Springer Proceedings in Mathematics & Statistics

Andreas H. Hamel
Frank Heyde
Andreas Löhne
Birgit Rudloff
Carola Schrage *Editors*

Set Optimization and Applications - The State of the Art

From Set Relations to Set-Valued Risk Measures

Springer

Springer Proceedings in Mathematics & Statistics

Volume 151

Springer Proceedings in Mathematics & Statistics

This book series features volumes composed of selected contributions from workshops and conferences in all areas of current research in mathematics and statistics, including operation research and optimization. In addition to an overall evaluation of the interest, scientific quality, and timeliness of each proposal at the hands of the publisher, individual contributions are all refereed to the high quality standards of leading journals in the field. Thus, this series provides the research community with well-edited, authoritative reports on developments in the most exciting areas of mathematical and statistical research today.

More information about this series at http://www.springer.com/series/10533

Andreas H. Hamel · Frank Heyde
Andreas Löhne · Birgit Rudloff
Carola Schrage
Editors

Set Optimization and Applications - The State of the Art

From Set Relations to Set-Valued Risk Measures

Springer

Editors
Andreas H. Hamel
Free University Bozen-Bolzano
Bolzano
Italy

Frank Heyde
University of Technology Bergakademie
Freiberg
Germany

Andreas Löhne
Friedrich-Schiller-University Jena
Jena
Germany

Birgit Rudloff
Vienna University of Economics
and Business
Vienna
Austria

Carola Schrage
Free University Bozen-Bolzano
Bolzano
Italy

ISSN 2194-1009 ISSN 2194-1017 (electronic)
Springer Proceedings in Mathematics & Statistics
ISBN 978-3-662-48668-9 ISBN 978-3-662-48670-2 (eBook)
DOI 10.1007/978-3-662-48670-2

Library of Congress Control Number: 2015952992

Mathematics Subject Classification (2010): 49-06, 90-06, 91Gxx, 06-xx, 49J53, 49N15, 49M29

Springer Heidelberg New York Dordrecht London

Printed on acid-free paper

Springer-Verlag GmbH Berlin Heidelberg is part of Springer Science+Business Media
(www.springer.com)

Foreword

I finished an Oxford D.Phil. on multi-criteria optimisation in 1974 and so have been observing and participating in the field for over four decades.

My research interests have included efficiency and tangency, lattice structure, set-valued mappings, convex analysis and convex programming. So it is a delight to see these old friends given a new life and a new purpose in this volume.

Let me turn to some remarks on the specifics of this valuable collection. Early in my research career, I discovered the power of set-valued analysis, whether for more efficient proofs of classical results such as the open mapping theorem, or for new approaches to current research as with the heading results of this volume.

Set Optimization connotes the study of an optimization problem with a set-valued objective. Why should one do this? What are the prospects?

1. A theory for set optimization problems can only be developed if it is accompanied by a convex analysis for set-valued functions; concepts like subdifferentials, Legendre–Fenchel conjugates, dual optimization problems are just too important and too relevant for all kinds of applications to be ignored; however, a "canonical convex analysis" for vector-valued functions did not exist so far (not to speak of set-valued ones), for example, there are many different "conjugates" for vector-valued functions which work under different—more or less restrictive—assumptions; recent developments summarized in this volume may fill this gap by means of set-valued approaches to duality, leading in particular to conjugation, for vector- as well as set-valued functions;
2. the concepts "infimum" and "supremum" are not relevant in the overwhelming part of the existing literature on vector- and set-valued optimization problems; the reason: infima/suprema with respect to vector orders may not exist for many sets (e.g., if the ordering cone does not generate a lattice) or do not provide useful solution concepts whence non-dominated (minimal or maximal) points are usually looked for in multi-criteria decision-making; set relations (order relations on the power set of an ordered set) not only open the way for a revival of the infimum/supremum but also trigger investigation of new solution concepts in vector/set optimization which show a split into infimum/supremum

attainment on the one side and minimality/maximality on the other side (a totally new idea);

3. this new solution concept (Heyde, Löhne 2011, see already Hamel 2004 and also Löhne 2011) provides "a fresh look" (title of Heyde, Löhne 2011) to multi-criteria decision-making problems: the infimum/supremum attainment property provides the decision maker with overall information about what can be potentially achieved whereas the minimal/maximal solutions provide non-dominated outcomes; in this way it is not necessary to look at all "efficient" points but only at enough of them to make a well-informed decision; this helps to answer the question of what is actually understood by a solution of a vector-valued optimization problem. The latter question is rarely answered in a satisfactory way in the many papers on such problems (one sees discussion of one or all "weak" or "efficient" or "properly efficient" minimizers/maximizers, one or all 'minimal/maximal' solutions or non-dominated image points; an evenly distributed subset of minimal/maximal image points, etc.);
4. recent developments in mathematical finance produced sets of superhedging portfolios and, more generally, set-valued risk measures which turned out to be appropriate tools for risk evaluation in markets with "frictions" (bid-ask price spreads, transaction costs etc.); dual representation results (such as Kabanov's 1999 superhedging theorem) have been identified as special cases of general set-valued duality theorems; optimization problems involving set-valued risk measures (optimal risk allocation, risk minimization, and hedging under constraints) are highly desirable subjects of study; the overall question—in statistics, math finance, as well as the mathematics of insurance—how to deal with multivariate risks—is a topical one which may profit from set-valued approaches;
5. the role of scalarization procedures has been clarified; as it is immediately clear and already known for decades, a convex vector- or set-valued function has an equivalent representation by a family of extended real-valued functions (take the collection of support functions of the images, for example). The set-valued approach can be seen as just another (efficient and elegant) calculus for such families; compared to the scalar case additional dual variables appear along with a new dependence of the classical dual variable ("Lagrange multipliers") on this new one which captures the order structure in the image space. This helps answer the question of what to choose as dual variables in vector/set optimization problems which is usually not answered clearly by the many papers on such problems (linear operators, some special type of nonlinear functions, etc. are possible answers); in mathematical finance applications these new dual variables could be interpreted as "consistent price processes," exactly as had been obtained earlier in finance papers (Kabanov 1999, Schachermayer 2004, among others); Set-valued duality in terms of scalarizations also paves the way to efficient algorithms for set-/vector optimization problems–along with a new 'geometric duality' which turned out to be extremely useful in particular for linear and polyhedral vector optimization problems; for such problems,

algorithms could be constructed which produce a solutions in the set-valued sense (Löhne, Schrage 2013).

Finally, consideration of a set-valued function as a family of extended real-valued functions provides a link between set optimization theory and generalized convexity; this is an area which needs further exploration. An important application in economics might be utility maximization for incomplete preferences (that is, non-total reflexive and transitive relations); for such relations, multi-utility representations are available (due to Aumann, Evren, Kannai, Marinacci, Ok, and a few others). That is, families of scalar functions which represent the preference; however, the problem of maximizing (expected) utility for such preferences has not yet been addressed. The current set-valued approach could well provide the missing tools.

Newcastle NSW
July 2015

Jonathan Borwein

Preface

In 1998, a special issue of the journal Mathematical Methods of Operations Research was published, edited by Guang-Ya Chen and Johannes Jahn. It was devoted solely to optimization problems with set-valued objective functions.

Since then, major breakthroughs have been made including new "set relations," new solution concepts for set optimization problems and a new framework for a set-valued convex analysis.

The area has been pushed further by the discovery of its relevance for financial mathematics: risk evaluation in markets with "frictions" such as transaction costs or illiquidity effects is best done using set-valued functions. It turns out that results such as the superhedging theorem of Y. Kabanov from 1999 are essentially set-valued duality results, and the dual variables in this superhedging theorem are precisely what the recent theory expects them to be.

Finally, development of algorithms was initiated that can deal with the sometimes scaring complexity of a set-valued objective function and can deliver results which are useful in applications. As a side effect, the theory of vector optimization is not what it used to be: set-valued approaches produced new insides, extensions and in many cases provide methods for repairing unsatisfactory "vector results." Examples of the latter include duality for linear vector optimization problems and Benson's now famous algorithm. The latter method was designed for (linear) vector optimization problems, but appropriate extensions allow the computation of infima and even solutions of set optimization problems.

All of this gave rise to a need to summarize the development. This is what motivated the compilation of this volume. The reader may find both surveys with extended bibliographies and original research articles, which provide evidence for the claims above, as well as open questions. The area of "set optimization" is under rapid development, and it is the opinion of the editors that it is becoming a field in its own right: new tools for example from lattice theory (residuation) and new algebraic structures (conlinear spaces of sets) enter the picture. These even shed new light on scalar optimization theory (the objective function is extended real-valued).

Looking at a bigger picture, there are two common denominators in many of the relevant developments in optimization theory. The first is the departure from linear structures in particular on the “image” side. Conlinear spaces of sets are not linear since there is no inverse addition, a feature that is already shared by the extended reals. Modules over L^0 turn out to be fundamental for capturing features of conditional risk measurement in a dynamic framework. We are, therefore, happy to include a contribution from this new field. The second is the utilization of order-complete lattices which leads to a comeback of the notions “infimum” and “supremum”—in particular in vector optimization where the infimum with respect to a vector order is not very useful or does not even exist. This “complete lattice approach” to set optimization complements the “set relation approach” initiated by D. Kuroiwa in the 1990s.

