x)$, called the impulse function, is a function not in the sense of mathematical analysis, as being zero everywhere except the origin, but that its integral is null and not equal to unity.

Also, the relations $x\delta(x) = 0, \mathrm{d}H(x)/\mathrm{d}x = \delta(x)$ do not make sense in classical mathematical analysis, where

$$H(x) = \begin{cases} 0\,, & x < 0 \\ 1\,, & x \geq 0 \end{cases}$$

is the Heaviside function, introduced in 1898 by the English engineer Oliver Heaviside.

The created formalism regarding the use of the function δ and others, although it was in contradiction with the concepts of mathematical analysis, allowed for the study of discontinuous phenomena and led to correct results from a physical point of view.

All these elements constituted the source of the theory of distributions or of the generalized functions, a theory designed to justify the formalism of calculation used in various fields of physics, mechanics and related techniques.

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In 1936, S.L. Sobolev introduced distributions (generalized functions) in an explicit form, in connection with the study of the Cauchy problem for partial differential equations of hyperbolic type.

The next major event took place in 1950–1951, when L. Schwartz published a treatise in two volumes entitled “Théory des distributions” [2]. This book provided a unified and systematic presentation of the theory of distributions, including all previous approaches, thus justifying mathematically the calculation formalisms used in physics, mechanics and other fields.

Schwartz’s monograph, which was based on linear functionals and on the theory of locally convex topological vector spaces, motivated further development of many chapters of mathematics: the theory of differential equations, operational calculus (Fourier and Laplace transforms), the theory of Fourier series and others.

Properties in the sense of distributions, such as the existence of the derivative of any order of a distribution and in particular of the continuous functions, the convergence of Fourier series and the possibility of term by term derivation of the convergent series of distributions, led to important technical applications of the theory of distributions, thus removing some restrictions of classical analysis.

The distribution theory had a significant further development as a result of the works developed by J. Mikusiński and R. Sikorski [3], M.I. Guelfand and G.E. Chilov [4, 5], L. Hörmander [6, 7], A. H. Zemanian [8], and so on.

Unlike the linear and continuous functionals method used by Schwartz to define distributions, J. Mikusiński and R. Sikorski introduced the concept of distribution by means of fundamental sequences of continuous functions.

This method corresponds to the spirit of classical analysis and thus it appears clearly that the concept of distribution is a generalization of the notion of function, which justifies the term generalized function, mainly used by the Russian school.

Other mathematicians, such as H. König, J. Korevaar, Sebastiano e Silva, and I. Halperin have defined the notion of distribution by various means (axiomatic, derivatives method, and so on).

Today the notion of distribution is generalized to the concept of a hyperfunction, introduced by M. Sato, [9, 10], in 1958. The hyperdistributions theory contains as special cases the extensions of the notion of distribution approached by C. Roumieu, H. Komatsu, J.F. Colombeu and others.

1.2
Fundamental Concepts and Formulae

For the purpose of distribution theory and its applications in various fields, we consider some function spaces endowed with a convergence structure, called fundamental spaces or spaces of test functions.

1.2.1
Normed Vector Spaces: Metric Spaces

We denote by Γ either the body $\mathbb{R}$ of real numbers or the body $\mathbb{C}$ of complex numbers and by $\mathbb{R}_+$, $\mathbb{R}^+$, $\mathbb{N}_0$ the sets $\mathbb{R}_+ = [0,\infty), \mathbb{R}^+ = (0,\infty), \mathbb{N}_0 = \{0,1,2,\ldots,n,\ldots\}$.

Let E, F be sets of abstract objects. We denote by $E \times F$ the direct product (Cartesian) of those two sets; where the symbol "$\times$" represents the direct or Cartesian product.

Definition 1.1 The set E is called a vector space with respect to Γ, and is denoted by (E,Γ), if the following two operations are defined: the sum, a mapping $(x, y) \to x + y$ from $E \times F$ into E, and the product with scalars from Γ, the mapping $(\lambda, x) \to \lambda x$ from $\Gamma \times E$ into E, having the following properties:

1. $\forall x, y \in E\,, \quad x + y = y + x\,;$
2. $\forall x, y, z \in E\,, \quad (x + y) + z = x + (y + z)\,;$
3. $\exists 0 \in E\,, \quad \forall x \in E\,, \quad x + 0 = x$, (0 is the null element);
4. $\forall x \in E\,, \quad \exists x' = -x \in E\,, \quad x + (-x) = 0\,;$
5. $\forall x \in E\,, \quad 1 \cdot x = x\,;$
6. $\forall \lambda, \mu \in \Gamma\,, \quad \forall x \in E\,, \quad \lambda(\mu x) = (\lambda\mu)x\,;$
7. $\forall \lambda, \mu \in \Gamma\,, \quad \forall x \in E\,, \quad (\lambda + \mu)x = \lambda x + \mu x\,;$
8. $\forall \lambda \in \Gamma\,, \quad \forall x, y \in E\,, \quad \lambda(x + y) = \lambda x + \lambda y\,.$

The vector space (E,Γ) is real if $\Gamma = \mathbb{R}$ and it is complex if $\Gamma = \mathbb{C}$. The elements of (E,Γ) are called points or vectors.

Let X be an upper bounded set of real numbers, hence there is $M \in \mathbb{R}$ such that for all $x \in X$ we have $x \le M$. Then there exists a unique number $M^* = \sup X$, which is called the lowest upper bound of X, such that

1. $\forall x \in X\,, \quad x \le M^*\,;$
2. $\forall a \in \mathbb{R}\,, \quad a < M^*\,, \quad \exists x \in X$ such that $x \in (a, M^*]\,.$

Similarly, if Y is a lower bounded set of real numbers, that is, if there is $m \in \mathbb{R}$ such that for all $x \in Y$ we have $x \ge m$, then there exists a unique number $m^* = \inf X$, which is called the greatest lower bound of Y, such that

1. $\forall x \in Y\,, \quad x \ge m^*\,;$
2. $\forall b \in \mathbb{R}\,, \quad b > m^*\,, \quad \exists x \in Y$ such that $x \in [m^*, b)\,.$

Example 1.1 The vector spaces $\mathbb{R}^n$, $\mathbb{C}^n$, $n \ge 2$ Let us consider the n-dimensional space $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ (n times). Two elements $x, y \in \mathbb{R}^n$, $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_n)$, are said to be equal, $x = y$, if $x_i = y_i$, $i = \overline{1,n}$.

Denote $x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$, $\alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$, $\alpha \in \mathbb{R}$, then $\mathbb{R}^n$ is a real vector space, also called n-dimensional real arithmetic space.

The n-dimensional complex space $\mathbb{C}^n$ may be defined in a similar manner. The elements of this space are ordered systems of n complex numbers. The sum and product operations performed on complex numbers are defined similarly with those in $\mathbb{R}^n$.

