ATLANTIS PRESS

Atlantis Studies in Probability and Statistics
Series Editor: Chris P. Tsokos

Mohammad Ahsanullah

Extreme Value Distributions

Atlantis Studies in Probability and Statistics

Volume 8

Series editor

Chris P. Tsokos, Tampa, USA

Aims and scope of the series

The series 'Atlantis Studies in Probability and Statistics' publishes studies of high quality throughout the areas of probability and statistics that have the potential to make a significant impact on the advancement in these fields. Emphasis is given to broad interdisciplinary areas at the following three levels:

(I) Advanced undergraduate textbooks, i.e., aimed at the 3rd and 4th years of undergraduate study, in probability, statistics, biostatistics, business statistics, engineering statistics, operations research, etc.;
(II) Graduate-level books, and research monographs in the above areas, plus Bayesian, nonparametric, survival analysis, reliability analysis, etc.;
(III) Full Conference Proceedings, as well as selected topics from Conference Proceedings, covering frontier areas of the field, together with invited monographs in special areas.

All proposals submitted in this series will be reviewed by the Editor-in-Chief, in consultation with Editorial Board members and other expert reviewers.

For more information on this series and our other book series, please visit our website at: www.atlantis-press.com/Publications/books

AMSTERDAM—PARIS—BEIJING
ATLANTIS PRESS
Atlantis Press
29, avenue Laumière
75019 Paris, France

More information about this series at http://www.atlantis-press.com

Mohammad Ahsanullah

Extreme Value Distributions

ATLANTIS
PRESS

Mohammad Ahsanullah
Department of Management Sciences
Rider University
Lawrenceville, NJ
USA

ISSN 1879-6893 ISSN 1879-6907 (electronic)
Atlantis Studies in Probability and Statistics
ISBN 978-94-6239-221-2 ISBN 978-94-6239-222-9 (eBook)
DOI 10.2991/978-94-6239-222-9

Library of Congress Control Number: 2016947032

Printed on acid-free paper

Preface

Extreme values are extremely interesting. The maximum or minimum of large number observations when normalized can only converge to three types of extreme value distributions, Gumbel, Frechet and Weibull. Thus the maximum and minimum order statistics of n observations when normalized converges to the extreme value distributions as n tends to infinity. The local maximum or minimum (records) of a sequence of independent and identically distributed random variables are useful to estimate the parameters of the extreme value distributions. In Chap. 1 of this book some distributional properties of the largest and smallest order statistics from some important distributions are presented. In Chap. 2 some basic properties of record values and inferences based on the distributional properties are given. In Chap. 3 the necessary and sufficient conditions of maximum and minimum order statistics to converge to the extreme value distributions are derived. In Chap. 3 also the normalizing constants of several well-known distributions are derived. In Chap. 4 estimations of parameters of the extreme value distributions are derived. An extensive reference of papers related to ordered random variables is given. This book can be used as a textbook or as a consulting book.

In this book there may be some errors escaped our attention. However, I will be glad to receive any comments from the readers about it. I am grateful to the Atlantis press for publishing this book.

Lawrenceville, NJ, USA — Mohammad Ahsanullah

Contents

Chapter 1
Order Statistics

1.1 Distributional Properties

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (I, I, d) absolutely continuous random variables. Suppose that F(x) be their cumulative distribution function (cdf) and f(x) be the corresponding probability density function (pdf). Let $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$ be the corresponding order statistics. We denote $F_{k,n}(x)$ and $f_{k,n}(x)$ as the cdf and pdf respectively of $X_{k,n}$, k = 1, 2, …, n. We can write

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} (F(x))^{k-1} (1 - F(x))^{n-k} f(x),$$

The joint probability density function of order statistics $X_{1,n}, X_{2,n}, \ldots, X_{n,n}$ has the form

$$f_{1,2,\ldots,n:n}\left(x_1, x_2, \ldots, x_n\right) = n! \prod_{k=1}^{n} f(x_k), -\infty < x_1 < x_2 < \cdots < x_n < \infty$$

and

$$= 0, \ otherwise.$$

There are some simple formulae for distributions of maxima ($X_{n,n}$) and minimum ($X_{1,n}$) of the n random variables.