The editors joined this development at an early stage: two of us (Hamel, Löhne) started working on “set relations” in 2001, and were soon followed by the others. A workshop at Humboldt University Berlin, organized by A. Hamel and R. Henrion in 2003, witnessed the first talk about set-valued risk measures from a set optimization perspective, and two theses were completed in 2005 (Hamel’s habilitation, Löhne’s Ph.D.) at Martin Luther University Halle-Wittenberg which paved the way for the “lattice approach” to set optimization.

A regular conference is now devoted to set optimization and finance, see www.set-optimization.org. The first edition took place in Lutherstadt-Wittenberg, Germany, 2012 the second one in Brunico-Bruneck, Italy, 2014. The third one is planned for 2016.

We thank all contributors of this volume for their effort and their patience. We thank Springer for publishing this volume. We thank all referees who decisively contributed to the scientific quality of the articles. Last but not least, we thank Prof. J. Jahn because he not only contributed to the editorial work of this volume, but already in 2003[1] shared our vision of a new area in optimization emerging, and also gave us the opportunity to publish and to present our results whenever possible and appropriate.

Brunico	Andreas H. Hamel
Freiberg	Frank Heyde
Halle	Andreas Löhne
Princeton	Birgit Rudloff
Bolzano	Carola Schrage
December 2014	

[1] J. Jahn, “Grundlagen der Mengenoptimierung” (in German). Multi-Criteria-und Fuzzy-Systeme in Theorie und Praxis. Deutscher Universitätsverlag, 2003, 37–71.

Contents

Part I
Surveys

A Comparison of Techniques for Dynamic Multivariate Risk Measures

Zachary Feinstein and Birgit Rudloff

Abstract This paper contains an overview of results for dynamic multivariate risk measures. We provide the main results of four different approaches. We will prove under which assumptions results within these approaches coincide, and how properties like primal and dual representation and time consistency in the different approaches compare to each other.

Keywords Dynamic risk measures · Transaction costs · Set-valued risk measures · Multivariate risk

Mathematics Subject Classification (2010): 91B30 · 46N10 · 26E25

1 Introduction

The concept of coherent risk measures was introduced in an axiomatic way in [3, 4] to find the minimal capital required to cover the risk of a portfolio. The notion was relaxed by introducing convex risk measures in [24, 25]. In these papers the risk was measured only at time zero, in a frictionless market, for univariate claims, and with only a single eligible asset that can be used for the capital requirements and serves as the numéraire. We call this the static scalar framework. In this paper these four assumptions will be removed and different methods compared.

Research supported by NSF award DMS-1007938.

Z. Feinstein
Department of Electrical and Systems Engineering,
Washington University in St. Louis, St. Louis, MO 63130, USA
e-mail: zfeinstein@ese.wustl.edu

B. Rudloff (✉)
Institute for Statistics and Mathematics,
Vienna University of Economics and Business, 1020 Vienna, Austria
brudloff@wu.ac.at

A.H. Hamel et al. (eds.), *Set Optimization and Applications - The State of the Art*,
Springer Proceedings in Mathematics & Statistics 151,
DOI 10.1007/978-3-662-48670-2_1

The static assumptions were relaxed by considering dynamic risk measures, where the risk evaluation of a portfolio is updated as time progresses and new information become available. In the dynamic framework time consistency plays an important role and has been studied for example in [8, 14, 17, 45, 48].

Eliminating the assumption that the financial markets are frictionless required a new framework. Since the 'value' of a portfolio is not uniquely determined anymore when bid and ask prices or market illiquidity exist, it is natural to consider portfolios as vectors in physical units instead, i.e. a portfolio is specified by the number of each of the asset which is held as opposed to their value. But even in the absence of transaction costs multivariate claims might be of interest, e.g. when assets are denoted in different currencies with fluctuating exchange rates, or different business lines with no direct exchange or different regularity rules are considered, see [13]. In contrast to frictionless univariate models also the choice of the numéraire assets matters, which leads to different approaches: pick a numéraire and allow capital requirements to be in this numéraire, which allows a risk manager to work with scalar risk measures again (see e.g. [5, 18, 26, 39, 49]); or use the more general numéraire-free approach and allow risk compensation to be made in a basket of assets which leads to risk measures that are set-valued. This approach was first studied in Jouini et al. [36] in the coherent case. Several extensions have been made. In this paper we will introduce four approaches to deal with dynamic multivariate risk measures, and compare and relate them by giving conditions under which the results obtained in each approach coincide. The four approaches we discuss are

1. a set-optimization approach;
2. a measurable selector approach;
3. an approach utilizing set-valued portfolios; and
4. a family of multiple asset scalar risk measures.

The first three approaches correspond to the numéraire-free framework, whereas the last approach includes scalar risk measures where a numéraire asset is chosen.

In [28–31] the results of [36] were extended to the convex case and a stochastic market model. The extension of the dual representation results were made possible by an application of convex analysis for set-valued functions (set-optimization), see Hamel [27]. The dynamic case and time consistency was studied in [21, 23]. We will call this approach the set-optimization approach. The values of risk measures and its minimal elements in this framework have been studied and computed in [22, 32, 33, 41, 42] via Benson's algorithm for linear vector optimization (see e.g. [40]) in the coherent and polyhedral convex case, respectively via an approximation algorithm in the convex case, see [22, 42].

Tahar and Lépinette [52] extended the results of [36] for coherent risk measures to the dynamic case. We will call this the measurable selector approach as it considers the value of a risk measures as a random set, and then provides a primal and dual representation for the measurable selectors in that set. Time consistency properties were also introduced and some equivalent characterizations discussed.

Most recently, in [13], set-valued coherent risk measures were considered as functions from random sets into the upper sets. The transaction costs model, and other financial considerations like trading constraints, or illiquidity, are then embedded into the construction of "set-valued portfolios". A subclass of risk measures in this framework can be constructed using a vector of scalar risk measures and [13] gives upper and lower bounds as well as dual representations for this subclass. We will present here the dynamic extension of this approach. Time consistency properties have not yet been studied within this framework. However, by comparing and relating the different approaches we will see that a larger subclass can be obtained by using the set-valued risk measures of the set-optimization approach, which provides already a link to dual representations and time consistency properties for this larger subclass.

The fourth approach is to consider a family of dynamic scalar risk measures to evaluate the risk of a multivariate claim. This approach has not been studied so far in the dynamic case. In the special case of frictionless markets, the family of scalar risk measures coincides with scalar risk measures using multiple eligible assets as discussed in [5, 18, 26, 39, 49]. Also the scalar static risk measure of multivariate claims with a single eligible asset studied in [11]; the scalar liquidity adjusted risk measures in market with frictions as studied in [53]; and the scalar superhedging price in markets with transaction costs, see [7, 10, 35, 41, 44, 46, 47], are special cases of this approach. Thus, the family of dynamic scalar risk measures for portfolio vectors generalizes these special cases in a unified way to allow for frictions, multiple eligible assets, and multivariate portfolios in a dynamic framework. The connection to the set-optimization approach allows to utilize the dual representation and time consistency results deduced there.

Other papers in the context of set-valued risk measures are [51], where an extension of the tail conditional expectation to the set-valued framework of [36] was presented and a numerical approximation for calculation was given; and [12], where set-valued risk measures in a more abstract setting were studied and a consistent structure for scalar-valued, vector-valued, and set-valued risk measures (but for constant solvency cones) was created. Furthermore, in [12] distribution based risk measures were extended to the set-valued framework via depth-trimmed regions. More recently, vector-valued risk measures were studied in [6].

Section 2 introduces the four approaches mentioned above. In Sect. 3 these four approaches are compared by showing how the set-optimization approach corresponds to each of the other three. For each comparison, assumptions are given under which there is a one-to-one relationship between the approaches. These relations allow generalizations in most of the different approaches that go beyond the results obtained so far.

2 Dynamic Risk Measures

Consider a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P}\right)$ satisfying the usual conditions with $\mathcal{F}_0$ being the completed trivial sigma algebra and $\mathcal{F}_T = \mathcal{F}$. Let $|\cdot|$ be an arbitrary norm in $\mathbb{R}^d$. Denote $L_t^p := L^p(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$ for $p \in [0, +\infty]$ (with $L^p := L_T^p$). If $p = 0$, L_t^0 is the linear space of the equivalence classes of $\mathcal{F}_t$-measurable functions $X : \Omega \to \mathbb{R}^d$. For $p > 0$, L_t^p denotes the linear space of $\mathcal{F}_t$-measurable functions $X : \Omega \to \mathbb{R}^d$ such that $\|X\|_p = \left(\int_\Omega |X(\omega)|^p d\mathbb{P}\right)^{1/p} < +\infty$ for $p \in (0, +\infty)$, and $\|X\|_\infty = \text{ess sup}_{\omega\in\Omega}|X(\omega)| < +\infty$ for $p = +\infty$. For $p \in [1, +\infty]$ we will consider the dual pair $\left(L_t^p, L_t^q\right)$, where $\frac{1}{p} + \frac{1}{q} = 1$ (with $q = +\infty$ when $p = 1$ and $q = 1$ when $p = +\infty$), and endow it with the norm topology, respectively the weak* topology (that is the $\sigma\left(L_t^\infty, L_t^1\right)$-topology on L_t^∞) in the case $p = +\infty$ unless otherwise noted.

