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Topology and Geometric Group Theory

Ohio State University, Columbus, USA, 2010–2011

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Preface

During the academic year 2010–2011, the Ohio State University Mathematics Department hosted a special year on geometric group theory. Over the course of the year, four-week-long workshops, two weekend conferences, and a week-long conference were held, each emphasizing a different aspect of topology and/or geometric group theory. Overall, approximately 80 international experts passed through Columbus over the course of the year, and the talks covered a large swath of the current research in geometric group theory. This volume contains contributions from the workshop on "Topology and geometric group theory," held in May 2011.

One of the basic questions in manifold topology is the Borel Conjecture, which asks whether the fundamental group of a closed aspherical manifold determines the manifold up to homeomorphism. The foundational work on this problem was carried out in the late 1980s by Farrell and Jones, who reformulated the problem in terms of the K-theoretic and L-theoretic Farrell–Jones Isomorphism Conjectures (FJIC). In the mid-2000s, Bartels, Lück, and Reich were able to vastly extend the techniques of Farrell and Jones. Notably, they were able to establish the FJICs (and hence the Borel Conjecture) for manifolds whose fundamental groups were Gromov hyperbolic. Lück reported on this progress at the 2006 ICM in Madrid. At the Ohio State University workshop, **Arthur Bartels** gave a series of lectures explaining their joint work on the FJICs. The write-up of these lectures provides a gentle introduction to this important topic, with an emphasis on the techniques of proof.

Staying on the theme of the Farrell–Jones Isomorphism Conjectures, **Daniel Juan-Pineda** and **Jorge Sánchez Saldaña** contributed an article in which both the K- and L-theoretic FJIC are verified for the braid groups on surfaces. These are the fundamental groups of configuration spaces of finite tuples of points, moving on the surface. Braid groups have been long studied, both by algebraic topologists, and by geometric group theorists.

A major theme in geometric group theory is the study of the behavior "at infinity" of a space (or group). This is a subject that has been studied by geometric topologists since the 1960s. Indeed, an important aspect of the study of open manifolds is the topology of their ends. The lectures by **Craig Guilbault** present the state of the art on these topics. These lectures were subsequently expanded into a graduate course, offered in Fall 2011 at the University of Wisconsin (Milwaukee).

An important class of examples in geometric group theory is given by CAT(0) cubical complexes and groups acting geometrically on them. Interest in these has grown in recent years, due in large part to their importance in 3-manifold theory (e.g., their use in Agol and Wise's resolution of Thurston's virtual Haken conjecture). A number of foundational results on CAT(0) cubical spaces were obtained in Michah Sageev's thesis. In his contributed article **Daniel Farley** gives a new proof of one of Sageev's key results: any hyperplane in a CAT(0) cubical complex embeds and separates the complex into two convex sets.

One of the powers of geometric group theory lies in its ability to produce, through geometric or topological means, groups with surprising algebraic properties. One such example was Burger and Mozes' construction of finitely presented, torsion-free simple groups, which were obtained as uniform lattices inside the automorphism group of a product of two trees (a CAT(0) cubical complex!). The article by **Pierre-Emmanuel Caprace** and **Bertrand Rémy** introduces a geometric argument to show that some nonuniform lattices inside the automorphism group of a product of trees are also simple.

An important link between algebra and topology is provided by the cohomology functors. Our final contribution, by **Peter Kropholler**, contributes to our understanding of the functorial properties of group cohomology. He considers, for a fixed group G, the set of integers n for which the group cohomology functor $H^n(G, -)$ commutes with certain colimits of coefficient modules. For a large class of groups, he shows this set of integers is always either finite or cofinite.

We hope these proceedings provide a glimpse of the breadth of mathematics covered during the workshop. The editors would also like to take this opportunity to thank all the participants at the workshop for a truly enjoyable event.