M. Ahsanullah, *Extreme Value Distributions*, Atlantis Studies
in Probability and Statistics 8, DOI 10.2991/978-94-6239-222-9_1

The pdfs of the smallest and largest order statistics are given respectively as

$$f_{1,n}(x) = n(1 - F(x))^{n-1} f(x)$$

and

$$f_{n,n}(x) = n(F(x))^{n-1} f(x)$$

The joint pdf $f_{1,n,n}(x,y)$ of $X_{1,n}$ and $X_{n.n}$ is given by

$$f_{1,n,n}(x, y) = n(n - 1)(F(y) - F(x))^{n-2} f(x)f(y),$$
$$-\infty < x < y < \infty.$$

Example 1.1 Exponential distribution.

Suppose that X_1, X_2, …, X_n are n i.i.d. random variables with cdf F(x) as $F(x) = 1 - e^{-x}.x \geq 0$.

The pdfs $f_{1,n}(x)$ of $X_{1,n}$ and $f_{n.n}(x)$ are respectively

$$f_{1,n}(x) = ne^{-nx}, x \geq 0.$$

and

$$f_{n,n}(x) = n(1 - e^{-x})^{n-1} e^{-x}, \quad x \geq 0.$$

It can be seen that $nX_{1,n}$ has the exponential; distribution.

The pdfs of $X_{1.n}$ and $X_{n,n}$ are given respectively in Figs. 1.1 and 1.2 for n = 3, 5 and 10.

The limiting distributions of standardized asymptotic distributions of $X_{1,n}$ and $X_{m,m}$ are given in Chap. 3.

Example 1.2 Uniform distribution.

Suppose that X_1, X_2, …, X_n are n i.i.d. random variables with cdf F(x) as $F(x) = x, \ 0 > x < 1$. We have

The pdfs $f_{1,n}(x)$ of $X_{1,n}$ and $f_{n.n}(x)$ are respectively

$$f_{1,n}(x) = n(1 - x)^{n-1}, \quad 0 < x < 1,$$

and

$$f_{1,n}(x) = nx^{n-1}, \quad 0 < x < 1.$$

The pdfs pf $X_{1.n}$ and $X_{n,n}$ are given respectively in Figs. 1.3 and 1.4 for n = 2, 5 and 10.