We write $L_{t,+}^p = \left\{X \in L_t^p : X \in \mathbb{R}_+^d \ \mathbb{P}\text{-a.s.}\right\}$ for the closed convex cone of $\mathbb{R}^d$-valued $\mathcal{F}_t$-measurable random vectors with non-negative components. Similarly define $L_+^p := L_{T,+}^p$. We denote by $L_t^p(D_t)$ those random vectors in L_t^p that take $\mathbb{P}$-a.s. values in D_t. Let $1_D : \Omega \to \{0, 1\}$ be the indicator function of $D \in \mathcal{F}$ defined by $1_D(\omega) = 1$ if $\omega \in D$ and 0 otherwise. Throughout we will consider the summation of sets by Minkowski addition. To distinguish the spaces of random vectors from those of random variables, we will write $L_t^p(\mathbb{R}) := L^p(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$ for the linear space of the equivalence classes of p integrable $\mathcal{F}_t$-measurable random variables $X : \Omega \to \mathbb{R}$. Note that an element $X \in L_t^p$ has components $X_1, \ldots, X_d$ in $L_t^p(\mathbb{R})$.

(In-)equalities between random vectors are always understood componentwise in the $\mathbb{P}$-a.s. sense. The multiplication between a random variable $\lambda \in L_t^\infty(\mathbb{R})$ and a set of random vectors $D \subseteq L^p$ is understood in the elementwise sense, i.e. $\lambda D = \{\lambda Y : Y \in D\} \subseteq L^p$ with $(\lambda Y)(\omega) = \lambda(\omega)Y(\omega)$. The multiplication and division between (random) vectors is understood in the componentwise sense, i.e. $xy := (x_1 y_1, \ldots, x_d y_d)^\mathsf{T}$ and $x/y := (x_1/y_1, \ldots, x_d/y_d)^\mathsf{T}$ for $x, y \in \mathbb{R}^d$ ($x, y \in L_t^p$) and with $y_i \neq 0$ (almost surely) for every index $i \in \{1, \ldots, d\}$ for division.

As in [37] and discussed in [38, 50], the portfolios in this paper are in "physical units" of an asset rather than the value in a fixed numéraire, except where otherwise mentioned. That is, for a portfolio $X \in L_t^p$, the values of X_i (for $1 \leq i \leq d$) are the number of units of asset i in the portfolio at time t.

Let $\tilde{M}_t[\omega]$ denote the set of eligible portfolios, i.e. those portfolios which can be used to compensate for the risk of a portfolio, at time t and state ω. We assume $\tilde{M}_t[\omega]$ is a linear subspace of $\mathbb{R}^d$ for almost every $\omega \in \Omega$. It then follows that $M_t := L_t^p(\tilde{M}_t)$ is a closed (and additionally weak* closed if $p = +\infty$) linear subspace of L_t^p, see Sect. 5.4 and Proposition 5.5.1 in [38]. For example, $\tilde{M}_t[\omega]$ could specify a certain ratio of Euros and Dollars to be used for risk compensations. Another typical example is the case where a subset of assets are used for capital requirements, i.e. $\tilde{M}_t^n[\omega] = \left\{m \in \mathbb{R}^d : \forall i \in \{n+1, \ldots, d\} : m_i = 0\right\}$ and $M_t^n = L_t^p(\tilde{M}_t^n)$. We will denote $M_{t,+} := M_t \cap L_{t,+}^p$ to be the nonnegative elements of M_t. We will assume that $M_{t,+} \neq \{0\}$, i.e. $M_{t,+}$ is nontrivial.

In the first three methods discussed below the risk measures have set-valued images. In the set-optimization approach (Sect. 2.1) and the set-valued portfolio approach (Sect. 2.3) the image space is explicitly given by the upper sets, i.e. $\mathcal{P}\left(M_t; M_{t,+}\right)$ where $\mathcal{P}(\mathcal{Z}; C) := \{D \subseteq \mathcal{Z} : D = D + C\}$ for some vector space $\mathcal{Z}$ and an ordering cone $C \subset \mathcal{Z}$. Additionally, let $\mathcal{G}(\mathcal{Z}; C) := \{D \subseteq \mathcal{Z} : D = \text{cl}\,\text{co}\,(D + C)\} \subseteq \mathcal{P}(\mathcal{Z}; C)$ be the upper closed convex subsets. It seems natural to use upper sets as the values of risk measures since if one portfolio can cover the risk then any larger portfolio should also cover this risk. Alternatively, one could consider the set of "minimal elements" of the risk compensating portfolios. However, in contrast to the upper sets, the set of "minimal elements" is in general not a convex set when convex risk measure are considered.

2.1 Set-Optimization Approach

The set-optimization approach to dynamic risk measures is studied in [21, 23], where set-valued risk measures [29, 31] were extended to the dynamic case. A benefit of this method is that dual representations are obtained by a direct application of the set-valued duality developed in [27], which allowed for the first time to study not only conditional coherent, but also convex set-valued risk measures.

In this setting we consider risk measures that map a portfolio vector into the complete lattice $\mathcal{P}\left(M_t; M_{t,+}\right)$ of upper sets.

Set-valued conditional risk measures have been defined in [21]. Here we give a stronger property for finiteness at zero than in [21] to ease the comparison to the other approaches.

Definition 2.1 A **conditional risk measure** is a mapping $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$ which satisfies:

1. L^p_+-monotonicity: if $Y - X \in L^p_+$ then $R_t(Y) \supseteq R_t(X)$;
2. M_t-translativity: $R_t(X + m) = R_t(X) - m$ for any $X \in L^p$ and $m \in M_t$;
3. finiteness at zero: $R_t(0) \neq \emptyset$ and $R_t(0)[\omega] \neq \tilde{M}_t[\omega]$ for almost every $\omega \in \Omega$, where $R_t(0)[\omega] := \{u(\omega) : u \in R_t(0)\}$.

For finiteness at zero, and elsewhere in later sections, we consider the ω projection of the risk compensating set $R_t(X)$. We point out that $R_t(X)$ is a collection of random vectors and is *not* a random set; therefore $R_t(X)[\omega] := \{u(\omega) : u \in R_t(X)\}$ is the collection of risk covering portfolios at state ω. As $R_t(X)$ is not a random set, it is generally the case that $R_t(X) \neq L^p_t(R_t(X)) := \{u \in M_t : \mathbb{P}(\omega \in \Omega : u(\omega) \in R_t(X)[\omega]) = 1\}$.

Below we consider additional properties for conditional risk measures that have useful financial and mathematical interpretations. Note that the definition for K-compatibility below is more general than the one given in [21], and corresponds to the definition in [32]. A conditional risk measure R_t at time t is

- ***convex (conditionally convex)*** if for all $X, Y \in L^p$ and any $\lambda \in [0, 1]$ (respectively $\lambda \in L^\infty_t(\mathbb{R})$ such that $0 \leq \lambda \leq 1$)

$$R_t(\lambda X + (1 - \lambda)Y) \supseteq \lambda R_t(X) + (1 - \lambda)R_t(Y);$$

- ***positive homogeneous (conditionally positive homogeneous)*** if for all $X \in L^p$ and any $\lambda \in \mathbb{R}_{++}$ (respectively $\lambda \in L^\infty_t(\mathbb{R}_{++})$)

$$R_t(\lambda X) = \lambda R_t(X);$$

- ***coherent (conditionally coherent)*** if it is convex and positive homogeneous (respectively conditionally convex and conditionally positive homogeneous);
- ***normalized*** if $R_t(X) = R_t(X) + R_t(0)$ for every $X \in L^p$;
- ***local*** if for every $D \in \mathcal{F}_t$ and every $X \in L^p$, $1_D R_t(X) = 1_D R_t(1_D X)$;
- K***-compatible*** for some convex cone $K \subseteq L^p$ if $R_t(X) = \bigcup_{k \in K} R_t(X - k)$;
- ***closed*** if the graph of the risk measure

$$\operatorname{graph}(R_t) = \left\{(X, u) \in L^p \times M_t : u \in R_t(X)\right\}$$

  is closed in the product topology (with the weak* topology if $p = +\infty$);
- ***convex upper continuous*** if

$$R_t^{-1}(D) := \left\{X \in L^p : R_t(X) \cap D \neq \emptyset\right\}$$

  is closed (weak* closed if $p = +\infty$) for any closed convex set $D \in \mathcal{G}(M_t; M_{t,-})$.

(Conditional) convexity and coherence for a risk measure define a regulatory framework which promotes diversification. Set-valued normalization is a generalization of the scalar normalization (zero capital needed to compensate the risk of the 0 portfolio). The local property means that the risks at some state (in $\mathcal{F}_t$) only depend on the possible future values of the portfolio reachable from that state. K-compatibility is closely related to a market model; assume for the moment an investor can trade the initial portfolio 0 into any random vector in $-K$ by the terminal time T, then K-compatibility means considering the (minimal) risk of a portfolio when all possible trades are taken into account. The closure is the set-valued version of lower semicontinuity and is necessary for the dual representation to hold. Convex upper continuity is a stronger property than closure and is useful when characterizing or creating multi-portfolio time consistent risk measures, the details will be given below.