Definition 1.2 Let (X, Γ) be a real or complex vector space. A norm on (X, Γ) is a function $\|\cdot\| : X \to [0, \infty)$ satisfying the following three axioms:

1. $\forall x \in X\,, \quad \|x\| > 0$ for $x \neq 0$, $\|0\| = 0\,;$
2. $\forall \lambda \in \Gamma\,, \quad \forall x \in X\,, \quad \|\lambda x\| = |\lambda|\|x\|\,;$
3. $\forall x, y \in X\,, \quad \|x + y\| \leq \|x\| + \|y\|\,.$

The vector space (X, Γ) endowed with the norm $\|\cdot\|$ will be called a normed vector space and will be denoted as $(X, \Gamma, \|\cdot\|)$.

The following properties result from the definition of the norm:

$$\|x\| \geq 0\,, \quad \forall x \in X\,,$$
$$|\|x_1\| - \|x_2\|| \leq \|x_1 - x_2\|\,, \quad \forall x_1, x_2 \in X\,,$$
$$\forall \alpha_i \in \Gamma\,, \quad \forall x_i \in X\,, \quad \|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \leq |\alpha_1|\|x_1\| + \cdots + |\alpha_n|\|x_n\|\,.$$

Definition 1.3 Let (X, Γ) be a vector space. We call an inner product on (X, Γ) a mapping $\langle\cdot,\cdot\rangle : E \to \Gamma$ that satisfies the following properties:

1. Conjugate symmetry: $\forall x \in X$, $\langle x, y\rangle = \overline{\langle y, x\rangle}$;
2. Homogeneity: $\forall \alpha \in \Gamma, \forall x, y \in E, \langle \alpha x, y\rangle = \alpha\langle y, x\rangle$;
3. Additivity: $\forall x, y, z \in X, \langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$;
4. Positive-definiteness: $\forall x \in X, \langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \Leftrightarrow x = 0$.

An inner product space $(X, \langle\cdot,\cdot\rangle)$ is a space containing a vector space (X, Γ) and an inner product $\langle\cdot,\cdot\rangle$.

Conjugate symmetry and linearity in the first variable gives

$$\langle x, ay\rangle = \overline{\langle ay, x\rangle} = \overline{a}\overline{\langle y, x\rangle} = \overline{a}\langle x, y\rangle\,,$$
$$\langle x, y + z\rangle = \overline{\langle y + z, x\rangle} = \overline{\langle y, x\rangle} + \overline{\langle z, x\rangle} = \langle x, y\rangle + \langle x, z\rangle\,,$$

so an inner product is a sesquilinear form. Conjugate symmetry is also called Hermitian symmetry.

In the case of $\Gamma = \mathbb{R}$, conjugate-symmetric reduces to symmetric, and sesquilinear reduces to bilinear. Thus, an inner product on a real vector space is a positive-definite symmetric bilinear form.

Proposition 1.1 In any inner product space $(X, \langle\cdot,\cdot\rangle)$ the Cauchy–Schwarz inequality holds:

$$|\langle x, y\rangle| \leq \sqrt{\langle x, x\rangle} \cdot \sqrt{\langle y, y\rangle}\,, \quad \forall x, y \in X\,, \tag{1.2}$$

with equality if and only if x and y are linearly dependent.

This is also known in the Russian mathematical literature as the Cauchy–Bunyakowski–Schwarz inequality.

Lemma 1.1 The inner product is antilinear in the second variable, that is $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$ for all $x, y, z \in \Gamma$ and $\langle x, a y\rangle = \overline{a}\langle x, y\rangle$.

Note that the convention in physics is often different. There, the second variable is linear, whereas the first variable is antilinear.

Definition 1.4 Let X be a nonempty set. We shall call metric (distance) on X any function $d : X \times X \to \mathbb{R}$, which satisfies the properties:

D_1 $d(x, x) = 0, \forall x \in X; d(x, y) > 0, \forall x, y \in X, x \neq y$,
D_2 $\forall x, y \in X, d(x, y) = d(y, x)$,
D_3 $\forall x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z)$.

The real number $d(x, y) \geq 0$ represents the distance between x and y, and the ordered pair (X, d) a *metric space* (whose elements are called *points*).

Let (X, d) be a metric space. We shall call an *open ball* in X a ball of radius $r > 0$ centered at the point $x_0 \in X$, usually denoted $B_r(x_0)$ or $B(x_0; r)$, the set

$$B_r(x_0) = \{x \in X \mid d(x, x_0) < r\} . \tag{1.3}$$

The closed ball, which will be denoted by $\overline{B}_r(x_0)$ is defined by

$$\overline{B}_r(x_0) = \{x \in X \mid d(x, x_0) \leq r\} . \tag{1.4}$$

Note, in particular, that a ball (open or closed) always includes x_0 itself, since the definition requires $r > 0$. We shall call a sphere of radius $r > 0$ centered at the point $x_0 \in X$, usually denoted $S_r(x_0)$, the set

$$S_r(x_0) = \{x \in X \mid d(x, x_0) = r\} . \tag{1.5}$$

Proposition 1.2 Any normed vector space is a metric space by defining the distance by the formula

$$d(x, y) = \|x - y\| , \quad \forall x, y \in X . \tag{1.6}$$

Proposition 1.3 Any inner product space $(X, \langle\cdot,\cdot\rangle)$ is a normed vector space if we define the norm by

$$\|x\| = \sqrt{\langle x, x\rangle} , \quad \forall x \in X . \tag{1.7}$$

An inner product space is also called a pre-Hilbert space, since its completion with respect to the metric induced by its inner product, is a Hilbert space.

The real vector space $\mathbb{R}^n$ endowed with the inner product

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i, x = (x_1, \ldots, x_n) , \quad y = (y_1, \ldots, y_n) \in \mathbb{R}^n \tag{1.8}$$

is called the *n-dimensional Euclidean real space*.