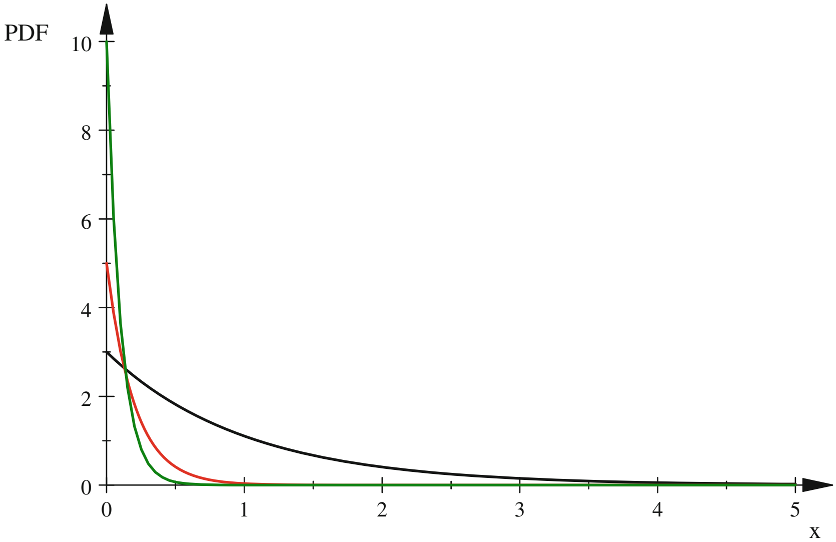

Fig. 1.1 PDFs $f_{1,3}(x)$—*black*, $f_{1,5}$ (x)—*red*, $f_{1.10}(x)$—*green*

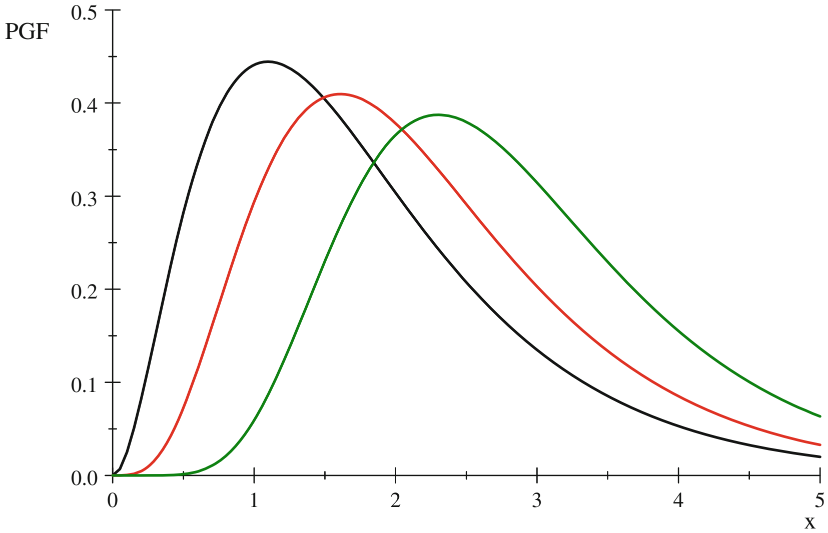

Fig. 1.2 PDFs $f_{3,3}(x)$—*black*, $f_{5,55}$ (x)—*red*, $f_{10.19}(x)$—*green*

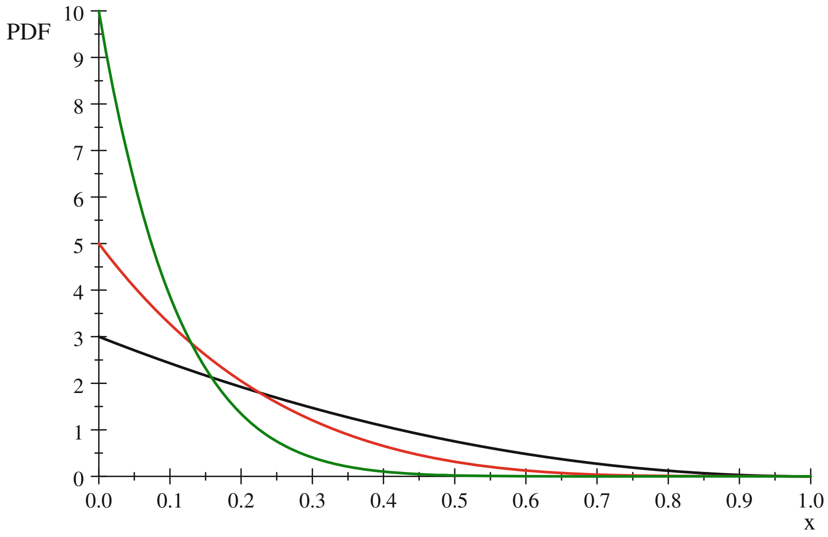

Fig. 1.3 PDFs $f_{1,3}(x)$—*black*, $f_{1,5}(x)$—*red*, $f_{1.10}(x)$—*green*

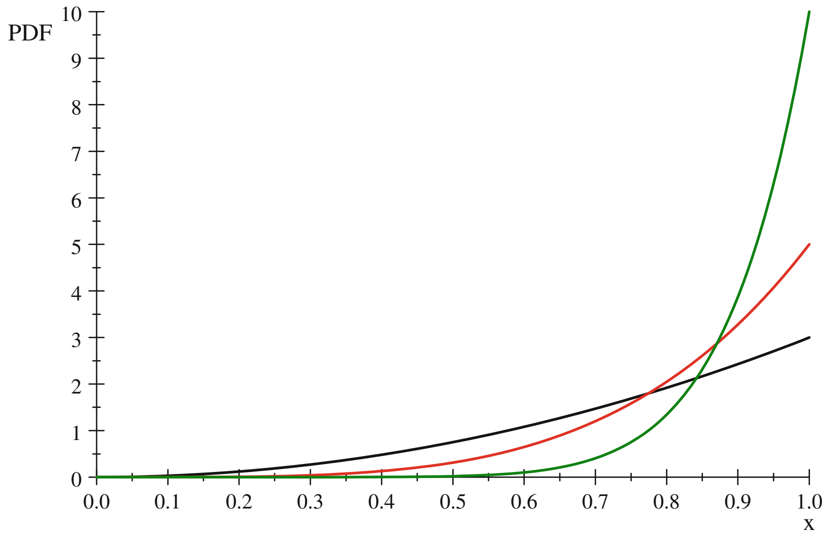

Fig. 1.4 PDFs $f_{3,3}(x)$—*black*, $f_{5,5}(x)$—*red*, $f_{10.19}(x)$—*green*

The limiting distributions of standardized $X_{1,n}$ and $X_{m,m}$ are given in Chap. 3.

Example 1.3 Rayleigh distribution.