A ***dynamic risk measure*** is a sequence $(R_t)_{t=0}^T$ of conditional risk measures. A dynamic risk measure is said to have a certain property if R_t has that property for all times t.

A static risk measure in the sense of [31] is a conditional risk measure at time 0. Note that for static risk measures convexity (positive homogeneity) coincides with conditional convexity (conditional positive homogeneity).

Any conditionally convex risk measure $R_t : L^p \to \mathcal{P}\left(M_t; M_{t,+}\right)$ is local, see Proposition 2.8 in [21].

Definition 2.2 A set $A_t \subseteq L^p$ is a **conditional acceptance set** at time t if it satisfies $A_t + L^p_+ \subseteq A_t$, $M_t \cap A_t \neq \emptyset$, and $\tilde{M}_t[\omega] \cap (\mathbb{R}^d \backslash A_t[\omega]) \neq \emptyset$ for almost every $\omega \in \Omega$, where $A_t[\omega] = \{X(\omega) : X \in A_t\}$.

The acceptance set of a conditional risk measure R_t is given by $A_t = \left\{X \in L^p : 0 \in R_t(X)\right\}$, which is the collection of "risk free" portfolios. For any conditional acceptance set A_t, the function defined by $R_t(X) = \{u \in M_t : X + u \in A_t\}$ is a conditional risk measure. This is the primal representation for conditional risk measures via acceptance sets, see [21]. This relation is one-to-one, i.e. we can consider an (R_t, A_t) pair or equivalently just one of the two. Given a risk measure and acceptance set pair (R_t, A_t) then the following properties hold, see Proposition 2.11 in [21].

- R_t is normalized if and only if $A_t + A_t \cap M_t = A_t$;
- R_t is (conditionally) convex if and only if A_t is (conditionally) convex;
- R_t is (conditionally) positive homogeneous if and only if A_t is a (conditional) cone;
- R_t has a closed graph if and only if A_t is closed.

For the duality results below we will consider $p \in [1, +\infty]$. Let $\mathcal{M}$ denote the set of d-dimensional probability measures absolutely continuous with respect to $\mathbb{P}$, and let $\mathcal{M}^e$ denote the set of d-dimensional probability measures equivalent to $\mathbb{P}$. We will say $\mathbb{Q} = \mathbb{P}|_{\mathcal{F}_t}$ for vector probability measures $\mathbb{Q}$ and some time $t \in [0, T]$ if for every $D \in \mathcal{F}_t$ it follows that $\mathbb{Q}_i(D) = \mathbb{P}(D)$ for all $i = 1, \ldots, d$. Consider $\mathbb{Q} \in \mathcal{M}$. We will use a $\mathbb{P}$-almost sure version of the $\mathbb{Q}$-conditional expectation of $X \in L^p$ given by

$$\mathbb{E}^{\mathbb{Q}}\left[X \middle| \mathcal{F}_t\right] := \mathbb{E}\left[\xi_{t,T}(\mathbb{Q})X \middle| \mathcal{F}_t\right],$$

where $\xi_{r,s}(\mathbb{Q}) = \left(\bar{\xi}_{r,s}(\mathbb{Q}_1), \ldots, \bar{\xi}_{r,s}(\mathbb{Q}_d)\right)^{\mathsf{T}}$ for any times $0 \le r \le s \le T$ with

$$\bar{\xi}_{r,s}(\mathbb{Q}_i)[\omega] := \begin{cases} \frac{\mathbb{E}\left[\frac{d\mathbb{Q}_i}{d\mathbb{P}} \middle| \mathcal{F}_s\right](\omega)}{\mathbb{E}\left[\frac{d\mathbb{Q}_i}{d\mathbb{P}} \middle| \mathcal{F}_r\right](\omega)} & \text{on } \mathbb{E}\left[\frac{d\mathbb{Q}_i}{d\mathbb{P}} \middle| \mathcal{F}_r\right](\omega) > 0 \\ 1 & \text{else} \end{cases}$$

for every $\omega \in \Omega$, see e.g. [15, 21]. For any probability measure $\mathbb{Q}_i \ll \mathbb{P}$ and any times $0 \le r \le s \le t \le T$, it follows that $\frac{d\mathbb{Q}_i}{d\mathbb{P}} = \bar{\xi}_{0,T}(\mathbb{Q}_i)$, $\bar{\xi}_{t,s}(\mathbb{Q}_i) = \bar{\xi}_{t,r}(\mathbb{Q}_i)\bar{\xi}_{r,s}(\mathbb{Q}_i)$, and $\mathbb{E}\left[\bar{\xi}_{r,s}(\mathbb{Q}_i) \middle| \mathcal{F}_r\right] = 1$ almost surely. The halfspace and the conditional "halfspace" in L_t^p with normal direction $w \in L_t^q \backslash \{0\}$ are denoted by

$$G_t(w) := \left\{u \in L_t^p : 0 \le \mathbb{E}\left[w^{\mathsf{T}} u\right]\right\}, \qquad \Gamma_t(w) := \left\{u \in L_t^p : 0 \le w^{\mathsf{T}} u \ \mathbb{P}\text{-a.s.}\right\}.$$

We will define the set of dual variables to be

$$\mathcal{W}_t := \left\{(\mathbb{Q}, w) \in \mathcal{M} \times \left(M_{t,+}^{+} \backslash M_t^{\perp}\right) : w_t^T(\mathbb{Q}, w) \in L_+^q, \mathbb{Q} = \mathbb{P}|_{\mathcal{F}_t}\right\},$$

where for any $0 \le t \le s \le T$

$$w_t^s(\mathbb{Q}, w) = w\xi_{t,s}(\mathbb{Q}),$$

$M_t^{\perp} = \left\{v \in L_t^q : \mathbb{E}\left[v^{\mathsf{T}} u\right] = 0 \ \forall u \in M_t\right\}$ and $C^+ = \left\{v \in L_t^q : \mathbb{E}\left[v^{\mathsf{T}} u\right] \ge 0 \ \forall u \in C\right\}$ denotes the positive dual cone of a cone $C \subseteq L_t^p$.

The set of dual variables $\mathcal{W}_t$ consists of two elements. The first component is a vector probability measure absolutely continuous to the physical measure $\mathbb{P}$ and corresponds to the dual element in the traditional scalar theory. The second component reflects the order relation in the image space as the w's are the collection of possible relative weights between the eligible portfolios. This component is not needed in the scalar case. The coupling condition $w_t^T(\mathbb{Q}, w) \in L_+^q$ guarantees that the probability measure $\mathbb{Q}$ and the ordering vector w are "consistent" in the following sense. If a portfolio X is component-wise ($\mathbb{P}$-)almost surely greater than or equal to another portfolio Y, then the $\mathbb{Q}$-conditional expectation keeps that relationship with respect to the order relation defined by w, that is $w^{\mathsf{T}}\mathbb{E}^{\mathbb{Q}}[X|\mathcal{F}_t] \geq w^{\mathsf{T}}\mathbb{E}^{\mathbb{Q}}[Y|\mathcal{F}_t]$ ($\mathbb{P}$-)almost surely. In the following, we review the duality results from [23]. Note that since we are only considering closed (conditionally) convex risk measures we can restrict the image space to $\mathcal{G}(M_t; M_{t,+}) := \left\{D \subseteq M_t : D = \operatorname{cl}\operatorname{co}\left(D + M_{t,+}\right)\right\}$.

Corollary 2.3 (Corollary 2.4 of [23]) *A conditional risk measure $R_t : L^p \to \mathcal{G}(M_t; M_{t,+})$ is closed and conditionally convex if and only if*

$$R_t(X) = \bigcap_{(\mathbb{Q},w)\in\mathcal{W}_t} \left[-\alpha_t^{\min}(\mathbb{Q}, w) + \left(\mathbb{E}^{\mathbb{Q}}[-X|\mathcal{F}_t] + \Gamma_t(w)\right) \cap M_t\right], \tag{2.1}$$

where $-\alpha_t^{\min}$ is the minimal conditional penalty function given by

$$-\alpha_t^{\min}(\mathbb{Q}, w) = \operatorname{cl} \bigcup_{Z\in A_t} \left(\mathbb{E}^{\mathbb{Q}}[Z|\mathcal{F}_t] + \Gamma_t(w)\right) \cap M_t. \tag{2.2}$$

R_t is additionally conditionally coherent if and only if

$$R_t(X) = \bigcap_{(\mathbb{Q},w)\in\mathcal{W}_t^{\max}} \left(\mathbb{E}^{\mathbb{Q}}[-X|\mathcal{F}_t] + \Gamma_t(w)\right) \cap M_t \tag{2.3}$$

with

$$\mathcal{W}_t^{\max} = \left\{(\mathbb{Q}, w) \in \mathcal{W}_t : w_t^T(\mathbb{Q}, w) \in A_t^+\right\}. \tag{2.4}$$

The more general convex and coherent case reads analogously to Corollary 2.3, just with $\Gamma_t(w)$ replaced by $G_t(w)$ in Eqs. (2.1), (2.2) and (2.3), see Theorem 2.3 in [23]. As shown in [21, 31], the G_t-version of the minimal penalty function $-\alpha_t^{\min}$ is the set-valued (negative) convex conjugate in the sense of [27] and the dual representation is the biconjugate, both with infimum and supremum defined for the image space $\mathcal{G}(M_t; M_{t,+})$.