Columbus, OH, USA
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James Fowler
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Chapter 1
On Proofs of the Farrell–Jones Conjecture

Arthur Bartels

Abstract These notes contain an introduction to proofs of Farrell–Jones Conjecture for some groups and are based on talks given in Ohio, Oxford, Berlin, Shanghai, Münster and Oberwolfach in 2011 and 2012.

Keywords K-theory · L-theory · Controlled topology · Controlled algebra · Geodesic flow · CAT(0)-Geometry

Introduction

Let R be a ring and G be a group. The Farrell–Jones Conjecture [25] is concerned with the K- and L-theory of the group ring $R[G]$. Roughly it says that the K- and L-theory of $R[G]$ is determined by the K- and L-theory of the rings $R[V]$ where V varies over the family of virtually cyclic subgroups of G and group homology. The conjecture is related to a number of other conjectures in geometric topology and K-theory, most prominently the Borel Conjecture. Detailed discussions of applications and the formulation of this conjecture (and related conjectures) can be found in [10, 32–35].

These notes are aimed at the reader who is already convinced that the Farrell–Jones Conjecture is a worthwhile conjecture and is interested in recent proofs [3, 6, 9] of instances of this Conjecture. In these notes I discuss aspects or special cases of these proofs that I think are important and illustrating. The discussion is based on talks given over the last two years. It will be much more informal than the actual proofs in the cited papers, but I tried to provide more details than I usually do in talks. I took the liberty to express opinion in some remarks; the reader is encouraged to disagree with me. The cited results all build on the seminal work of Farrell and Jones surrounding their conjecture, in particular, their introduction of the geodesic flow as a tool in K- and L-theory [23]. Nevertheless, I will not assume that the reader is already familiar with the methods developed by Farrell and Jones.

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A brief summary of these notes is as follows. Section 1.1 contains a brief discussion of the statement of the conjecture. The reader is certainly encouraged to consult [10, 32–35] for much more details, motivation and background. Section 1.2 contains a short introduction to geometric modules that is sufficient for these notes. Three axiomatic results, labeled Theorems A, B and C, about the Farrell–Jones Conjecture are formulated in Sect. 1.3. Checking for a group G the assumptions of these results is never easy. Nevertheless, the reader is encouraged to find further applications of them. In Sect. 1.4 an outline of the proof of Theorem A is given. Section 1.5 describes the role of flows in proofs of the Farrell–Jones Conjecture. It also contains a discussion of the flow space for CAT(0)-groups. Finally, in Sect. 1.6 an application of Theorem C to some groups of the form $\mathbb{Z}^n \rtimes \mathbb{Z}$ is discussed.

1.1 Statement of the Farrell–Jones Conjecture

Classifying Spaces for Families

Let G be a group. A family of subgroups of G is a non-empty collection $\mathcal{F}$ of subgroups of G that is closed under conjugation and taking subgroups. Examples are the family Fin of finite subgroups, the family Cyc of cyclic subgroups, the family of virtually cyclic subgroups VCyc, the family Ab of abelian subgroups, the family $\{1\}$ consisting of only the trivial subgroup and the family All of all subgroups. If $\mathcal{F}$ is a family, then the collection $V\mathcal{F}$ of all $V \subseteq G$ which contain a member of $\mathcal{F}$ as a finite index subgroup is also a family. All these examples are closed under abstract isomorphism, but this is not part of the definition. If G acts on a set X then $\{H \leq G \mid X^H \neq \emptyset\}$ is a family of subgroups.

Definition 1.1.1 A G-CW-complex E is called a classifying space for the family $\mathcal{F}$, if E^H is non-empty and contractible for all $H \in \mathcal{F}$ and empty otherwise.

Such a G-CW-complex always exists and is unique up to G-equivariant homotopy equivalence. We often say such a space E is a model for $E_{\mathcal{F}}G$; less precisely we simply write $E = E_{\mathcal{F}}G$ for such a space.