Suppose that $X_1, X_2, \ldots, X_n$ are n i.i.d. random variables with cdf

$$F(x) = 1 - e^{-\frac{x^2}{2}}, \quad x \geq 0.$$

We have the pdfs $f_{1,n}(x)$ of $X_{1,n}$ and $f_{n.n}(x)$ are respectively

$$f_{1,n}(x) = nxe^{-\frac{nx^2}{2}}, \quad x \geq 0.,$$

and

$$f_{1,n}(x) = nx(1 - e^{-\frac{x^2}{2}},)^{n-1}e^{-\frac{x^2}{2}}, \quad x \geq 0.$$

The pdfs pf $X_{1.n}$ and $X_{n,n}$ are given respectively in Figs. 1.5 and 1.6 for n = 3, 5 and 10.

Example 1.4 Pareto distribution.

Suppose that $X_1, X_2, \ldots, X_n$ are n i.i.d. random variables with cdf $F(x) = 1 - \frac{1}{x^2}$, $x \geq 1$. We have the pdfs $f_{1,n}(x)$ of $X_{1,n}$ and $f_{n.n}(x)$ are respectively

$$f_{1,n}(x) = \frac{2n}{x^{2n+1}} \quad x \geq 1,$$

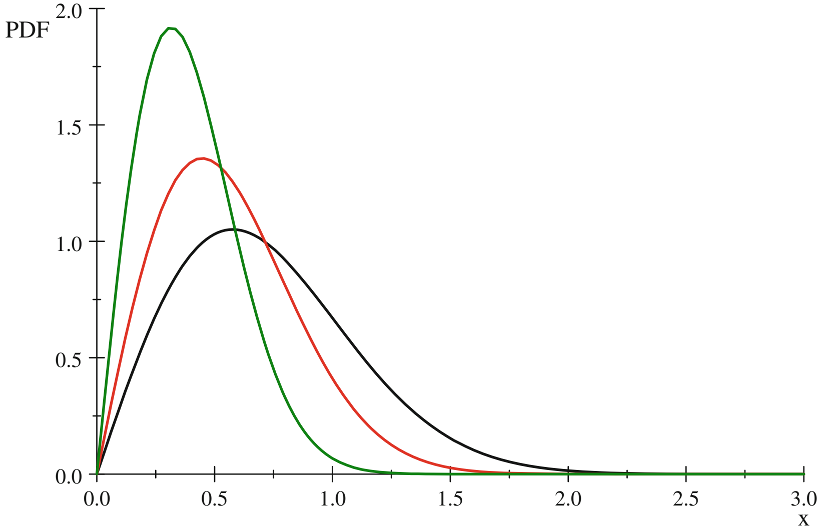

Fig. 1.5 PDFs $f_{3,3}(x)$—*black*, $f_{3,5}(x)_f$—*red*, $f_{1.19}(x)$—*green*

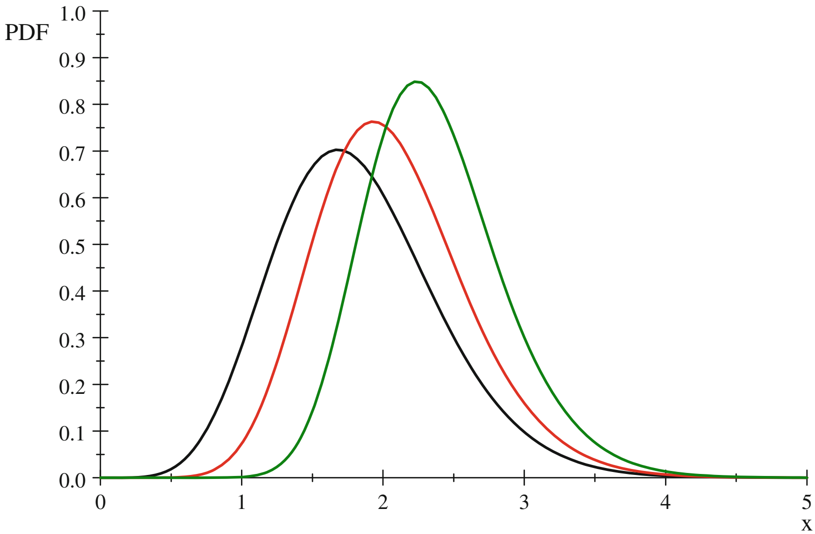

Fig. 1.6 PDFs $f_{3,3}(x)$—*black*, $f_{5,5}(x)$—*red*, $f_{10.19}(x)$—*green*

and

$$f_{n,n}(x) = \frac{2n}{x^3}\left(1 - \frac{1}{x^2},\right)^{n-1}, \quad x \geq 1$$

The pdfs pf $X_{1.n}$ and $X_{n,n}$ are given respectively in Figs. 1.7 and 1.8 for n = 3, 5 and 10.