Remark 2.4 The dual representation given in [36] for the static coherent case (and in [52] for the dynamic case, see Sect. 2.2 below) uses a single dual variable. This set of dual variables from [36] is equivalent to $\left\{w_t^T(\mathbb{Q}, w) : (\mathbb{Q}, w) \in \mathcal{W}_t\right\}$, and as discussed in the Proof of theorem 2.3 in [23], the dual representation (2.1) and (2.3) can be given by this set alone. This means, the results presented in this section include the previously known dual representation results.

We conclude this section by giving a brief description and equivalent characterizations of a time consistency property for set-valued risk measures in the set-optimization approach. The property we will discuss is multi-portfolio time consistency, which was proposed in [21] and further studied in [23]. We also return to the general case with $p \in [0, +\infty]$.

Definition 2.5 A dynamic risk measure $(R_t)_{t=0}^T$ is called **multi-portfolio time consistent** if for all times $t, s \in [0, T]$ with $t < s$, all portfolios $X \in L^p$ and all sets $\mathbf{Y} \subseteq L^p$ the implication

$$R_s(X) \subseteq \bigcup_{Y \in \mathbf{Y}} R_s(Y) \Rightarrow R_t(X) \subseteq \bigcup_{Y \in \mathbf{Y}} R_t(Y) \tag{2.5}$$

is satisfied.

Multi-portfolio time consistency means that if at some time any risk compensating portfolio for X also compensates the risk of some portfolio Y in the set $\mathbf{Y}$, then at any prior time the same relation should hold true. Implicitly within the definition, the choice of eligible portfolios can have an impact on the multi-portfolio time consistency of a risk measure.

In [21], (set-valued) time consistency was also introduced. This property is defined by

$$R_s(X) \subseteq R_s(Y) \Rightarrow R_t(X) \subseteq R_t(Y)$$

for any time $t, s \in [0, T]$ with $t < s$ and any portfolios $X, Y \in L^p$. It is weaker than multi-portfolio time consistency, though in the scalar case both properties coincide.

Before we give some equivalent characterizations for multi-portfolio time consistency, we must give a few additional definitions. These definitions are used for defining the stepped risk measures $R_{t,s} : M_s \to \mathcal{P}(M_t; M_{t,+})$ for $t \leq s$, as discussed in [23, Appendix C]. We denote and define the stepped acceptance set by $A_{t,s} := A_t \cap M_s$. And akin to Corollary 2.3, for the closed conditionally convex and closed (conditionally) coherent stepped risk measures we will define the minimal stepped penalty function (for the conditionally convex case with $M_t \subseteq M_s$) by $-\alpha_{t,s}^{\min}(\mathbb{Q}, w) := \operatorname{cl} \bigcup_{X \in A_{t,s}} \left(\mathbb{E}^{\mathbb{Q}}[X | \mathcal{F}_t] + \Gamma_t(w)\right) \cap M_t$ for every $(\mathbb{Q}, w) \in \mathcal{W}_{t,s}$ and the maximal stepped dual set (for the (conditionally) coherent case with $M_t \subseteq M_s$) by $\mathcal{W}_{t,s}^{\max} := \left\{(\mathbb{Q}, w) \in \mathcal{W}_{t,s} : w_t^s(\mathbb{Q}, w) \in A_{t,s}^+\right\}$. As can be seen, both the stepped penalty function and the stepped maximal dual set are with respect to dual elements $\mathcal{W}_{t,s}$, which in general differ from $\mathcal{W}_t$. In the case that $\tilde{M}_t = M_0$ almost surely, it holds $\mathcal{W}_{t,s} \supseteq \mathcal{W}_t$ for all times $t \leq s \leq T$; if $\tilde{M}_s = \mathbb{R}^d$ almost surely then $\mathcal{W}_{t,s} = \mathcal{W}_t$.

In the below theorem, for the convex upper continuous (conditionally) coherent case we introduce two more definitions. We define the mapping $H_t^s : 2^{\mathcal{W}_s} \to 2^{\mathcal{W}_t}$ for times $t \leq s \leq T$ by $H_t^s(D) := \left\{(\mathbb{Q}, w) \in \mathcal{W}_t : \left(\mathbb{Q}, w_t^s(\mathbb{Q}, w)\right) \in D\right\}$ for $D \subseteq \mathcal{W}_s$.

Additionally, for $\mathbb{Q}, \mathbb{R} \in \mathcal{M}$ we denote by $\mathbb{Q} \oplus^s \mathbb{R}$ the pasting of $\mathbb{Q}$ and $\mathbb{R}$ in s, i.e. the vector probability measures $\mathbb{S} \in \mathcal{M}$ defined via

$$\frac{d\mathbb{S}}{d\mathbb{P}} = \xi_{0,s}(\mathbb{Q})\xi_{s,T}(\mathbb{R}).$$

The following theorem gives equivalent characterizations of multi-portfolio time consistency: a recursion in the spirit of Bellman's principle (property 2 below), an additive property for the acceptance sets (property 3), the so called cocyclical property (property 4) and stability (property 6). The properties are important for the construction of multi-portfolio time consistent risk measures.

Theorem 2.6 (Theorem 3.4 of [21], Corollaries 3.5 and 4.3 and Theorem 4.6 of [23]) *For a normalized dynamic risk measure $(R_t)_{t=0}^T$ the following are equivalent:*

1. $(R_t)_{t=0}^T$ *is multi-portfolio time consistent,*
2. R_t *is recursive, that is for every time* $t, s \in [0, T]$ *with* $t < s$

$$R_t(X) = \bigcup_{Z \in R_s(X)} R_t(-Z) =: R_t(-R_s(X)). \tag{2.6}$$

If additionally $M_t \subseteq M_s$ for every time $t, s \in [0, T]$ with $t < s$ then all of the above is also equivalent to

3. *for every time* $t, s \in [0, T]$ *with* $t < s$

$$A_t = A_s + A_{t,s}. \tag{2.7}$$

If additionally $p \in [1, +\infty]$, $\tilde{M}_t = \mathbb{R}^n \times \{0\}^{d-n}$ almost surely for some $n \leq d$ for every time $t \in [0, T]$, $(R_t)_{t=0}^T$ is a c.u.c. conditionally convex risk measure and

$$R_t(X) = \bigcap_{(\mathbb{Q},w) \in \mathcal{W}_t^e} \left[-\alpha_t^{\min}(\mathbb{Q}, w) + \left(\mathbb{E}^{\mathbb{Q}}\left[-X|\,\mathcal{F}_t\right] + \Gamma_t(w)\right) \cap M_t\right]$$

for every $X \in L_T^p$ where $\mathcal{W}_t^e = \{(\mathbb{Q}, w) \in \mathcal{W}_t : \mathbb{Q} \in \mathcal{M}^e\}$, then all of the above is also equivalent to

4. *for every time* $t, s \in [0, T]$ *with* $t < s$

$$-\alpha_t^{\min}(\mathbb{Q}, w) = \operatorname{cl}\left(-\alpha_{t,s}^{\min}(\mathbb{Q}, w) + \mathbb{E}^{\mathbb{Q}}\left[-\alpha_s^{\min}(\mathbb{Q}, w_t^s(\mathbb{Q}, w))\middle|\,\mathcal{F}_t\right]\right) \tag{2.8}$$

for every $(\mathbb{Q}, w) \in \mathcal{W}_t^e$.

If additionally $p \in [1, +\infty]$, $\tilde{M}_t = \mathbb{R}^n \times \{0\}^{d-n}$ almost surely for some $n \leq d$ for every time $t \in [0, T]$ and $(R_t)_{t=0}^T$ is a c.u.c. (conditionally) coherent risk measure then all of the above is also equivalent to

5. *for every time* $t, s \in [0, T]$ *with* $t < s$

$$\mathcal{W}_t^{\max} = \mathcal{W}_{t,s}^{\max} \cap H_t^s\left(\mathcal{W}_s^{\max}\right), \tag{2.9}$$

which in turn is equivalent to

6. *for every time* $t, s \in [0, T]$ *with* $t < s$

$$\mathcal{W}_t^{\max} = \left\{(\mathbb{Q} \oplus^s \mathbb{R}, w) : (\mathbb{Q}, w) \in \mathcal{W}_{t,s}^{\max}, (\mathbb{R}, w_t^s(\mathbb{Q}, w)) \in \mathcal{W}_s^{\max}\right\}. \tag{2.10}$$

2.2 Measurable Selector Approach

The measurable selector approach was proposed in [52] and is an extension of [36] to the dynamic framework. Only coherent risk measures are considered in this approach as the technique used to deduce the dual representation relies on coherency. The risk measures are assumed to be compatible to a conical market model at the final time T, i.e. portfolios are compared based on the final "values". In so doing, a new pre-image space denoted by $B_{K_T,n}$ is introduced, which will be defined below and is discuss in Remark 3.1. In [52], the space of eligible assets is $M_t^n = L_t^0(\tilde{M}_t^n)$ with $\tilde{M}_t^n[\omega] = \left\{m \in \mathbb{R}^d : \forall i \in \{n+1, \ldots, d\} : m_i = 0\right\}$, i.e. $n \leq d$ of the d assets can be used to cover risk.