The norm in $\mathbb{R}^n$ is called *the Euclidean norm* and is defined as

$$\|x\| = \langle x, x\rangle^{1/2} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} , \tag{1.9}$$

whereas the metric associated to this norm is given by

$$d(x, y) = \|x - y\| = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2} . \tag{1.10}$$

1.2.2
Spaces of Test Functions

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ be a generic point in the n-dimensional Euclidean real space and let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ be a multiindex of order n; we denote by $|\alpha| = \alpha_1 + \cdots + \alpha_n$ the length of the multiindex. If $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$, then we use the following notations:

$$\alpha \leq \beta \text{ if } \alpha_i \leq \beta_i, i = \overline{1, n} ; \tag{1.11}$$

$$\binom{\beta}{\alpha} = \frac{\beta!}{\alpha!(\beta - \alpha)!} , \quad \text{where } \alpha! = \alpha_1!\alpha_2!\ldots\alpha_n! , \tag{1.12}$$

$$x^\alpha = {x_1}^{\alpha_1}{x_2}^{\alpha_2}\ldots{x_n}^{\alpha_n} . \tag{1.13}$$

We denote by $\mathrm{D}^\alpha f$ the partial derivative of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$ of a function $f : \Omega \subset \mathbb{R}^n \to \Gamma$,

$$\mathrm{D}^\alpha f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}} f , \quad \mathrm{D}^\alpha = \mathrm{D}_1^{\alpha_1} \mathrm{D}_2^{\alpha_2} \ldots \mathrm{D}_n^{\alpha_n} , \quad \mathrm{D}_j = \frac{\partial}{\partial x_j} , \quad j = \overline{1, n} .$$

If $|\alpha| = 0$, then $\alpha_i = 0$, $i = \overline{1, n}$, that is, $\mathrm{D}^0 f = f$.

If the function f has continuous partial derivatives up to the order $|\alpha + \beta|$ inclusively, then

$$\mathrm{D}^{\alpha+\beta} f = \mathrm{D}^\alpha(D^\beta f) = \mathrm{D}^\beta(\mathrm{D}^\alpha f).$$

We shall denote by $C^m(\Omega)$ the set of functions $f : \Omega \subset \mathbb{R}^n \to \Gamma$ with continuous derivatives of order m, that is, $\mathrm{D}^\alpha f$ is continuous on Ω for every α with $|\alpha| \leq m$. When $m = 0$ we have the set $C^0(\Omega)$ of continuous functions on Ω; $C^\infty(\Omega)$ is the set of functions on Ω with continuous derivatives of all orders. Clearly, we have $C^\infty(\Omega) \subset C^m(\Omega) \subset C^0(\Omega)$.

These sets are vector spaces over Ω with respect to the usual definition of addition of functions and multiplication by scalars from Ω. The null element of these spaces is the identically zero function on Ω and it will be denoted by 0.

Definition 1.5 We call the support of the function $f : \mathbb{R}^n \to \Gamma$ the set

$$\text{supp}(f) = \overline{\{x \in \mathbb{R}^n, f(x) \neq 0\}}\,, \tag{1.14}$$

hence the closure of the set of points where the function is not zero.

If $x_0 \in \text{supp}(f)$, then $\forall B_{x_0}(r)$, $\exists x \in \mathbb{R}^n$ thus that $f(x) \neq 0$. In particular, if supp(f) is bounded, then, since supp(f) is a closed set, it is also compact.

Proposition 1.4 If $f, g : \mathbb{R}^n \to \Gamma$, then:

$$\text{supp}(f + g) \subset \text{supp}(f) \cup \text{supp}(g)\,, \tag{1.15}$$

$$\text{supp}(f \cdot g) \subset \text{supp}(f) \cap \text{supp}(g)\,, \tag{1.16}$$

$$\text{supp}(\lambda f) = \text{supp}(f)\,, \quad \lambda \neq 0\,. \tag{1.17}$$

Proof: If $x_0 \in \text{supp}(f + g)$, then $\forall B_r(x_0) \subset \mathbb{R}^n, \exists x \in B_r(x_0)$ such that $(f + g)(x) \neq 0$, from which results $f(x) \neq 0$ or $g(x) \neq 0$. Consequently, either $x_0 \in \text{supp}(f)$ or $x_0 \in \text{supp}(g)$, hence $x_0 \in \text{supp}(f) \cup \text{supp}(g)$. Regarding relation (1.16), we notice that $x_0 \in \text{supp}(f \cdot g)$ implies $(fg)(x) \neq 0, x \in B_r(x_0)$; hence $f(x) \neq 0$ and $g(x) \neq 0$. Consequently, $x_0 \in \text{supp}(f)$ and $x_0 \in \text{supp}(g)$, hence supp$(f) \cap$supp(g). Because relation (1.17) is obvious, the proof is complete. □

Proposition 1.5 If the functions $f, g \in C^p(\Omega)$, $\Omega \subset \mathbb{R}^n$, then $fg \in C^p(\Omega)$ and we have

$$\mathrm{D}^\alpha(f \cdot g) = \sum_{\beta+\gamma} \frac{\alpha!}{\beta!\gamma!} \mathrm{D}^\beta f \cdot \mathrm{D}^\gamma g\,, \quad \mathrm{D}^\alpha = \mathrm{D}_1^{\alpha_1} \mathrm{D}_2^{\alpha_2} \ldots \mathrm{D}_n^{\alpha_n}\,, \tag{1.18}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n, |\alpha| \leq p$.
The proof of this formula is accomplished through induction.

Definition 1.6 A function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is said to be uniformly continuous on A if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any $x, y \in A$ satisfying the condition $\|x - y\| < \delta(\varepsilon)$ the inequality $|f(x) - f(y)| < \varepsilon$ holds.

We mention that a uniformly continuous function on $A \subset \mathbb{R}^n$ is continuous at each point of the set A. It follows that the continuity is a local (more precisely, pointwise) property of a function f, while the uniform continuity is a global property of f. In the study of the properties of spaces of test functions, the notion of uniformly convergent sequence plays an important role.

Definition 1.7 We consider the sequence of functions $(f_n)_{n\geq 1}$, $f_n : A \subset \mathbb{R}^n \to \mathbb{R}$ and the function $f : A \subset \mathbb{R}^n \to \mathbb{R}$. We say that the sequence of functions $(f_n)_{n\geq 1}$, $x \in A$ is uniformly convergent towards f, $x \in A$, and we write $f_n \stackrel{u}{\to} f$, $x \in A \subset \mathbb{R}^n$, if for every $\varepsilon > 0$ there exists a natural number $N(\varepsilon)$ such that for all $x \in A$ and all $n \geq N(\varepsilon)$ the inequality $|f_n(x) - f(x)| < \varepsilon$ holds.

In the case of uniform convergence, the natural number $N(\varepsilon)$ depends only on $\varepsilon > 0$, being the same for all $x \in A$, while in the case of pointwise convergence the natural number N depends on ε and $x \in A$. Therefore the uniform convergence implies pointwise convergence $f_n \xrightarrow{s} f$. The converse is not always true.

Definition 1.8 We say that the function $f : A \subset \mathbb{R}^n \to \mathbb{C}$ is absolutely integrable on A if the integral $\int_A |f(x)|\mathrm{d}x$ is finite, hence $\int_A |f(x)|\mathrm{d}x < \infty$. The integral can be considered either in the sense of Riemann, or in the sense of Lebesgue.

If the integral is considered in the sense of Lebesgue, then the existence of the integral $\int_A |f(x)|\mathrm{d}x$ implies the existence of the integral $\int_A f(x)\mathrm{d}x$.

The set of the Lebesgue integrable functions on A will be denoted $L^1(A)$.