Example 1.1.2 Let $\mathcal{F}$ be a family of subgroups. Consider the G-set $S := \coprod_{F \in \mathcal{F}} G/F$. The full simplicial complex $\Delta(S)$ spanned by S (i.e., the simplicial complex that contains a simplex for every non-empty finite subset of S) carries a simplicial G-action. The isotropy groups of vertices of $\Delta(S)$ are all members of $\mathcal{F}$, but for an arbitrary point of $\Delta(S)$ the isotropy group will only contain a member of $\mathcal{F}$ as a finite index subgroup. The first barycentric subdivision of $\Delta(S)$ is a G-CW-complex and it is not hard to see that it is a model for $E_{V\mathcal{F}}G$.

This construction works for any G-set S such that $\mathcal{F} = \{H \leq G \mid S^H \neq \emptyset\}$.

More information about classifying spaces for families can be found in [31].

Statement of the Conjecture

The original formulation of the Farrell–Jones Conjecture [25] used homology with coefficients in stratified and twisted Ω-spectra. We will use the elegant formulation of the conjecture developed by Davis and Lück [21]. Given a ring R and a group G Davis–Lück construct a homology theory for G-spaces

$$X \mapsto H_*^G(X; \mathbf{K}_R)$$

with the property that $H_*^G(G/H; \mathbf{K}_R) = K_*(R[H])$.

Definition 1.1.3 Let $\mathcal{F}$ be a family of subgroups of G. The projection $E_{\mathcal{F}}G \twoheadrightarrow G/G$ to the one-point G-space G/G induces the $\mathcal{F}$-assembly map

$$\alpha_{\mathcal{F}} \colon H_*^G(E_{\mathcal{F}}G; \mathbf{K}_R) \to H_*^G(G/G; \mathbf{K}_R) = K_*(R[G]).$$

Conjecture 1.1.4 (Farrell–Jones Conjecture) *For all groups G and all rings R the assembly map α_{VCyc} is an isomorphism.*

Remark 1.1.5 Farrell–Jones really only conjectured this for $R = \mathbb{Z}$. Moreover, they wrote (in 1993) that they regard this and related conjectures *only as estimates which best fit the known data at this time*. It still fits all known data today.

For arbitrary rings the conjecture was formulated in [2]. The proofs discussed in this article all work for arbitrary rings and it seems unlikely that the conjecture holds for $R = \mathbb{Z}$ and all groups, but not for arbitrary rings.

Remark 1.1.6 Let $\mathcal{F}$ be a family of subgroups of G. If R is a ring such that $K_*R[F] = 0$ for all $F \in \mathcal{F}$, then $H_*^G(E_{\mathcal{F}}G; \mathbf{K}_R) = 0$.

In particular, the Farrell–Jones Conjecture predicts the following: if R is a ring such that $K_*(R[V]) = 0$ for all $V \in \mathrm{VCyc}$ then $K_*(R[G]) = 0$ for all groups G.

Transitivity Principle

The family in the Farrell–Jones Conjecture is fixed to be the family of virtually cyclic groups. Nevertheless, it is beneficial to keep the family flexible, because of the following transitivity principle [25, A. 10].

Proposition 1.1.7 *Let $\mathcal{F} \subseteq \mathcal{H}$ be families of subgroups of G. Write $\mathcal{F} \cap H$ for the family of subgroups of H that belong to $\mathcal{F}$. Assume that*

(a) $\alpha_{\mathcal{H}} \colon H_*^G(E_{\mathcal{H}}G; \mathbf{K}_R) \to K_*(R[G])$ is an isomorphism,
(b) $\alpha_{\mathcal{F}\cap H} \colon H_*^H(E_{\mathcal{F}\cap H}H; \mathbf{K}_R) \to K_*(R[H])$ is an isomorphism for all $H \in \mathcal{H}$.

Then $\alpha_{\mathcal{F}} \colon H_*^G(E_{\mathcal{F}}G; \mathbf{K}_R) \to K_*(R[G])$ is an isomorphism.

Remark 1.1.8 The following illustrates the transitivity principle.

