

Atlantis Studies in Probability and Statistics
Series Editor: Chris P. Tsokos

Mohammad Ahsanullah

Extreme Value Distributions

Atlantis Studies in Probability and Statistics

Volume 8

Series editor

Chris P. Tsokos, Tampa, USA

Aims and scope of the series

The series ‘Atlantis Studies in Probability and Statistics’ publishes studies of high quality throughout the areas of probability and statistics that have the potential to make a significant impact on the advancement in these fields. Emphasis is given to broad interdisciplinary areas at the following three levels:

- (I) Advanced undergraduate textbooks, i.e., aimed at the 3rd and 4th years of undergraduate study, in probability, statistics, biostatistics, business statistics, engineering statistics, operations research, etc.;
- (II) Graduate-level books, and research monographs in the above areas, plus Bayesian, nonparametric, survival analysis, reliability analysis, etc.;
- (III) Full Conference Proceedings, as well as selected topics from Conference Proceedings, covering frontier areas of the field, together with invited monographs in special areas.

All proposals submitted in this series will be reviewed by the Editor-in-Chief, in consultation with Editorial Board members and other expert reviewers.

For more information on this series and our other book series, please visit our website at: www.atlantis-press.com/Publications/books

AMSTERDAM—PARIS—BEIJING

ATLANTIS PRESS

Atlantis Press

29, avenue Laumière

75019 Paris, France

More information about this series at <http://www.atlantis-press.com>

Mohammad Ahsanullah

Extreme Value Distributions



Mohammad Ahsanullah
Department of Management Sciences
Rider University
Lawrenceville, NJ
USA

ISSN 1879-6893 ISSN 1879-6907 (electronic)
Atlantis Studies in Probability and Statistics
ISBN 978-94-6239-221-2 ISBN 978-94-6239-222-9 (eBook)
DOI 10.2991/978-94-6239-222-9

Library of Congress Control Number: 2016947032

© Atlantis Press and the author(s) 2016

This book, or any parts thereof, may not be reproduced for commercial purposes in any form or by any means, electronic or mechanical, including photocopying, recording or any information storage and retrieval system known or to be invented, without prior permission from the Publisher.

Printed on acid-free paper

Preface

Extreme values are extremely interesting. The maximum or minimum of large number observations when normalized can only converge to three types of extreme value distributions, Gumbel, Frechet and Weibull. Thus the maximum and minimum order statistics of n observations when normalized converges to the extreme value distributions as n tends to infinity. The local maximum or minimum (records) of a sequence of independent and identically distributed random variables are useful to estimate the parameters of the extreme value distributions. In Chap. 1 of this book some distributional properties of the largest and smallest order statistics from some important distributions are presented. In Chap. 2 some basic properties of record values and inferences based on the distributional properties are given. In Chap. 3 the necessary and sufficient conditions of maximum and minimum order statistics to converge to the extreme value distributions are derived. In Chap. 3 also the normalizing constants of several well-known distributions are derived. In Chap. 4 estimations of parameters of the extreme value distributions are derived. An extensive reference of papers related to ordered random variables is given. This book can be used as a textbook or as a consulting book.

In this book there may be some errors escaped our attention. However, I will be glad to receive any comments from the readers about it. I am grateful to the Atlantis press for publishing this book.

Lawrenceville, NJ, USA

Mohammad Ahsanullah

Contents

1	Order Statistics	1
1.1	Distributional Properties	1
1.2	Minimum Variance Linear Unbiased Estimates	8
2	Record Statistics	23
2.1	Introduction and Examples of Record Values	23
2.1.1	Definition of Record Values and Record Times	24
2.2	The Exact Distribution of Record Values	24
2.3	Moments of Record Values	29
2.4	Entropies of Record Values	41
2.5	Estimation of Parameters and Predictions of Records	43
2.5.1	Exponential Distribution	43
2.5.2	Generalized Pareto Distribution	48
2.5.3	Power Function Distribution	52
2.5.4	Rayleigh Distribution	56
2.5.5	Two Parameter Uniform Distribution	61
2.5.6	Minimum Variance Linear Unbiased Estimate of θ_1 and θ_2	62
2.5.7	One Parameter Uniform Distribution	64
2.5.8	Prediction of Record Values	67
2.6	Weibull Distribution	68
2.6.1	Minimum Variance Linear Unbiased Estimators of μ and σ	68
3	Extreme Value Distributions	73
3.1	Introduction	73
3.2	The Pdfs of the Extreme Values Distributions	74
3.2.1	Type 1 Extreme Value for $X_{n,n}$	74
3.2.2	Type 2 Extreme Value Distribution for $X_{n,n}$	75
3.2.3	Type 3 Extreme Value Distribution for $X_{n,n}$	75