If f is absolutely integrable on any bounded domain $A \subset \mathbb{R}^n$, then we say that f is a *locally integrable function*. We shall use $L^1_{\text{loc}}(A)$ to denote the space of locally integrable functions on A.

The set $A \subset \mathbb{R}^n$ is said to be *negligible* or of *null Lebesgue measure* if for any $\varepsilon > 0$ there is a sequence $(B_i)_{i\geq 1}$, $B_i \subset \mathbb{R}^n$, such that $\cup_{i=1}^{\infty} B_i \supset A$ and the summed volume of the open ball B_i is less than ε.

The function $f : A \subset \mathbb{R}^n \to \Gamma$ is said to be *null a.e.* (almost everywhere) on the set A if the set $\{x \in A,\ f(x) \neq 0\}$ is of null Lebesgue measure.

Thus, the functions $f, g : A \subset \mathbb{R}^n \to \Gamma$ are *a.e. equal* (almost everywhere equal), denoted by $f = g$ *a.e.*, $x \in A$, if the set $\{x \in A,\ f(x) \neq g(x)\}$ is of null Lebesgue measure.

The function $f : A \subset \mathbb{R}^n \to \Gamma$ is *p-integrable* on A, $1 \leq p < \infty$, if $|f|^p \in L^1(A)$. The set of *p-integrable* functions on A is denoted by $\mathcal{L}^p(A)$. In this set we can introduce the equivalence relation $f \sim g$ if $f(x) = g(x)$ a.e. The set of all the equivalence classes is denoted by $L^p(A)$.

The space $L^p(A)$ is a vector space over Γ. The spaces $L^p(A)$ and $L^q(A)$ for which we have $p^{-1} + q^{-1} = 1$ are called *conjugate*. For these spaces, we have Hölder's inequality

$$\int_A |f(x)g(x)|\mathrm{d}x \leq \left(\int_A |f(x)|^p \mathrm{d}x\right)^{1/p} \cdot \left(\int_A |g(x)|^q \mathrm{d}x\right)^{1/q} . \tag{1.19}$$

In particular, for $p = 2$, we have $q = 2$, that is, $L^2(A)$ is self-conjugated and Schwarz's inequality holds

$$\int_A |f(x)g(x)|\mathrm{d}x \leq \left(\int_A |f(x)|^2 \mathrm{d}x\right)^{1/2} \cdot \left(\int_A |g(x)|^2 \mathrm{d}x\right)^{1/2} . \tag{1.20}$$

The norm of the space $L^p(A)$ is defined as

$$\|f\|_p = \left(\int_A |f(x)|^p \mathrm{d}x\right)^{1/p} . \tag{1.21}$$

We notice that the space $L^p(A)$ is normed.

1.2.2.1 The Space $\mathcal{D}^m(\Omega)$

Definition 1.9 Let $\Omega \subset \mathbb{R}^n$ be a given compact set and consider the functions $\varphi : \mathbb{R}^n \to \Gamma$. The set of functions $\mathcal{D}^m(\Omega) = \{\varphi | \varphi \in C^m(\mathbb{R}^n), \mathrm{supp}(\varphi) \subset \Omega\}$ is called the space of test functions $\mathcal{D}^m(\Omega)$.

We notice that $\varphi \in C^m(\mathbb{R}^n)$ with $\mathrm{supp}(\varphi) \subset \Omega$ implies $\mathrm{supp}(\mathrm{D}^\alpha \varphi(x)) \subset \mathrm{supp}(\varphi) \subset \Omega, |\alpha| \leq m$. Consequently, all functions $\varphi \in C^m(\Omega)$ together with all their derivatives up to order m inclusive are null outside the compact Ω. We notice that $\mathcal{D}^m(\Omega)$ is a vector space with respect to Γ. The null element of this space is the identically null function, denoted by 0, $\forall x \in \mathbb{R}^n, \varphi(x) = 0$.

Definition 1.10 We say that the sequence of functions $(\varphi_i)_{i\geq 1} \subset \mathcal{D}^m(\Omega)$ converges towards $\varphi \in \mathcal{D}^m(\Omega)$, and we write $\varphi_i \xrightarrow{\mathcal{D}^m(\Omega)} \varphi$ if the sequence of functions $(\mathrm{D}^\alpha \varphi_i(x))_{i\geq 1}$ converges uniformly towards $\mathrm{D}^\alpha\, \varphi(x)$ in Ω, hence $\mathrm{D}^\alpha\, \varphi_i(x) \overset{u}{\to} \mathrm{D}^\alpha \varphi(x), 0 \leq |\alpha| \leq m, \forall x \in \Omega$.

We note that the space $\mathcal{D}^m(\Omega)$ becomes a normed vector space if we define the norm by

$$\|\varphi\|_{\mathcal{D}^m} = \sup_{|\alpha|\leq m,\, x\in\Omega} |\mathrm{D}^\alpha \varphi(x)| = \sup_{0\leq|\alpha|\leq m} \sup_{x\in\Omega} |\mathrm{D}^\alpha \varphi(x)|, \alpha \in \mathbb{N}_0^n \,. \tag{1.22}$$

In particular, for $m = 0$, the space $\mathcal{D}^0(\Omega)$ will be denoted by $C_C^0(\Omega)$. This is the space of complex (real) functions of class $C^0(\mathbb{R}^n)$, the supports of which are contained in the compact set $\Omega \subset \mathbb{R}^n$. The test functions space $C_C^0(\Omega)$ is a normed vector space with the norm

$$\|\varphi\|_{C_C^0} = \sup_{x\in\Omega} |\varphi(x)| \,. \tag{1.23}$$

The sequence $(\varphi_i)_{i\geq 1} \subset C_C^0(\Omega)$ converges towards $\varphi \in C_C^0(\Omega)$ if $\lim_i \sup_{x\in\Omega} |\varphi_i - \varphi| = 0$, that is, if $(\varphi_i)_{i\geq 1}$ converges uniformly towards φ in Ω.

An example of functions from the space $\mathcal{D}^m(\Omega)$ is the function

$$\varphi(x) = \begin{cases} \prod_{i=1}^{n} \sin^{m+1} \dfrac{x_i - a_i}{b_i - a_i}\pi \,, & x \in [a_1, b_1] \times \cdots \times [a_n, b_n] = \prod_{i=1}^{n} \times [a_i, b_i] \\ 0 \,, & x \notin [a_1, b_1] \times \cdots \times [a_n, b_n] \end{cases}$$

where

$$\Omega \supset \prod_{i=1}^{n} \times [a_i, b_i] \,.$$

It is immediately verified that $\varphi \in C^m(\mathbb{R}^n)$ and $\mathrm{supp}(\varphi) = \prod_{i=1}^n \times [a_i, b_i]$.