Assume that R is a ring such that $K_*(R[F]) = 0$ for all $F \in \mathcal{F}$. Assume moreover that the assumptions of Proposition 1.1.7 are satisfied. Combining Remark 1.1.6 with (b) we conclude $K_*(R[H]) = 0$ for all $H \in \mathcal{H}$. Then combining Remark 1.1.6 with (a) it follows that $K_*(R[G]) = 0$.

Remark 1.1.9 The transitivity principle can be used to prove the Farrell–Jones Conjecture for certain classes by induction. For example the proof of the Farrell–Jones Conjecture for $\mathrm{GL_n}(\mathbb{Z})$ uses an induction on n [11]. Of course the hard part is still to prove in the induction step that $\alpha_{\mathcal{F}_{n-1}}$ is an isomorphism for $\mathrm{GL_n}(\mathbb{Z})$ where the family $\mathcal{F}_{n-1}$ contains only groups that can be build from $\mathrm{GL_{n-1}}(\mathbb{Z})$ and poly-cyclic groups. The induction step uses Theorem B from Sect. 1.3. See also Remark 1.5.18.

More General Coefficients

Farrell and Jones also introduced a generalization of their conjecture now called the fibered Farrell–Jones Conjecture. This version of the conjecture is often not harder to prove than the original conjecture. Its advantage is that it has better inheritance properties. An alternative to the fibered conjecture is to allow more general coefficients where the group can act on the ring. As K-theory only depends on the category of finitely generated projective modules and not on the ring itself, it is natural to also replace the ring by an additive category. We briefly recall this generalization from [13].

Let $\mathcal{A}$ be an additive category with a G-action. There is a construction of an additive category $\mathcal{A}[G]$ that generalizes the twisted group ring for actions of G on a ring R. (In the notation of [13, Definition 2.1] this category is denoted as $\mathcal{A} *_G G/G$; $\mathcal{A}[G]$ is a more descriptive name for it.) There is also a homology theory $H^G_*(-; \mathbf{K}_{\mathcal{A}})$ for G-spaces such that $H^G_*(G/H; \mathbf{K}_{\mathcal{A}}) = K_*(\mathcal{A}[H])$. Therefore there are assembly maps

$$\alpha_{\mathcal{F}} \colon H^G_*(E_{\mathcal{F}}G; \mathbf{K}_{\mathcal{A}}) \to H^G_*(G/G; \mathbf{K}_{\mathcal{A}}) = K_*(\mathcal{A}[G]).$$

Conjecture 1.1.10 (Farrell–Jones Conjecture with coefficients) *For all groups G and all additive categories $\mathcal{A}$ with G-action the assembly map α_{VCyc} is an isomorphism.*

An advantage of this version of the conjecture is the following inheritance property.

Proposition 1.1.11 *Let $N \rightarrowtail G \twoheadrightarrow Q$ be an extension of groups. Assume that Q and all preimages of virtually cyclic subgroups under $G \twoheadrightarrow Q$ satisfies the Farrell–Jones Conjecture with coefficients 1.1.10. Then G satisfies the Farrell–Jones Conjecture with coefficients 1.1.10.*

Remark 1.1.12 Proposition 1.1.11 can be used to prove the Farrell–Jones Conjecture with coefficients for virtually nilpotent groups using the conjecture for virtually abelian groups, compare [10, Theorem 3.2].

It can also be used to reduce the conjecture for virtually poly-cyclic groups to irreducible special affine groups [3, Sect. 3]. The latter class consists of certain groups G for which there is an exact sequence $\Delta \to G \to D$, where D is infinite cyclic or the infinite dihedral group and Δ is a crystallographic group.