3.3	Domain of Attraction	76
3.3.1	Domain of Attraction of Type I Extreme Value Distribution for $X_{n,n}$	77
3.3.2	Domain of Attraction of Type 2 Extreme Value Distribution for $X_{n,n}$	79
3.3.3	Domain of Attraction of Type 3 Extreme Value Distribution for $X_{n,n}$	80
3.4	Domain of Attraction for $X_{1,n}$	86
3.4.1	Domain of Attraction for Type 1 Extreme Value Distribution for $X_{1,n}$	86
3.4.2	Domain of Attraction of Type 2 Distribution for $X_{1,n}$	87
3.4.3	Domain of Attraction of Type 3 Extreme Value Distribution	87
4	Inferences of Extreme Value Distributions	93
4.1	Type I Extreme Value (Gumbel) Distribution.	93
4.1.1	Minimum Variance Linear Unbiased Estimates (MVLUE)	94
4.1.2	Best Invariant Estimates (BLIE)	96
4.2	Maximum Likelihood Estimates (MLE)	97
4.2.1	Characterization.	98
4.2.2	Applications	100
4.3	Type II and Type II Distributions.	101
4.3.1	Distributional Properties	102
4.3.2	Estimation of Parameters.	103
4.4	Characterizations.	109
	References	113
	Index	137

Chapter 1

Order Statistics

1.1 Distributional Properties

Let $X_1, X_2, \dots, X_n$ be independent and identically distributed (I, I, d) absolutely continuous random variables. Suppose that $F(x)$ be their cumulative distribution function (cdf) and $f(x)$ be the corresponding probability density function (pdf). Let $X_{1,n} \leq X_{2,n} \leq \dots \leq X_{n,n}$ be the corresponding order statistics. We denote $F_{k,n}(x)$ and $f_{k,n}(x)$ as the cdf and pdf respectively of $X_{k,n}$, $k = 1, 2, \dots, n$. We can write

$$f_{k,n}(x) = \frac{n!}{(k-1)!(n-k)!} (F(x))^{k-1} (1 - F(x))^{n-k} f(x),$$

The joint probability density function of order statistics $X_{1,n}, X_{2,n}, \dots, X_{n,n}$ has the form

$$f_{1,2,\dots,n,n}(x_1, x_2, \dots, x_n) = n! \prod_{k=1}^n f(x_k), -\infty < x_1 < x_2 < \dots < x_n < \infty$$

and

$$= 0, \text{ otherwise.}$$

There are some simple formulae for distributions of maxima ($X_{n,n}$) and minimum ($X_{1,n}$) of the n random variables.

The pdfs of the smallest and largest order statistics are given respectively as

$$f_{1,n}(x) = n(1 - F(x))^{n-1}f(x)$$

and

$$f_{n,n}(x) = n(F(x))^{n-1}f(x)$$

The joint pdf $f_{1,n,n}(x,y)$ of $X_{1,n}$ and $X_{n,n}$ is given by

$$f_{1,n,n}(x,y) = n(n-1)(F(y) - F(x))^{n-2}f(x)f(y), \\ -\infty < x < y < \infty.$$

Example 1.1 Exponential distribution.

Suppose that $X_1, X_2, \dots, X_n$ are n i.i.d. random variables with cdf $F(x)$ as $F(x) = 1 - e^{-x}, x \geq 0$.

The pdfs $f_{1,n}(x)$ of $X_{1,n}$ and $f_{n,n}(x)$ are respectively

$$f_{1,n}(x) = ne^{-nx}, x \geq 0.$$

and

$$f_{n,n}(x) = n(1 - e^{-x})^{n-1}e^{-x}, \quad x \geq 0.$$

It can be seen that $nX_{1,n}$ has the exponential distribution.