Also the function $\varphi : \mathbb{R} \to \mathbb{R}$, where

$$\varphi = \begin{cases} (x-a)^\alpha (b-x)^\beta \,, & x \in [a, b] \\ 0 \,, & x \notin [a, b] \end{cases} \,, \quad \alpha, \beta > m \,,$$

is a function from $\mathcal{D}^m([c, d])$, $[c, d] \supset [a, b]$, because $\varphi \in C^m([c, d])$ and $\text{supp}(\varphi) = [a, b]$.

Let us consider the sequence of functions $(\varphi_n)_{n\geq 1} \subset \mathcal{D}^m(\Omega)$, defined by

$$\varphi_n(x) = \begin{cases} \frac{1}{n}\sin^{m+1}\frac{x+a}{2a}\pi \,, & x \in [-a, a] \,, \\ 0 \,, & x \notin [-a, a] \,. \end{cases}$$

We have $\text{supp}\,\varphi_n(x) = [-a, a] = \Omega$ for any n. This sequence, with its derivatives up to order m inclusive, converges uniformly towards zero in Ω. So we can write $\varphi_n(x) \xrightarrow{\mathcal{D}(\Omega)} 0$ in Ω.

Even if the sequence of functions

$$\varphi_n(x) = \begin{cases} \frac{1}{n}\sin^{m+1}\frac{a + x/n}{2a}\pi \,, & \frac{x}{n} \in [-a, a] \,, \\ 0 \,, & \frac{x}{n} \notin [-a, a] \,, \end{cases}$$

converges uniformly towards zero, together with all their derivatives up to order m inclusive, it is not convergent towards zero in the space $\mathcal{D}^m(\Omega)$. This is because $\text{supp}[\varphi_n(x)] = [-na, na]$, thus the supports of the functions $\varphi_n(x)$ are not bounded when $n \to \infty$, hence $\varphi_n(x)$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, are not test functions from $\mathcal{D}^m(\Omega)$.

1.2.2.2 The Space $\mathcal{D}(\Omega)$

Definition 1.11 Let $\Omega \subset \mathbb{R}^n$ be a given compact set and consider the functions $\varphi : \mathbb{R}^n \to \Gamma$. The set of functions

$$\mathcal{D}(\Omega) = \{\varphi | \varphi \in C^\infty(\mathbb{R}^n), \text{supp}(\varphi) \subset \Omega\}$$

is called the space of test functions $\mathcal{D}(\Omega)$.

The space $\mathcal{D}(\Omega)$ is a vector space over Γ like $\mathcal{D}^m(\Omega)$.

Definition 1.12 We say that the sequence $(\varphi_i)_{i\geq 1} \subset \mathcal{D}(\Omega)$ converges towards $\varphi \in \mathcal{D}(\Omega)$, and we write $\varphi_i \xrightarrow{\mathcal{D}(\Omega)} \varphi$, if the sequence of derivative $(\mathrm{D}^\alpha \varphi_i(x))_{i\geq 1}$ converges uniformly towards $\mathrm{D}^\alpha \varphi(x)$ in Ω, $\forall \alpha \in \mathbb{N}_0^n$, hence $\mathrm{D}^\alpha \varphi_i(x) \xrightarrow{u} \mathrm{D}^\alpha \varphi(x)$, $\forall x \in \Omega$, $\forall \alpha \in \mathbb{N}_0^n$.

We remark that the test space $\mathcal{D}(\Omega)$ is not a normed vector space.

Example 1.2 If $\Omega = \{x | x \in \mathbb{R}^n, \|x\| \leq 2a\}$, then the function $\varphi : \mathbb{R}^n \to \mathbb{R}$, having the expression

$$\varphi(x) = \begin{cases} \exp\left(-\frac{a^2}{a^2 - \|x\|^2}\right) \,, & \|x\| < a \\ 0 \,, & \|x\| \geq a \end{cases} \,, \quad a > 0 \,, \tag{1.24}$$

is an element of the space $\mathcal{D}(\Omega)$, since $\varphi \in C^\infty(\mathbb{R}^n)$ and $\text{supp}(\varphi) = \{x | x \in \mathbb{R}^n, \|x\| \leq a\} \subset \Omega$.

The sets Ω and $\text{supp}(\varphi)$ are compact sets of $\mathbb{R}^n$, representing closed balls with centers at the origin and radii $2a$ and a, respectively.

Unlike the function φ, the function $\psi : \mathbb{R} \to \mathbb{R}$,

$$\psi(x) = \begin{cases} 0\,, & x \leq 0\,, \\ \exp(-x^2)\,, & x > 0\,, \end{cases} \tag{1.25}$$

does not belong to the space $\mathcal{D}(\Omega)$.

This function is infinitely differentiable, so $\psi \in C^\infty(\mathbb{R}^n)$, but the support is not a compact set because $\text{supp}(\psi) = (0, \infty)$.

1.2.2.3 The Space $\mathcal{E}$

Definition 1.13 The functions set

$$\mathcal{E} = \{\varphi | \varphi : \mathbb{R}^n \to \Gamma, \varphi \in C^\infty(\mathbb{R}^n)\}\,. \tag{1.26}$$

having arbitrary support is called the space of test functions $\mathcal{E} = \mathcal{E}(\mathbb{R}^n)$.

With respect to the usual sum and scalar product operation, the space $\mathcal{E}$ is a vector space over Γ.

Thus, the functions $\varphi(x) = 1, \varphi(x) = x^2, \varphi(x) = \exp(x^2), x \in \mathbb{R}$ are elements of $\mathcal{E}(\mathbb{R}^n)$.

As regards the convergence in the space $\mathcal{E}$ this is given:

Definition 1.14 The sequence $(\varphi_i)_{i\geq 1} \subset \mathcal{E}$ is said to converge towards $\varphi \in \mathcal{E}$, and we write $\varphi_i \overset{\mathcal{E}}{\to} \varphi$, if the sequence of functions $(\mathrm{D}^\alpha \varphi_i)_{i\geq 1} \subset \mathcal{E}$ converges uniformly towards $\mathrm{D}^\alpha\, \varphi(x) \in \mathcal{E}$ on any compact of $\mathbb{R}^n$, $\forall \alpha \in \mathbb{N}_0^n$, that is, $\mathrm{D}^\alpha\, \varphi_i \overset{u}{\to} \mathrm{D}^\alpha \varphi$.

The function (1.25) belongs to the space $\mathcal{E}$ since $\psi \in C^\infty(\mathbb{R}^n)$, its supports being the unbounded set $(0, \infty)$.