Let $\mathcal{S}_t^d$ be the set of $\mathcal{F}_t$-measurable random sets in $\mathbb{R}^d$. Recall that a mapping $\Gamma : \Omega \to 2^{\mathbb{R}^d}$ is an $\mathcal{F}_t$-measurable random set if

$$\text{graph } \Gamma = \left\{(\omega, x) \in \Omega \times \mathbb{R}^d : x \in \Gamma(\omega)\right\}$$

is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$-measurable (where $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-algebra). The random set Γ is closed (convex, conical) if $\Gamma(\omega)$ is closed (convex, conical) for almost every $\omega \in \Omega$.

Let $K_T \in \mathcal{S}_T^d$ satisfy the following assumptions:

k1. for almost every $\omega \in \Omega$: $K_T(\omega)$ is a closed convex cone in $\mathbb{R}^d$;
k2. for almost every $\omega \in \Omega$: $\mathbb{R}_+^d \subseteq K_T(\omega) \neq \mathbb{R}^d$;
k3. for almost every $\omega \in \Omega$: $K_T(\omega)$ is a proper cone, i.e. $K_T(\omega) \cap -K_T(\omega) = \{0\}$.

It is then possible to create a partial ordering in L^0 defined by K_T such that $X \geq_{K_T} Y$ if and only if $\mathbb{P}(X - Y \in K_T) = 1$. The solvency cones with friction, see e.g. [37, 38, 50], satisfy the conditions given above for K_T.

Let $n \leq d$, then we define $B_{K_T,n} := \left\{X \in L^0 : \exists c \in \mathbb{R}_+ : c1_{d,n} \geq_{K_T} X \geq_{K_T} -c1_{d,n}\right\}$ where the i-th component of $1_{d,n} \in \mathbb{R}^d$ is $1_{d,n}^i = \begin{cases} 1 & \text{if } i \in \{1, \ldots, n\} \\ 0 & \text{else} \end{cases}$. Then we can define a norm on $B_{K_T,n}$ by $\|X\|_{K_T,n} := \inf\left\{c \in \mathbb{R}_+ : c1_{d,n} \geq_{K_T} X \geq_{K_T} -c1_{d,n}\right\}$, and $(B_{K_T,n}, \|\cdot\|_{K_T,n})$ defines a Banach space.

Let $\mathcal{S}_t^{d,n} \subseteq \mathcal{S}_t^d$ be such that $\Gamma \in \mathcal{S}_t^{d,n}$ if $\Gamma \in \mathcal{S}_t^d$ and $\Gamma(\omega) \subseteq \tilde{M}_t^n[\omega]$ for almost every $\omega \in \Omega$.

Definition 2.7 A **risk process** is a sequence $(\tilde{R}_t)_{t=0}^T$ of mappings $\tilde{R}_t : B_{K_T,n} \to \mathcal{S}_t^{d,n}$ satisfying

1. $\tilde{R}_t(X)$ is a closed $\mathcal{F}_t$-measurable random set for any $X \in B_{K_T,n}$, $\tilde{R}_t(0) \neq \emptyset$, and $\tilde{R}_t(0)[\omega] \neq \tilde{M}_t^n[\omega]$ for almost every $\omega \in \Omega$.
2. For any $X, Y \in B_{K_T,n}$ with $Y \geq_{K_T} X$ it holds $\tilde{R}_t(Y) \supseteq \tilde{R}_t(X)$.
3. $\tilde{R}_t(X + m) = \tilde{R}_t(X) - m$ for any $X \in B_{K_T,n}$ and $m \in M_t^n$.

A risk process is **conditionally convex** at time t if for all $X, Y \in B_{K_T,n}$ and $\lambda \in L_t^\infty(\mathbb{R})$ with $0 \leq \lambda \leq 1$ almost surely it holds $\lambda\tilde{R}_t(X) + (1-\lambda)\tilde{R}_t(Y) \subseteq \tilde{R}_t(\lambda X + (1-\lambda)Y)$.

A risk process is **conditionally positive homogeneous** at time t if for all $X \in B_{K_T,n}$ and $\lambda \in L_t^0(\mathbb{R}_{++})$ with $\lambda X \in B_{K_T,n}$ it holds $\tilde{R}_t(\lambda X) = \lambda\tilde{R}_t(X)$.

A risk process is **conditionally coherent** at time t if it is both conditionally convex and conditionally positive homogeneous at time t.

A risk process is **normalized** at time t if $\tilde{R}_t(X) + \tilde{R}_t(0) = \tilde{R}_t(X)$ for every $X \in B_{K_T,n}$.

Thus, the values $\tilde{R}_t(X)$ of a risk process are $\mathcal{F}_t$-measurable random sets in $\mathbb{R}^d$. Primal and dual representations can be provided for the measurable selectors of this set. Recall that γ is a $\mathcal{F}_t$-measurable selector of a $\mathcal{F}_t$-random set Γ if $\gamma(\omega) \in \Gamma(\omega)$ for almost every $\omega \in \Omega$. Then using the notation from above, the measurable selectors in L^p are given by $L_t^p(\Gamma) = \{\gamma \in L_t^p : \mathbb{P}(\gamma \in \Gamma) = 1\}$.

Definition 2.8 Given a risk process $(\tilde{R}_t)_{t=0}^T$, then $S_{\tilde{R}} : [0,T] \times B_{K_T,n} \to 2^{M_t^n}$ is a **selector risk measure** if $S_{\tilde{R}}(t, X) := L_t^0(\tilde{R}_t(X))$ for every time t and portfolio $X \in B_{K_T,n}$. The **bounded selector risk measure** is defined by $S_{\tilde{R}}^\infty(t, X) := S_{\tilde{R}}(t, X) \cap B_{K_T,n}$.

Definition 2.9 A set $A_t \subseteq B_{K_T,n}$ is a **conditional acceptance set** at time t if:

1. A_t is closed in the $(B_{K_T,n}, \|\cdot\|_{K_T,n})$ topology.
2. If $X \in B_{K_T,n}$ such that $X \geq_{K_T} 0$ then $X \in A_t$.
3. $B_{K_T,n} \cap M_t^n \nsubseteq A_t$.
4. A_t is $\mathcal{F}_t$-decomposable, i.e. if for any finite partition $(\Omega_t^n)_{n=1}^N \subseteq \mathcal{F}_t$ of Ω and any family $(X_n)_{n=1}^N \subseteq A_t$, then $\sum_{n=1}^N 1_{\Omega_t^n} X_n \in A_t$.
5. A_t is a conditionally convex cone.

Remark 2.10 Note that the definition for $\mathcal{F}_t$-decomposability above differs from that in [52], as in that paper $\mathcal{F}_t$-decomposability is considered with respect to countable rather than finite partitions. We weakened the condition by adapting the Proof of theorem 1.6 of Chap. 2 from [43] when $p = +\infty$ to the space $B_{K_T,n}$.

Proposition 2.11 (Proposition 3.4 of [52]) *Given a conditionally coherent risk process* $\tilde{R}_t$ *at time* t*, then* $A_t := \left\{X \in B_{K_T,n} : 0 \in \tilde{R}_t(X)\right\}$ *is a conditional acceptance set at time* t.

A primal representation of the selector risk measure is given as follows.

Theorem 2.12 (Theorem 3.3 in [52]) *Let* A_t *be a closed subset of* $(B_{K_T,n}, \|\cdot\|_{K_T,n})$. *Then* A_t *is a conditional acceptance set if and only if there exists some conditionally coherent risk process* $\tilde{R}_t$ *at time* t *such that the associated bounded selector risk measure* $S^{\infty}_{\tilde{R}}$ *satisfies* $S^{\infty}_{\tilde{R}}(t, X) = \left\{m \in M^n_t : X + m \in A_t\right\}$ *for all* $X \in B_{K_T,n}$.

Below, we give the dual representation for coherent selector risk measures as done in Theorems 4.1 and 4.2 of [52]. This dual representation can be viewed as the intersection of supporting halfspaces for the selector risk measure, which is the reason that coherence is needed in this approach.

From [52], it is known that $(B_{K_T,n}, \|\cdot\|_{K_T,n})$ is a Banach space, we will let $ba_{K_T,n}$ be the topological dual of $B_{K_T,n}$, and let $ba^+_{K_T,n}$ denote the positive linear forms, that is

$$ba^+_{K_T,n} := \left\{\phi \in ba_{K_T,n} : \phi(X) \geq 0 \; \forall X \geq_{K_T} 0\right\}.$$

Definition 2.13 (*Definition 4.1 of* [52]) A set $\Lambda \subseteq ba_{K_T,n}$ is called $\mathcal{F}_t$**-stable** if for all $\lambda \in L^{\infty}_t(\mathbb{R}_+)$ and $\phi \in \Lambda$, the linear form $\phi^{\lambda} : X \ni B_{K_T,n} \mapsto \phi(\lambda X)$ is an element of Λ.