The pdfs of $X_{1,n}$ and $X_{n,n}$ are given respectively in Figs. 1.1 and 1.2 for $n = 3, 5$ and 10.

The limiting distributions of standardized asymptotic distributions of $X_{1,n}$ and $X_{m,m}$ are given in Chap. 3.

Example 1.2 Uniform distribution.

Suppose that $X_1, X_2, \dots, X_n$ are n i.i.d. random variables with cdf $F(x)$ as $F(x) = x, 0 < x < 1$. We have

The pdfs $f_{1,n}(x)$ of $X_{1,n}$ and $f_{n,n}(x)$ are respectively

$$f_{1,n}(x) = n(1 - x)^{n-1}, \quad 0 < x < 1,$$

and

$$f_{1,n}(x) = nx^{n-1}, \quad 0 < x < 1.$$

The pdfs of $X_{1,n}$ and $X_{n,n}$ are given respectively in Figs. 1.3 and 1.4 for $n = 2, 5$ and 10.

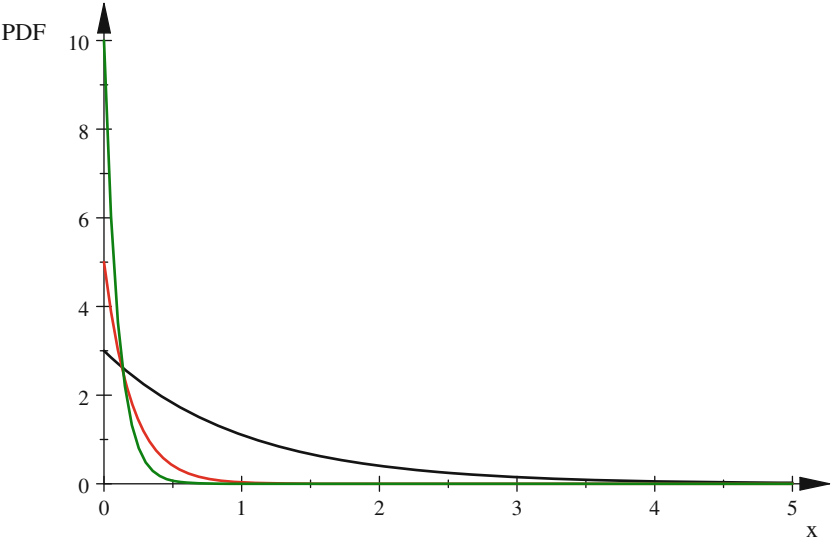


Fig. 1.1 PDFs $f_{1,3}(x)$ —black, $f_{1,5}(x)$ —red, $f_{1,10}(x)$ —green

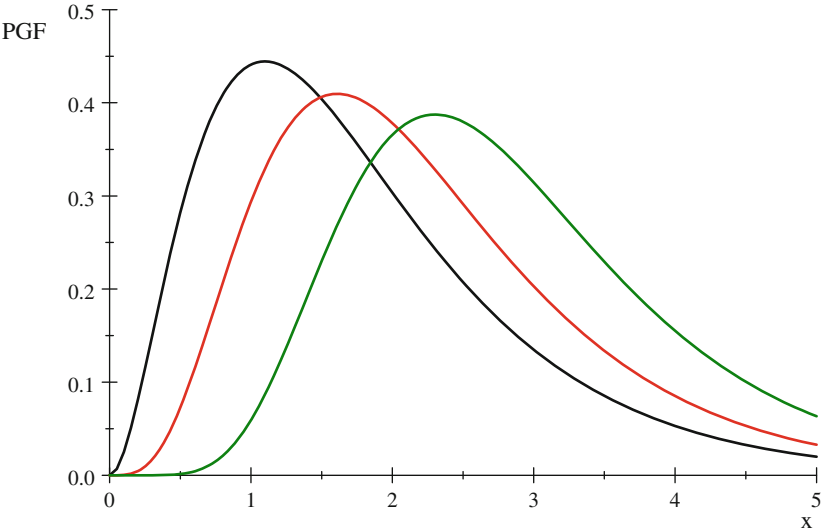


Fig. 1.2 PDFs $f_{3,3}(x)$ —black, $f_{5,55}(x)$ —red, $f_{10,19}(x)$ —green