1.2.2.4 The Space $\mathcal{D}$ (the Schwartz Space)

Definition 1.15 The space $\mathcal{D} = \mathcal{D}(\mathbb{R}^n)$ consists of the set of functions

$$\mathcal{D} = \{\varphi | \varphi : \mathbb{R}^n \to \Gamma, \varphi \in C^\infty(\mathbb{R}^n), \text{supp}(\varphi) = \Omega = \text{compact}\}\,. \tag{1.27}$$

Since $\forall \varphi \in \mathcal{D}$, it belongs to a certain $\mathcal{D}(\Omega)$, it follows that $\mathcal{D}$ is the reunion of spaces $\mathcal{D}(\Omega)$ over the compacts $\Omega \subset \mathbb{R}^n$. Consequently, we can write the following relations:

$$\mathcal{D} = \bigcup_{\Omega} \mathcal{D}(\Omega), \mathcal{D}(\Omega) \subset \mathcal{D} \subset \mathcal{E}\,.$$

With respect to the usual sum and scalar product operations, $\mathcal{D}$ is a vector space on Γ, its null element being the identically zero function. The support of this function is the empty set.

The convergence in the space $\mathcal{D}$ is defined as:

Definition 1.16 The sequence of functions $(\varphi_i)_{i\geq 1} \subset \mathcal{D}$ converges towards $\varphi \in \mathcal{D}$, and we write $\varphi_i \stackrel{\mathcal{D}}{\longrightarrow} \varphi$, if the following conditions are satisfied:

1. $\forall i \in \mathbb{N}$, there is a compact $\Omega \subset \mathbb{R}^n$ such that $\operatorname{supp}(\varphi_i), \operatorname{supp}(\varphi) \subset \Omega$;
2. $\forall \alpha \in \mathbb{N}_0^n$, $\mathrm{D}^\alpha \varphi_i$ converges uniformly towards $\mathrm{D}^\alpha \varphi$ on Ω, that is, $\mathrm{D}^\alpha \varphi_i \stackrel{u}{\rightarrow} \mathrm{D}^\alpha \varphi$ on Ω.

Thus, the convergence in the space $\mathcal{D}$ is reduced to the convergence in the space $\mathcal{D}(\Omega)$.
The vector space $\mathcal{D}(\mathbb{R}^n)$ endowed with the convergence structure defined above is called the space of test functions or the Schwartz space. Every element of the space $\mathcal{D}$ will be called a test function.

Example 1.3 The function $\varphi_a : \mathbb{R}^n \to \mathbb{R}$, $a > 0$, defined by

$$\varphi_a(x) = \begin{cases} \exp\left(-\dfrac{a^2}{a^2 - \|x\|^2}\right) , & \|x\| < a , \\ 0 , & \|x\| \geq a , \end{cases} \tag{1.28}$$

is an element of $\mathcal{D}(\mathbb{R}^n)$, since $\varphi_a \in C^\infty(\mathbb{R}^n)$ and $\operatorname{supp}(\varphi_a) = \{x | x \in \mathbb{R}^n, \|x\| \leq a\} = \text{compact}$.

Example 1.4 Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\varphi(x) = \begin{cases} \exp\left(-\dfrac{|ab|}{(x-a)(b-x)}\right) , & x \in (a, b) , \\ 0 , & x \notin (a, b) . \end{cases} \tag{1.29}$$

It is noted that $\varphi \in C^\infty(\mathbb{R})$ has compact support $[a, b]$. At the points a and b, the function φ and with its derivatives of any order are zero. Consequently, $\varphi \in \mathcal{D}(\mathbb{R})$. The graph of the function is shown in Figure 1.1.

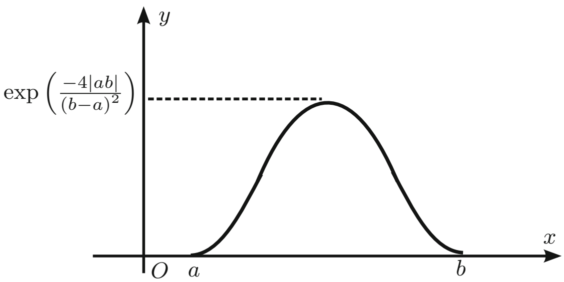

Figure 1.1

Also, the function $\varphi : \mathbb{R}^n \to \mathbb{R}$, where

$$\varphi(x_1, \dots, x_n) = \begin{cases} \prod_{i=1}^{n} \exp\left(-\frac{|a_i b_i|}{(x_i - a_i)(b_i - x_i)}\right) , & x_i \in (a_i, b_i) , \\ 0 , & x_i \notin (a_i, b_i) , \end{cases} \tag{1.30}$$

is a function of the space $\mathcal{D}(\mathbb{R}^n)$, with the compact support $\Omega_n = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$.

Example 1.5 Let $(\varphi_n)_{n \geq 1} \subset \mathcal{D}(\mathbb{R})$ be a sequence of functions

$$\varphi_n(x) = \frac{1}{n} \varphi_a(x) = \begin{cases} \frac{1}{n} \exp\left(-\frac{a^2}{a^2 - x^2}\right) , & |x| < a, a > 0 , \\ 0 , & |x| \geq a, a > 0 . \end{cases} \tag{1.31}$$

We have $\varphi_n \xrightarrow{\mathcal{D}(\mathbb{R})} 0$, that is, the sequence $(\varphi_n)_{n \geq 1} \subset \mathcal{D}(\mathbb{R})$ converges towards $\varphi = 0 \in \mathcal{D}(\mathbb{R})$ in the space $\mathcal{D}(\mathbb{R})$, because $\forall n \in \mathbb{N}, \text{supp}(\varphi_n) \subset \text{supp}(\varphi_a) =$ compact and $(\mathrm{d}^\alpha / \mathrm{d}x^\alpha)\varphi_n(x) \xrightarrow{u} 0, \forall \alpha \in \mathbb{N}_0, |x| \leq a$.

Definition 1.17 We say that the function $\psi : \mathbb{R}^n \to \Gamma$ is a multiplier for the space $\mathcal{D}$ if $\forall \varphi \in \mathcal{D}$ the mapping $\varphi \to \psi\varphi$ is continuous from $\mathcal{D}$ in $\mathcal{D}$.

Hence, if ψ is a multiplier for space $\mathcal{D}$, then $\psi\varphi \in \mathcal{D}, \forall \varphi \in \mathcal{D}$ and $\varphi_i \xrightarrow{\mathcal{D}} \varphi$ implies $\psi\varphi_i \xrightarrow{\mathcal{D}} \psi\varphi$.

We can easily check that any function $\psi \in C^\infty(\mathbb{R}^n)$ is a multiplier for space $\mathcal{D}$.

Indeed, since $\psi \in C^\infty(\mathbb{R}^n)$ and $\varphi \in C^\infty(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we apply formula (1.18) and have

$$\mathrm{D}^\alpha(\psi\varphi) \in \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \mathrm{D}^\beta \psi \mathrm{D}^\gamma \varphi, \mathrm{D}^\alpha = \mathrm{D}_1^{\alpha_1} \dots \mathrm{D}_n^{\alpha_n}, \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n , \tag{1.32}$$

from which it results that $\psi\varphi \in C^\infty(\mathbb{R}^n)$.