Remark 1.1.13 For additive categories with G-action the consequence from Remark 1.1.6 becomes an equivalent formulation of the conjecture: A group G satisfies the Farrell–Jones Conjecture with coefficients 1.1.10 if and only if for additive categories $\mathscr{B}$ with G-action we have

$$K_*(\mathscr{B}[V]) = 0 \text{ for all } V \in \mathrm{VCyc} \implies \mathrm{K}_*(\mathscr{B}[\mathrm{G}]) = 0.$$

(This follows from [9, Proposition 3.8] because the obstruction category $\mathscr{O}^G(E_{\mathscr{F}}G; \mathscr{A})$ is equivalent to $\mathscr{B}[G]$ for some $\mathscr{B}$ with $K_*(\mathscr{B}[F]) = 0$ for all $F \in \mathscr{F}$.)

In particular, surjectivity implies bijectivity for the Farrell–Jones Conjecture with coefficients.

Remark 1.1.14 The Farrell–Jones Conjecture 1.1.4 should be viewed as a conjecture about finitely generated groups. If it holds for all finitely generated subgroups of a group G, then it holds for G. The reason for this is that the conjecture is stable under directed unions of groups [27, Theorem 7.1].

With coefficients the situation is even better. This version of the conjecture is stable under directed colimits of groups [4, Corollary 0.8]. Consequently the Farrell–Jones Conjecture with coefficients holds for all groups if and only if it holds for all *finitely presented* groups, compare [1, Corollary 4.7]. It is therefore a conjecture about finitely presented groups.

Despite the usefulness of this more general version of the conjecture I will mostly ignore it in this paper to keep the notation a little simpler.

L-Theory

There is a version of the Farrell–Jones Conjecture for L-Theory. For some applications this is very important. For example the Borel Conjecture asserting the rigidity of closed aspherical topological manifolds follows in dimensions ≥ 5 via surgery theory from the Farrell–Jones Conjecture in K- and L-theory. The L-theory version of the conjecture is very similar to the K-theory version. Everything said so far about the K-theory version also holds for the L-theory version.

For some time proofs of the L-theoretic Farrell–Jones conjecture have been considerably harder than their K-theoretic analoga. Geometric transfer arguments used in L-theory are considerably more involved than their counterparts in K-theory. A change that came with considering arbitrary rings as coefficients in [2], is that transfers became more algebraic. It turned out [6] that this more algebraic point of view allowed for much easier L-theory transfers. (In essence, because the world of chain complexes with Poincaré duality is much more flexible than the world of manifolds.) This is elaborated at the end of Sect. 1.4.

I think that it is fair to say, as far as proofs are concerned, there is as at the moment no significant difference between the K-theoretic and the L-theoretic Farrell–Jones Conjecture. For this reason L-theory is not discussed in much detail in these notes.

1.2 Controlled Topology

The Thin h-Cobordism Theorem

An *h-cobordism* W is a compact manifold whose boundary is a disjoint union $\partial W = \partial_0 W \amalg \partial_1 W$ of closed manifolds such that the inclusions $\partial_0 W \to W$ and $\partial_1 W \to W$ are homotopy equivalences. If $M = \partial_0 W$, then we say W is an h-cobordism over M. If W is homeomorphic to $M \times [0, 1]$, then W is called trivial.

Definition 1.2.1 Let M be a closed manifold with a metric d. Let $\varepsilon \geq 0$.

An h-cobordism W over M is said to be ε-controlled over M if there exists a retraction $p\colon W \to M$ for the inclusion $M \to W$ and a homotopy $H\colon \mathrm{id_W} \to \mathrm{p}$ such that for all $x \in W$ the track

$$\{p(H(t, x)) \mid t \in [0, 1]\} \subseteq M$$

has diameter at most ε.

Remark 1.2.2 Clearly, the trivial h-cobordism is 0-controlled. Thus it is natural to think of being ε-controlled for small ε as being close to the trivial h-cobordism.

The following theorem is due to Quinn [39, Theorem 2.7]. See [18, 19, 28] for closely related results by Chapman and Ferry.

Theorem 1.2.3 (Thin ***h***-cobordism theorem) *Assume* $\dim M \geq 5$. *Fix a metric d on M (generating the topology of M).*

Then there is $\varepsilon > 0$ such that all ε-controlled h-cobordisms over M are trivial.