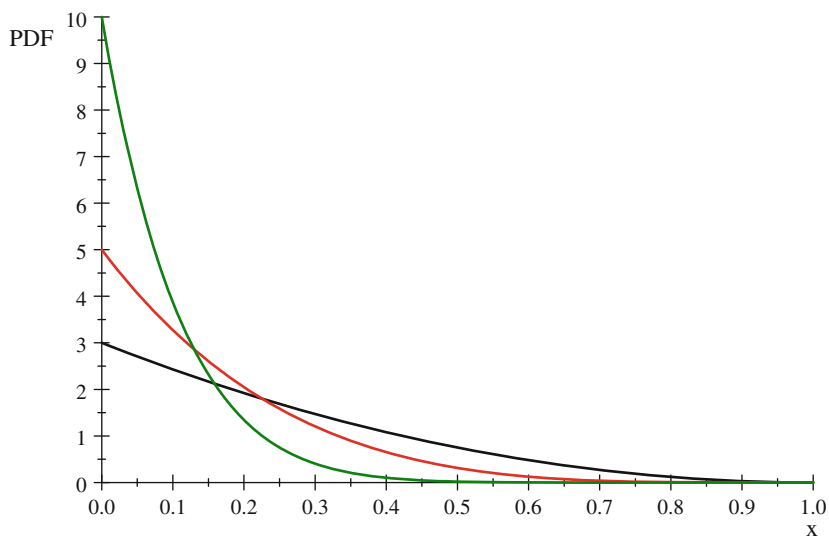


Fig. 1.3 PDFs $f_{1,3}(x)$ —black, $f_{1,5}(x)$ —red, $f_{1,10}(x)$ —green

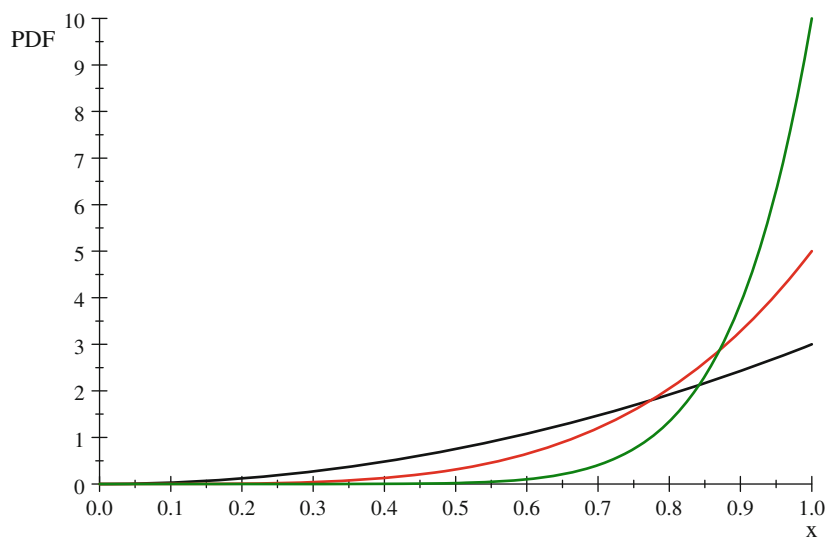


Fig. 1.4 PDFs $f_{3,3}(x)$ —black, $f_{5,5}(x)$ —red, $f_{10,19}(x)$ —green

The limiting distributions of standardized $X_{1,n}$ and $X_{m,m}$ are given in Chap. 3.

Example 1.3 Rayleigh distribution.

Suppose that $X_1, X_2, \dots, X_n$ are n i.i.d. random variables with cdf

$$F(x) = 1 - e^{-\frac{x^2}{2}}, \quad x \geq 0.$$

We have the pdfs $f_{1,n}(x)$ of $X_{1,n}$ and $f_{n,n}(x)$ are respectively

$$f_{1,n}(x) = nxe^{-\frac{nx^2}{2}}, \quad x \geq 0.,$$

and

$$f_{1,n}(x) = nx(1 - e^{-\frac{x^2}{2}})^{n-1}e^{-\frac{x^2}{2}}, \quad x \geq 0.$$

The pdfs pf $X_{1,n}$ and $X_{n,n}$ are given respectively in Figs. 1.5 and 1.6 for $n = 3, 5$ and 10.

Example 1.4 Pareto distribution.