Theorem 2.14 (Theorem 4.1 of [52]) *Let* $(\tilde{R}_t)_{t=0}^T$ *be a sequence of* $(\mathcal{S}^{d,n}_t)_{t=0}^T$*-valued mappings on* $B_{K_T,n}$. *Then the following are equivalent:*

1. $(\tilde{R}_t)_{t=0}^T$ *is a conditionally coherent risk process.*
2. *There exists a nonempty* $\sigma(ba_{K_T,n}, B_{K_T,n})$*-closed subset* $\mathcal{Q}_t \neq \{0\}$ *of* $ba^+_{K_T,n}$ *which is* $\mathcal{F}_t$*-stable and satisfies the equality*

$$S^{\infty}_{\tilde{R}}(t, X) = \left\{u \in M^n_t \cap B_{K_T,n} : \phi(X + u) \geq 0 \; \forall \phi \in \mathcal{Q}_t\right\}. \tag{2.11}$$

We finish the discussion of the dual representation by considering the case when the risk process additionally satisfies a "Fatou property" as defined below.

Definition 2.15 A sequence $(\tilde{R}_t)_{t=0}^T$ of $(\mathcal{S}^{d,n}_t)_{t=0}^T$-valued mappings on $B_{K_T,n}$ is said to satisfy the **Fatou property** if for all $X \in B_{K_T,n}$ and all times t

$$\limsup_{n \to +\infty} S^{\infty}_{\tilde{R}}(t, X_n) \subseteq S^{\infty}_{\tilde{R}}(t, X)$$

for any bounded sequence $(X_m)_{m \in \mathbb{N}} \subseteq B_{K_T,n}$ which converges to X in probability.

Note that in the above definition the limit superior is defined to be $\limsup_{n \to +\infty} B_n = \operatorname{cl} \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} B_m$ for a sequence of sets $(B_n)_{n \in \mathbb{N}}$.

For the following theorem we assume two additional properties on the convex cone K_T:

k4. for almost every $\omega \in \Omega$: $\mathbb{R}^d_+\backslash\{0\} \subseteq \operatorname{int}[K_T(\omega)]$ or equivalently $K_T(\omega)^+\backslash\{0\} \subseteq \operatorname{int}[\mathbb{R}^d_+]$;

k5. K_T and K_T^+ are both generated by a finite number of linearly independent and bounded generators denoted respectively by $(\xi_i)_{i=1}^N$ and $(\xi_i^+)_{i=1}^{N^+}$.

Let $L^{1,n}(K_T^+) := \left\{Z \in L^0(K_T^+) : 1_{d,n}^\mathsf{T} Z \in L^1(\mathbb{R})\right\}$. In the following theorem we will use $L^{1,n}(K_T^+)$ as a dual space for $B_{K_T,n}$. For $Z \in L^{1,n}(K_T^+)$, the linear form $\phi_Z(X) := \mathbb{E}\left[Z^\mathsf{T} X\right]$ belongs to $ba^+_{K_T,n}$. The norm $\|Z\|_{d,n} := \sup\left\{|\mathbb{E}\left[Z^\mathsf{T} X\right]| : X \in B_{K_T,n}, \|X\|_{K_T,n} \leq 1\right\}$ is the dual norm for any $Z \in L^{1,n}(K_T^+)$.

Theorem 2.16 (Theorem 4.2 of [52]) *Let $(\tilde{R}_t)_{t=0}^T$ be a conditionally coherent risk process on $B_{K_T,n}$ and let K_T satisfy property $k1 - k5$. The following are equivalent:*

1. *For every time $t \in [0, T]$, there exists a closed conditional cone $\{0\} \neq \mathcal{Q}_t^1 \subseteq L^{1,n}(K_T^+)$ (in the norm topology, with norm $\|\cdot\|_{d,n}$) such that for any $X \in B_{K_T,n}$*

$$S^\infty_{\tilde{R}}(t, X) = \left\{u \in M_t^n \cap B_{K_T,n} : \forall Z \in \mathcal{Q}_t^1 : \mathbb{E}\left[Z^\mathsf{T}(X + u)\right] \geq 0\right\}. \tag{2.12}$$

2. *$(\tilde{R}_t)_{t=0}^T$ satisfies the Fatou property.*
3. *$C_t := \left\{X \in B_{K_T,n} : 0 \in \tilde{R}_t(X)\right\}$ is $\sigma(B_{K_T,n}, L^{1,n}(K_T^+))$-closed.*

We conclude this section by discussing time consistency properties as they were defined in the measurable selector approach in [52]. As in the set-optimization approach in the previous section one would like to define a property that is equivalent to a recursive form. For this reason we will extend the risk process to be a function of a set. For a set $\mathbf{X} \subseteq B_{K_T,n}$, let us define $\tilde{R}_t(\mathbf{X}) \in \mathcal{S}_t^{d,n}$ via its selectors, that is

$$L_t^0(\tilde{R}_t(\mathbf{X})) \cap B_{K_T,n} = \operatorname{cl}\operatorname{env}_{\mathcal{F}_t} \bigcup_{X \in \mathbf{X}} S^\infty_{\tilde{R}}(t, X) =: S^\infty_{\tilde{R}}(t, \mathbf{X}),$$

where, for any $\Gamma \subseteq B_{K_T,n}$, $\operatorname{env}_{\mathcal{F}_t}\Gamma$ denotes the smallest $\mathcal{F}_t$-decomposable set (see Definition 2.9) which contains Γ. This means that the measurable selectors of the risk process of a set are defined by the closed and $\mathcal{F}_t$-decomposable version of the pointwise union. Note that if $\mathbf{X} = \{X\}$ then this reduces to the prior definition on portfolios. The risk process of a set is defined in this way because the selection risk measure must be closed and $\mathcal{F}_t$-decomposable-valued to guarantee the existence of an $\mathcal{F}_t$-measurable random set $\tilde{R}_t(\mathbf{X})$ such that $S^\infty_{\tilde{R}}(t, \mathbf{X}) = L_t^0(\tilde{R}_t(\mathbf{X})) \cap B_{K_T,n}$.

Definition 2.17 A risk process $(\tilde{R}_t)_{t=0}^T$ is called **consistent in time** if for any $t, s \in [0, T]$ with $t < s$ and $X \in B_{K_T,n}$, $\mathbf{Y} \subseteq B_{K_T,n}$

$$\tilde{R}_s(X) \subseteq \tilde{R}_s(\mathbf{Y}) \Rightarrow \tilde{R}_t(X) \subseteq \tilde{R}_t(\mathbf{Y}).$$

The following theorem gives equivalent characterizations of consistency in time, the last one being a recursion in the spirit of Bellman's principle.

Theorem 2.18 (Theorem 5.1 of [52]) *A normalized risk process* $(\tilde{R}_t)_{t=0}^T$ *on* $B_{K_T,n}$ *is consistent in time if any of the following equivalent conditions hold:*

1. *If* $\tilde{R}_s(X) \subseteq \tilde{R}_s(\mathbf{Y})$ *for* $X \in B_{K_T,n}$ *and* $\mathbf{Y} \subseteq B_{K_T,n}$*, then* $\tilde{R}_t(X) \subseteq \tilde{R}_t(\mathbf{Y})$ *for* $t \leq s \leq T$.
2. *If* $\tilde{R}_s(X) = \tilde{R}_s(\mathbf{Y})$ *for* $X \in B_{K_T,n}$ *and* $\mathbf{Y} \subseteq B_{K_T,n}$*, then* $\tilde{R}_t(X) = \tilde{R}_t(\mathbf{Y})$ *for* $t \leq s \leq T$.
3. *For all* $X \in B_{K_T,n}$, $S^{\infty}_{\tilde{R}}(t, X) = S^{\infty}_{\tilde{R}}(t, -S^{\infty}_{\tilde{R}}(s, X))$ *for* $t \leq s \leq T$.

2.3 Set-Valued Portfolio Approach

The approach for considering sets of portfolios, so called set-valued portfolios, as the argument of a set-valued risk measure was proposed in [13]. The reasoning for considering set-valued portfolios is to take the risk, not only of a portfolio X, but of every possible portfolio that X can be traded for in the market, into account. We will denote by $\mathbf{X}$ the random set of portfolios for which $X \in L^p$ can be exchanged. The concept of set-valued portfolios appears naturally when trading opportunities in the market are taken into account. Below we provide two examples, one in which no trading is allowed and another in which any possible trade can be used. There are other examples provided in [13] on how a set-valued portfolio can be obtained, and the definition of the risk measure is independent of the method used to construct set-valued portfolios.

Example 2.19 The random mapping $\mathbf{X} = X + \mathbb{R}^d_-$ for a random vector $X \in L^p$ describes the case when no exchanges are allowed.

Example 2.20 (Example 2.2 of [13]) The random mapping $\mathbf{X} = X + \mathbf{K}$ for a random vector $X \in L^p$ and a lower convex (random) set $\mathbf{K}$, such that $L^p(\mathbf{K})$ is closed, defines the set-valued portfolios related to the exchanges defined by $\mathbf{K}$. If K is a solvency cone (see e.g. [37, 38, 50]) or the sum of solvency cones at different time points, then $\mathbf{K} = -K$ is an exchange cone, and the associated random mapping defines a set-valued portfolio. The setting of Example 2.19 corresponds to the case where $\mathbf{K} = \mathbb{R}^d_-$.