Remark 1.2.4 Farrell–Jones used the thin h-cobordism Theorem 1.2.3 and generalizations thereof to study $K_*(\mathbb{Z}[G])$, $* \leq 1$. For example in [23] they used the geodesic flow of a negatively curved manifold M to show that any element in $\mathrm{Wh}(\pi_1 M)$ could be realized by an h-cobordism that in turn had to be trivial by an application of (a generalization of) the thin h-cobordism theorem. Thus $\mathrm{Wh}(\pi_1 M) = 0$. In later papers they replaced the thin h-cobordism theorem by controlled surgery theory and controlled pseudoisotopy theory.

The later proofs of the Farrell–Jones Conjecture that we discuss here do not depend on the thin h-cobordism theorem, controlled surgery theory or controlled pseudoisotopy theory, but on a more algebraic control theory that we discuss in the next subsection.

An Algebraic Analog of the Thin h-Cobordism Theorem

Geometric groups (later also called geometric modules) were introduced by Connell-Hollingsworth [20]. The theory was developed much further by, among others, Quinn and Pedersen and is sometimes referred to as controlled algebra. A very pleasant introduction to this theory is given in [37].

Let R be a ring and G be a group.

Definition 1.2.5 Let X be a free G-space and $p\colon X \to Z$ be a G-map to a metric space with an isometric G-action.

(a) A *geometric* $R[G]$*-module over* X is a collection $(M_x)_{x\in X}$ of finitely generated free R-modules such that the following two conditions are satisfied.

- $M_x = M_{gx}$ for all $x \in X$, $g \in G$.
- $\{x \in X \mid M_x \neq 0\} = G \cdot S_0$ for some finite subset S_0 of X.

(b) Let M and N be geometric $R[G]$-modules over X. Let $f\colon \bigoplus_{x\in X} M_x \to \bigoplus_{x\in X} N_x$ be an $R[G]$-linear map (for the obvious $R[G]$-module structures). Write $f_{x'',x'}$ for the composition

$$M_{x'} \rightarrowtail \bigoplus_{x\in X} M_x \xrightarrow{f} \bigoplus_{x\in X} N_x \twoheadrightarrow N_{x''}.$$

The *support* of f is defined as $\operatorname{supp} f := \{(x'', x') \mid f_{x'',x'} \neq 0\} \subseteq X \times X$. Let $\varepsilon \geq 0$. Then f is said to be ε*-controlled over* Z if

$$d_Z(p(x''), p(x')) \leq \varepsilon \quad \text{for all} \quad (x'', x') \in \operatorname{supp} f.$$

(c) Let M be a geometric $R[G]$-module over X. Let $f\colon \bigoplus_{x\in X} M_x \to \bigoplus_{x\in X} M_x$ be an $R[G]$-automorphism. Then f is said to be an ε*-automorphism over* Z if both f and f^{-1} are ε-controlled over Z.

Remark 1.2.6 Geometric $R[G]$-modules over X are finitely generated free $R[G]$-modules with an additional structure, namely an G-equivariant decomposition into R-modules indexed by points in X. This additional structure is not used to change the notion of morphisms which are still $R[G]$-linear maps. But this structure provides an additional point of view for $R[G]$-linear maps: the set of morphisms between two geometric $R[G]$-modules now carries a filtration by control.

A good (and very simple) analog is the following. Consider finitely generated free R-modules. An additional structure one might be interested in are bases for such modules. This additional information allows us to view R-linear maps between them as matrices.

Controlled algebra is really not much more than working with (infinite) matrices whose index set is a (metric) space. Nevertheless this theory is very useful and flexible.

It is a central theme in controlled topology that sufficiently controlled obstructions (for example Whitehead torsion) are trivial. Another related theme is that assembly maps can be constructed as *forget-control* maps. In this paper we will use a variation of this theme for K_1 of group rings over arbitrary rings. Before we can state it we briefly fix some conventions for simplicial complexes.