On the other hand, we have $\text{supp}(\psi\varphi) \subset \text{supp}(\psi) \cap \text{supp}(\varphi) \subset \text{supp}(\varphi) = \Omega =$ compact.

Next, we show that $\varphi_i \xrightarrow{\mathcal{D}} \varphi$ implies $\psi\varphi_i \xrightarrow{\mathcal{D}} \psi\varphi$. From the expression of the derivative $\mathrm{D}^\alpha(\psi\varphi)$ it results

$$|\mathrm{D}^\alpha \psi(\varphi_i - \varphi)| \leq \sum_{\|\gamma\| \leq \|\alpha\|} A_\gamma |\mathrm{D}^\gamma(\varphi_i - \varphi)| , \quad A_\gamma > 0 \text{ constants} .$$

Since $\mathrm{D}^\alpha(\varphi_i - \varphi) \xrightarrow{\mathcal{D}} 0$, we obtain $|\mathrm{D}^\alpha \psi(\varphi_i - \varphi)| \xrightarrow{\mathcal{D}} 0$, hence $\psi\varphi_i \xrightarrow{\mathcal{D}} \psi\varphi$.

Theorem 1.1 The partition of unity If $\varphi \in \mathcal{D}$ and U_i, $i = 1, 2, \dots, p$, are open and bounded sets, which form a finite covering of the support function φ, then there exist the functions $e_i \in \mathcal{D}$, $i = 1, 2, \dots, p$, with the properties:

1. $e_i(x) \in [0,1], \operatorname{supp}(e_i) \subset U_i$;
2. $\sum_{i=1}^{p} e_i(x) = 1, x \in \operatorname{supp}(\varphi)$;
3. $\varphi(x) = \sum_{i=1}^{p} e_i(x)\varphi(x)$.

We note that the partition theorem is frequently used to demonstrate the local properties of distributions, as well as the operations with them.

1.2.2.5 The Space $\mathcal{S}$ (the Space Functions which Decrease Rapidly)

Definition 1.18 We call the test function space $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ the set of functions $\varphi : \mathbb{R}^n \to \Gamma$, infinitely differentiable, which for $\|x\| \to \infty$ approach zero together with all their derivatives of any order, faster than any power of $\|x\|^{-1}$.

If $\varphi \in \mathcal{S}$, then $\forall k \in \mathbb{N}$ and $\forall \beta \in \mathbb{N}_0^n$ we have

$$\lim_{\|x\|\to\infty} \|x\|^k \mathrm{D}^\beta \varphi = 0 \, .$$

This means that $\forall \varphi \in \mathcal{S}$, we have $\varphi \in C^\infty(\mathbb{R}^n)$ and $\forall \alpha, \beta \in \mathbb{N}_0^n$, $\lim_{\|x\|\to\infty} |x^\alpha \mathrm{D}^\beta \varphi| = 0$, that is, $|x^\alpha \mathrm{D}^\beta \varphi| < C_{\alpha,\beta}$, where $C_{\alpha,\beta}$ are constants.

Example 1.6 An example of a function in $\mathcal{S}$ is $\varphi(x) = \exp(-a\|x\|^2)$, $a > 0$, $x \in \mathbb{R}^n$. On the other hand, the function $\varphi(x) = \exp(-x)$, $x \in \mathbb{R}$, does not belong to the space $\mathcal{S}(\mathbb{R})$, since $\lim_{\|x\|\to-\infty} |x^\alpha \varphi^{(n)}(x)| = \lim_{x\to-\infty} |x|^\alpha \exp(-x) = \infty, \forall \alpha \in \mathbb{N}$, although $\lim_{\|x\|\to+\infty} |x^\alpha \varphi^{(n)}(x)| = \lim_{x\to+\infty} |x|^\alpha \exp(-x) = 0, \forall \alpha \in \mathbb{N}_0$.

Also, the functions $\varphi_1(x) = \exp(x), \varphi_2(x) = \exp(-|x|)$, $x \in \mathbb{R}$ do not belong to the space $\mathcal{S}(\mathbb{R})$ because the function $\varphi_1(x)$ does not tend to zero when $x \to \infty$, and the function $\varphi_2(x)$ is not differentiable at the origin.

Obviously, the space $\mathcal{S}$ is a vector space over Γ, having as null element $\varphi = 0, \forall x \in \mathbb{R}^n$. Between the spaces $\mathcal{D}, \mathcal{S}, \mathcal{E}$ there exist the relations $\mathcal{D} \subset \mathcal{S} \subset \mathcal{E}$.

Definition 1.19 Let $\varphi \in \mathcal{S}$ and consider the sequence $(\varphi_i)_{i\geq 1} \subset \mathcal{S}$. We say that the sequence of functions $(\varphi_i)_{i\geq 1}$ converges towards φ and write $\varphi_i \overset{\mathcal{S}}{\to} \varphi$ if

$$\forall \alpha, \beta \in \mathbb{N}_0^n, x^\beta \mathrm{D}^\alpha \varphi_i \overset{u}{\to} x^\beta \mathrm{D}^\alpha \varphi \, , \quad x \in \mathbb{R}^n \, . \tag{1.33}$$

Consequently, if $\varphi_i \overset{S}{\to} \varphi$, then $\forall \alpha, \beta \in \mathbb{N}_0^n$ on any compact from $\mathbb{R}^n$ we have $x^\beta \mathrm{D}^\alpha \varphi_i \overset{u}{\to} x^\beta \mathrm{D}^\alpha \varphi$.

Comparing the convergence of the spaces $\mathcal{D}$ and $\mathcal{S}$, $\mathcal{D} \subset \mathcal{S}$, we can state:

Proposition 1.6 The convergence in space $\mathcal{D}$ is stronger than the convergence in space $\mathcal{S}$.

Indeed, if $\varphi_i \overset{\mathcal{D}}{\longrightarrow} \varphi$, then there is $\mathcal{D}(\Omega) \subset \mathcal{D}$ so that $\varphi_i \xrightarrow{\mathcal{D}(\Omega)} \varphi$, hence $x^\beta \mathrm{D}^\alpha \varphi_i$ converges uniformly towards $x^\beta \mathrm{D}^\alpha \varphi$ on any compact from $\mathbb{R}^n$, that is, $\varphi_i \overset{\mathcal{S}}{\to} \varphi$.

Proposition 1.7 The space $\mathcal{D}$ is dense in $\mathcal{S}$.

This means that $\forall\varphi \in \mathcal{S}$ there is $(\varphi_i)_{i\geq 1} \subset \mathcal{D}$ such that $\varphi_i \xrightarrow{\mathcal{S}} \varphi$.