Suppose that $X_1, X_2, \dots, X_n$ are n i.i.d. random variables with cdf $F(x) = 1 - \frac{1}{x^2}$, $x \geq 1$. We have the pdfs $f_{1,n}(x)$ of $X_{1,n}$ and $f_{n,n}(x)$ are respectively

$$f_{1,n}(x) = \frac{2n}{x^{2n+1}} \quad x \geq 1,$$

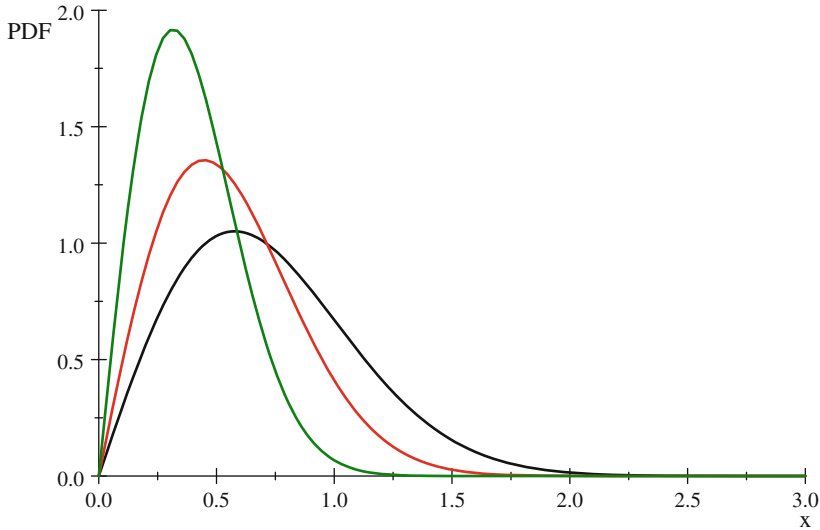


Fig. 1.5 PDFs $f_{3,3}(x)$ —black, $f_{3,5}(x)$ —red, $f_{1,19}(x)$ —green

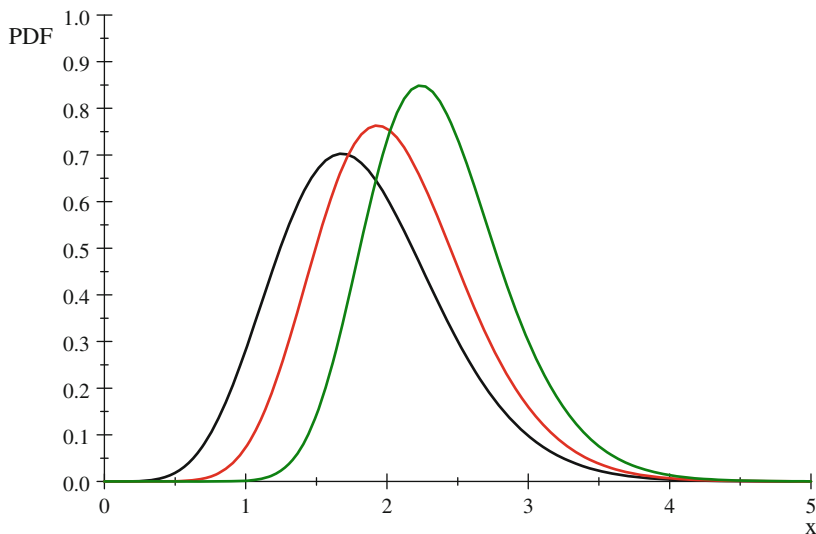


Fig. 1.6 PDFs $f_{3,3}(x)$ —black, $f_{5,5}(x)$ —red, $f_{10,19}(x)$ —green

and

$$f_{n,n}(x) = \frac{2n}{x^3} \left(1 - \frac{1}{x^2}\right)^{n-1}, \quad x \geq 1$$

The pdfs of $X_{1,n}$ and $X_{n,n}$ are given respectively in Figs. 1.7 and 1.8 for $n = 3, 5$ and 10.

The joint pdf $f_{r,s,n}$ of two order statistics $X_{1,n}$ and $X_{n,n}$ ($1 \leq r < s \leq n$) is given by

$$f_{r,s,n}(x, y) = c_{r,s,n} (F(x))^{r-1} (F(y) - F(x))^{s-r-1} (1 - F(y))^{n-s} f(x) f(y)$$

where $c_{r,s,n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$, $1 \leq r < s \leq n$.