We will slightly adjust the definitions given in [13] to include the dynamic extension of such risk measures, to incorporate the set of eligible portfolios M_t, and go beyond the coherent case.

Let $\mathcal{S}^d_T$ denote the set of $\mathcal{F}$-random sets in $\mathbb{R}^d$ (as in Sect. 2.2 above). Let $\bar{\mathcal{S}}^d_T \subseteq \mathcal{S}^d_T$ be those random sets that are nonempty, closed, convex and lower, that is for $X \in \mathbf{X}$ also $Y \in \mathbf{X}$ whenever $X - Y \in \mathbb{R}^d_+$ $\mathbb{P}$-a.s. As in [13], we will consider set-valued portfolios $\mathbf{X} \in \bar{\mathcal{S}}^d_T$. By Proposition 2.1.5 and Theorem 2.1.6 in [43], the collection of p-integrable selectors of $\mathbf{X}$, that is $L^p(\mathbf{X})$, is a nonempty, closed, ($\mathcal{F}$-)conditionally convex, lower and $\mathcal{F}$-decomposable set, which is an element of $\mathcal{G}(L^p; L^p_-)$. In [13], $\bar{\mathcal{S}}^d_T$ is used as the pre-image set, one could also use the family of sets of selectors $\{L^p(\mathbf{X}) : \mathbf{X} \in \bar{\mathcal{S}}^d_T\} \subset \mathcal{G}(L^p; L^p_-)$ as the pre-image set, which is particular useful when dynamic risk measures are considered and recursions due to multi-portfolio time consistency become important. Recall that $\mathcal{P}(M_t; M_{t,+}) := \left\{D \subseteq M_t : D = D + M_{t,+}\right\}$ denotes the set of upper sets, which will be used as the image space for the risk measures. Closed (conditionally) convex risk measures map into $\mathcal{G}(M_t; M_{t,+})$.

In the following definition for convex risk measures we consider a modified version of set-addition used in [13] which is denoted by $\oplus$. For two random sets $\mathbf{X}, \mathbf{Y} \in \mathcal{S}^d_T$, $\mathbf{X} \oplus \mathbf{Y} \in \mathcal{S}^d_T$ is the random set defined by the closure of $\mathbf{X}[\omega] + \mathbf{Y}[\omega]$ for all $\omega \in \Omega$. Note that, by Proposition 2.1.4 in [43], if the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is non-atomic and $p \in [1, +\infty)$ then $L^p(\mathbf{X} \oplus \mathbf{Y}) = \mathrm{cl}\left[L^p(\mathbf{X}) + L^p(\mathbf{Y})\right]$.

Definition 2.21 (*Definition 2.9 of* [13]) A function $\mathbf{R}_t : \bar{\mathcal{S}}^d_T \to \mathcal{P}(M_t; M_{t,+})$ is called a **set-valued conditional risk measure** if it satisfies the following conditions.

1. Cash invariance: $\mathbf{R}_t(\mathbf{X} + m) = \mathbf{R}_t(\mathbf{X}) - m$ for any $\mathbf{X}$ and $m \in M_t$.
2. Monotonicity: Let $\mathbf{X} \subseteq \mathbf{Y}$ almost surely, then $\mathbf{R}_t(\mathbf{Y}) \supseteq \mathbf{R}_t(\mathbf{X})$.

The risk measure $\mathbf{R}_t$ is said to be **closed-valued** if its values are closed sets.

The risk measure $\mathbf{R}_t$ is said to be **(conditionally) convex** if for every set-valued portfolio $\mathbf{X}, \mathbf{Y}$ and $\lambda \in [0, 1]$ (respectively $\lambda \in L^\infty_t(\mathbb{R})$ such that $0 \leq \lambda \leq 1$)

$$\mathbf{R}_t(\lambda\mathbf{X} \oplus (1-\lambda)\mathbf{Y}) \supseteq \lambda\mathbf{R}_t(\mathbf{X}) + (1-\lambda)\mathbf{R}_t(\mathbf{Y}).$$

The risk measure $\mathbf{R}_t$ is said to be **(conditionally) positive homogeneous** if for every $\mathbf{X}$ and $\lambda > 0$ (respectively $\lambda \in L^\infty_t(\mathbb{R}_{++})$)

$$\mathbf{R}_t(\lambda\mathbf{X}) = \lambda\mathbf{R}_t(\mathbf{X}).$$

The risk measure $\mathbf{R}_t$ is said to be **(conditionally) coherent** if it is (conditionally) convex and (conditionally) positive homogeneous.

The closed-valued variant of R_t is denoted by $\bar{\mathbf{R}}_t(\mathbf{X}) = \mathrm{cl}(\mathbf{R}_t(\mathbf{X}))$ for every set-valued portfolio $\mathbf{X} \in \bar{\mathcal{S}}^d_T$.

A set-valued portfolio $\mathbf{X}$ is acceptable if $0 \in \mathbf{R}_t(\mathbf{X})$, i.e. we can define the acceptance set $\mathbf{A}_t \subseteq \bar{\mathcal{S}}^d_T$ by $\mathbf{A}_t := \{\mathbf{X} : 0 \in \mathbf{R}_t(\mathbf{X})\}$. And a primal representation for the risk measures can be given by the usual definition $\mathbf{R}_t(\mathbf{X}) = \{u \in M_t : \mathbf{X} + u \in \mathbf{A}_t\}$ due to cash invariance.

We will now consider a subclass of set-valued conditional risk measures presented in [13, Sect. 3] that are constructed using a scalar dynamic risk measure for each component. For the remainder of this section we will consider the case when $M_t = L^p_t$. In [13], only (scalar) law invariant coherent risk measures were considered for this approach, we will consider the more general case.

Let $\rho_t^1, \dots, \rho_t^d$ be dynamic risk measures defined on $L^p(\mathbb{R})$ with values in $L_t^p(\mathbb{R} \cup \{+\infty\})$. For a random vector $X = (X_1, \dots, X_d)^\mathsf{T} \in L^p$ we define

$$\boldsymbol{\rho}_t(X) = \left(\rho_t^1(X_1), \dots, \rho_t^d(X_d)\right)^\mathsf{T}.$$

We say the vector $X \in L^p$ is **acceptable** if $\boldsymbol{\rho}_t(X) \leq 0$, i.e. $\rho_t^i(X_i) \leq 0$ for all $i = 1, \dots, d$. We say the set-valued portfolio $\mathbf{X}$ is **acceptable** if there exists a $Z \in L^p(\mathbf{X})$ such that $\boldsymbol{\rho}_t(Z) \leq 0$.

Definition 2.22 (*Definition 3.3 of* [13]) The **constructive conditional risk measure** $\mathbf{R}_t : \bar{\mathcal{S}}_T^d \to \mathcal{P}(L_t^p; L_{t,+}^p)$ is defined for any set-valued portfolio $\mathbf{X}$ by

$$\mathbf{R}_t(\mathbf{X}) = \left\{u \in L_t^p : \mathbf{X} + u \text{ is acceptable}\right\},$$

which is equivalently to

$$\mathbf{R}_t(\mathbf{X}) = \bigcup_{Z \in L^p(\mathbf{X})} (\boldsymbol{\rho}_t(Z) + L_{t,+}^p). \tag{2.13}$$

The closed-valued variant is defined by $\bar{\mathbf{R}}_t(\mathbf{X}) := \mathrm{cl}(\mathbf{R}_t(\mathbf{X}))$ for every $\mathbf{X} \in \bar{\mathcal{S}}_T^d$.

In [13], the constructive (static) risk measures have been called selection risk measures, we modified the name here in accordance to the title of the paper [13] to avoid confusion with the measurable selector approach from Sect. 2.2.

Example 2.23 Consider the no-exchange set-valued portfolios from Example 2.19. Then the constructive conditional risk measure associated with any vector of scalar conditional risk measures is given by

$$\mathbf{R}_t(\mathbf{X}) = \boldsymbol{\rho}_t(X) + L_{t,+}^p.$$

Theorem 2.24 (Theorem 3.4 of [13]) *Let $\boldsymbol{\rho}_t$ be a vector of dynamic risk measures, then $\mathbf{R}_t$ and $\bar{\mathbf{R}}_t$ given in Definition 2.22 are both set-valued conditional risk measures.*

If $\boldsymbol{\rho}_t$ is convex (conditionally convex, positive homogeneous, conditionally positive homogeneous, law invariant convex on an non-atomic probability space), then $\mathbf{R}_t$ and $\bar{\mathbf{R}}_t$ are convex (conditionally convex, positive homogeneous, conditionally positive homogeneous, law invariant convex on an non-atomic probability space).

Furthermore, [13] gives conditions under which the constructive (static) risk measure $\mathbf{R}_0$ defined in (2.13) in the coherent case is closed, or Lipschitz and deduces upper and lower bounds for it and dual representations in certain special cases. Numerical examples for the calculation of upper and lower bounds are given.