Also, we can prove that the space $\mathcal{D}$ is dense in $\mathcal{E}$.

Regarding the multipliers of the space $\mathcal{S}$, we note that not every infinitely differentiable function is a multiplier.

Thus, the function $a(x) = \exp(\|x\|^2)$ belongs to the class $C^\infty(\mathbb{R}^n)$, but it is not a multiplier of the space $\mathcal{S}$, because considering $\varphi(x) = \exp(-\|x\|^2) \in \mathcal{S}$, we then have $a(x)\varphi(x) \equiv 1 \notin \mathcal{S}$.

We note O_M the functions of class $C^\infty(\mathbb{R}^n)$ such that the function and all its derivatives do not increase at infinity faster than a polynomial does, hence if $\psi \in O_M$, then we have

$$\forall\alpha \in \mathbb{N}_0^n, |\mathrm{D}^\alpha\psi| \leq c_\alpha(1 + \|x\|)^{m_\alpha} , \tag{1.34}$$

where $c_\alpha > 0$, $m_\alpha \geq 0$ are constants.

It follows that O_M is the space of multipliers for $\mathcal{S}$, because if $\psi \in O_M$ and $\forall\varphi \in \mathcal{S}$, then $\psi\varphi \in \mathcal{S}$ and $\varphi_i \xrightarrow{\mathcal{S}} \varphi$ involve $\psi\varphi_i \xrightarrow{\mathcal{S}} \psi\varphi$.

Thus, the functions $f_1(x) = \cos x$, $f_2(x) = \sin x$, $P(x)$ (polynomial in x), $x \in \mathbb{R}$, are multipliers for the space $S(\mathbb{R})$.

Consequently, if $\varphi \in \mathcal{S}$ then $\forall\alpha, \beta \in \mathbb{N}_0^n$, $x^\beta\mathrm{D}^\alpha\varphi \in \mathcal{S}$ is bounded and integrable on $\mathbb{R}^n$, hence $\mathcal{S} \subset L^p$, $p \geq 1$.

The spaces of functions with convergence $\mathcal{D}^m(\Omega), \mathcal{D}(\Omega), \mathcal{D}, \mathcal{E}$ and $\mathcal{S}$ will be called test function spaces, and the functions of these spaces, test functions.

Let Φ be a test function space, so $\Phi \in \{\mathcal{D}^m(\Omega), \mathcal{D}(\Omega), \mathcal{D}, \mathcal{E}, \mathcal{S}\}$.

We note that the function $h(x) = \mathrm{e}^x$, $x \in \mathbb{R}$ is not a multiplier of the space $\mathcal{S}(\mathbb{R})$, because it increases to infinity faster than a polynomial.

1.2.3 Spaces of Distributions

The concept by which one introduces the notion of distribution is the linear functional one. This method, used by Schwartz, has been proved useful, with wide applications in various fields of mathematics, mechanics, physics and technology.

Let $(E, \Gamma), (Y, \Gamma)$ be two vector spaces over the same scalar body Γ and let $X \subset E$ be a subspace of (E, Γ). We shall call the mapping $T : X \to Y$ operator defined on X with values in Y. The value of the operator T at the point $x \in X$ will be denoted by $(T, x) = T(x) = y \in Y$.

Definition 1.20 The operator $T : X \to Y$ is called linear if and only if

$$T(\alpha_1x_1 + \alpha_2x_2) = \alpha_1T(x_1) + \alpha_2T(x_2), \forall\alpha_1, \alpha_2 \in \Gamma, \forall x_1, x_2 \in X . \tag{1.35}$$

Thus, if we denote $E = C^n(\Omega)$ and $Y = C^0(\Omega)$, $\Omega \subset \mathbb{R}$, then the application $T : E \to Y$ defined by

$$(T, f) = a_0\mathrm{D}^n f + a_1\mathrm{D}^{n-1} f + \cdots + a_{n-1}\mathrm{D} f + a_n f , \tag{1.36}$$

where $f(x) \in E, \mathrm{D}^k = \mathrm{d}^k/\mathrm{d}x^k, a_k(x) \in C^0(\Omega), k = 0, 1, 2, \ldots, n$ is a linear operator on E.

The operator (1.36) expressed by means of derivatives D^j is called linear differential operator with variable coefficients or polynomial differential operator and we also note $P(\mathrm{D})$.

The operator $T : C^0[a, b] \to C^1[a, b]$ defined by

$$(T, f) = \int_a^x f(t)\mathrm{d}t \,, \quad x \in [a, b] \,, \tag{1.37}$$

is an integral operator. It is shown that it is an integral operator.

A particular class of operators is formed by functionals. Thus, if the domain Y in which the linear operator T takes values is Γ, $Y = \Gamma$, then the operator

$$T : X \subset E \to \Gamma \tag{1.38}$$

will be called functional.

The functional T will be called real or complex as its value (T, x) at the point $x \in X$ is a real or complex number.

We say that the functional (1.38) is linear if it satisfies the condition of linearity of an operator (1.35).

Definition 1.21 A continuous linear functional defined on a space of test functions $\Phi \in \{\mathcal{D}^m(\Omega), \mathcal{D}(\Omega), \mathcal{D}, \mathcal{E}, \mathcal{S}\}$ is called distribution.

This definition involves the fulfillment of the following conditions:

1. To any function $\varphi \in \Phi$ we associate according to some rule f, a complex number $(f, \varphi) \in \Gamma$;
2. $\forall \lambda_1, \lambda_2 \in \Gamma, \forall \varphi_1, \varphi_2 \in \Phi, (f, \lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1(f, \varphi_1) + \lambda_2(f, \varphi_2)$;
3. If $(\varphi_i)_{i \geq 1} \in \Phi, \varphi \in \Phi$ and $\varphi_i \xrightarrow{\Phi} \varphi$, then $\lim_i (f, \varphi_i) = (f, \varphi)$.

The first condition expresses the fact that it is a functional, the second condition corresponds to the linearity of the functional, whereas the third condition expresses its continuity.

The set of distributions defined on Φ is denoted by Φ' and can be organized as a vector space over the field of scalars Γ.

For this purpose, we define the sum of two distributions and the product of a distribution with a scalar as follows:

$$\forall f \,,\, g \in \Phi' \,, \quad \forall \varphi \in \Phi \,,\, (f + g, \varphi) = (f, \varphi) + (g, \varphi) \,, \tag{1.39}$$

$$\forall \alpha \in \Gamma \,,\, \forall \varphi \in \Phi \,, \quad \forall f \in \Phi' \,,\, (\alpha f, \varphi) = \alpha(f, \varphi) \,. \tag{1.40}$$

It can be verified immediately that the functional $\alpha f + \beta g$ is linear and continuous, hence it is a distribution from Φ'.