The conditional distribution of $X_{s,n}|X_{r,n}$.

Let $f_{s|r,n}$ be the pdf of $X_{s,n}|X_{r,n}$, then

$$\begin{aligned} f_{s|r,n}(y|x) &= \frac{f_{r,s,n}(x,y)}{F_{r,n}(x)} \\ &= \frac{c_{r,s,n} (F(y) - F(x))^{s-r-1} (1 - F(y))^{n-s} f(y)}{c_{r,n} (1 - F(x))^{n-r}}, \end{aligned}$$

where $1 < r < s < n$, $-\infty < x < y < \infty$ and $c_{r,n} = \frac{n!}{(r-1)!(n-r)!}$.

The distribution of the difference between two order statistics.

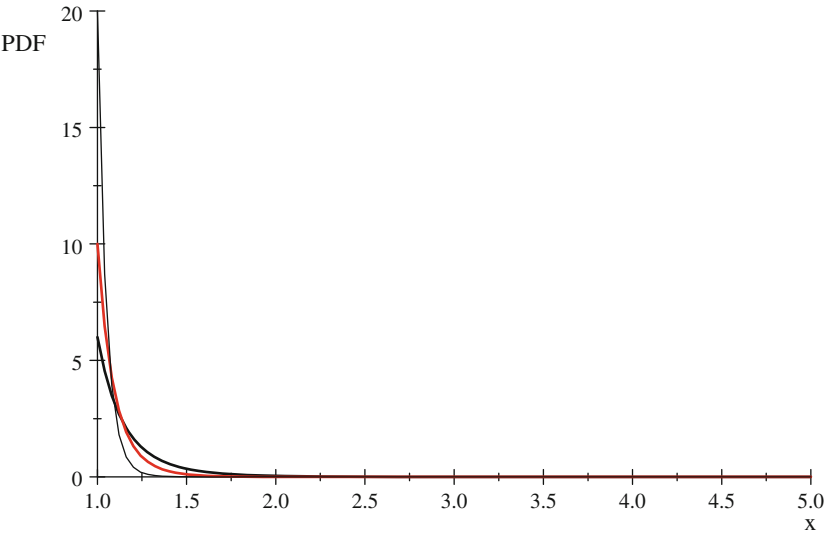


Fig. 1.7 PDFs $f_{3,3}(x)$ —black, $f_{3,5}(x)$ —red, $f_{1,19}(x)$ —green

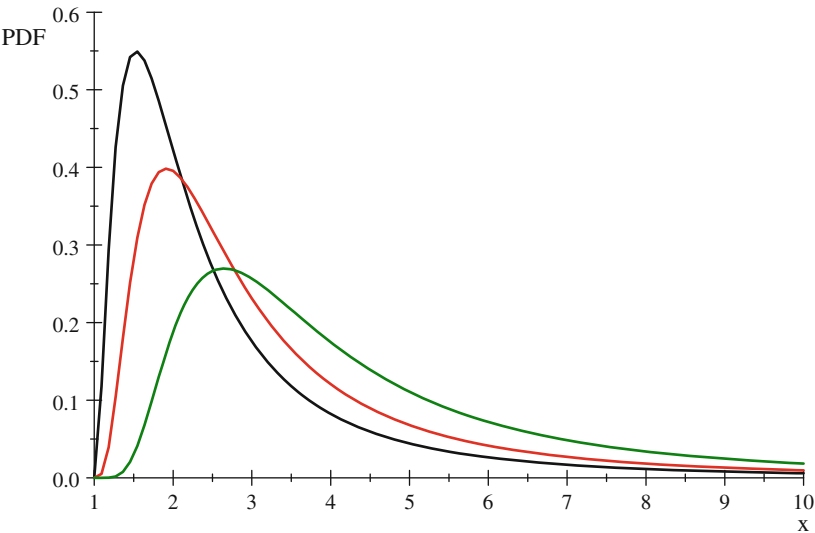


Fig. 1.8 PDFs $f_{3,3}(x)$ —black, $f_{5,5}(x)$ —red, $f_{10,19}(x)$ —green