The joint pdf $f_{r,s,n}$ of two order statistics $X_{1,n}$ and $X_{n,n}$ (1 ≤ r < s < n) is given by

$$f_{r,s,n}(x, y) = c_{r,s,n}(F(x))^{r-1}(F(y)\_F(x))^{s-r-1}(1 - F(y))^{n-s}f(x)f(y)$$

where $c_{r,s,n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}, 1 \leq r < s \leq n.$

The conditional distribution of $X_{s,n}|X_{r,n}$.

Let $f_{s|r,n}$ be the pdf of $X_{s,n}|X_{r,n}$, then

$$\begin{aligned} f_{s\,|\,r,n}(y\,|\,x) &= \frac{f_{r,s,n(x,y)}}{F_{r,n}(x)} \\ &= \frac{c_{r,s,n}}{c_{r,n}} \frac{(F(y) - F(x))^{s-r-1}(1 - F(y)^{n-s}f(y)}{(1 - F(x))^{n-r}}, \end{aligned}$$

where $1 < r < s < n$, $-\infty < x < y < \infty$ and $c_{r,n} = \frac{n!}{(r-1)!(n-r)!}$.

The distribution of the difference between two order statistics.

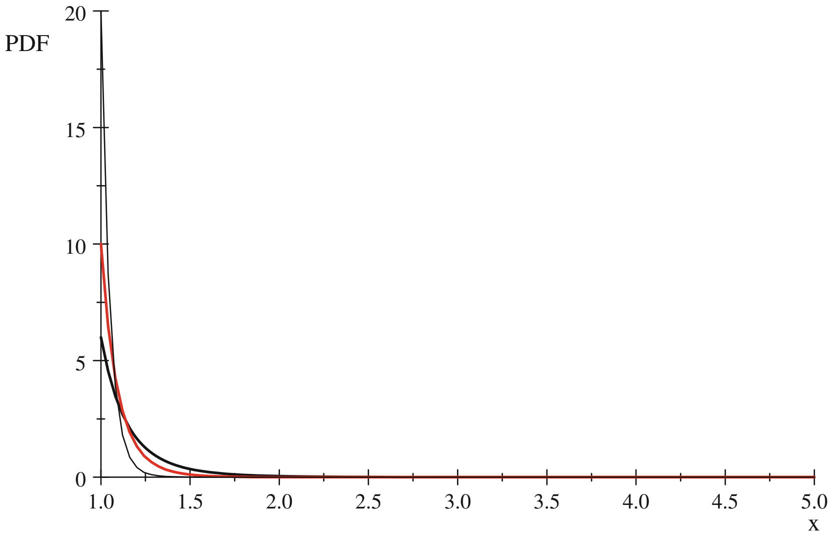

Fig. 1.7 PDFs $f_{3,3}(x)$—*black*, $f_{3,5}(x)_f$—*red*, $f_{1.19}(x)$—*green*

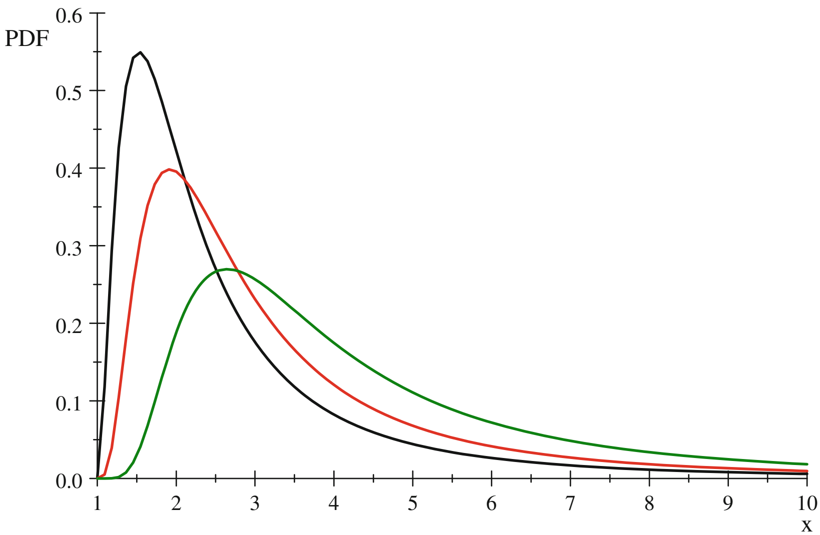

Fig. 1.8 PDFs $f_{3,3}(x)$—*black*, $f_{5,5}(x)$—*red*, $f_{10.19}(x)$—*green*