Convention 1.2.1 *Let $\mathcal{F}$ be a family of subgroups of G. By a simplicial $(G, \mathcal{F})$-complex we shall mean a simplicial complex E with a simplicial G-action whose isotropy groups $G_x = \{g \in G \mid g \cdot x = x\}$ belong to $\mathcal{F}$ for all $x \in E$.*

Convention 1.2.2 *We will always use the l^1-metric on simplicial complexes. Let $Z^{(0)}$ be the vertex set of the simplicial complex Z. Then every element $z \in Z$ can be uniquely written as $z = \sum_{v \in Z^{(0)}} z_v \cdot v$ where $z_v \in [0, 1]$, all but finitely many z_v are zero and $\sum_{v \in Z^{(0)}} z_v = 1$. The l^1-metric on Z is given by*

$$d^1_Z(z, z') = \sum_{v \in V} |z_v - z'_v|.$$

Remark 1.2.7 If E is a simplicial complex with a simplicial G-action such that the isotropy groups G_v belong to $\mathscr{F}$ for all vertices $v \in E^{(0)}$ of E, then E is a simplicial $(G, V\mathscr{F})$-complex, where $V\mathscr{F}$ consists of all subgroups H of G that admit a subgroup of finite index that belongs to $\mathscr{F}$.

Theorem 1.2.8 (Algebraic thin ***h***-cobordism theorem) *Given a natural number N, there is $\varepsilon_N > 0$ such that the following holds: Let*

(a) Z be a simplicial $(G, \mathscr{F})$-complex of dimension at most N,
(b) $p\colon X \to Z$ be a G-map, where X is a free G-space,
(c) M be a geometric $R[G]$-module over X,
(d) $f\colon M \to M$ be an ε_N-automorphism over Z (with respect to the l^1-metric on Z).

Then the K_1-class $[f]$ of f belongs to the image of the assembly map

$$\alpha_{\mathscr{F}}\colon H^G_1(E_{\mathscr{F}}G; \mathbf{K}_R) \to K_1(R[G]).$$

Remark 1.2.9 I called Theorem 1.2.8 the algebraic thin h-cobordism theorem here, because it can be used to prove the thin h-cobordism theorem. Very roughly, this works as follows. Let W be an ε-thin h-cobordism over M. Let $G = \pi_1 M = \pi_1 W$. The Whitehead torsion of W can be constructed using the singular chain complexes of the universal covers $\widetilde{W}$ and $\widetilde{M}$. This realizes the Whitehead torsion $\tau_W \in \mathrm{Wh}(G)$ of W by an $\widetilde{\varepsilon}$-automorphism f_W over $\widetilde{M}$, i.e. $[f_W]$ maps to τ_W under $K_1(\mathbb{Z}[G]) \to \mathrm{Wh}(G)$. Moreover, $\widetilde{\varepsilon}$ can be explicitly bounded in terms of ε, such that $\widetilde{\varepsilon} \to 0$ as $\varepsilon \to 0$. Because $\widetilde{M}$ is a free $G = \pi_1 M$-space it follows from Theorem 1.2.8 that $[f_W]$ belongs to the image of the assembly map $\alpha\colon H^G_1(EG, \mathbf{K}_{\mathbb{Z}}) \to K_1(\mathbb{Z}[G])$. But $\mathrm{Wh}(G)$ is the cokernel of α and therefore $\tau_W = 0$. This reduces the thin h-cobordism theorem to the s-cobordism theorem.

I believe that—at least in spirit—this outline is very close to Quinn's proof in [39].

Remark 1.2.10 If $f\colon M \to M'$ is ε-controlled over Z and $f'\colon M' \to M''$ is ε'-controlled over Z, then their composition $f' \circ f$ is $\varepsilon + \varepsilon'$-controlled. In particular, there is no category whose objects are geometric modules and whose morphisms are ε-controlled for fixed (small) ε. However, there are very elegant substitutes for this ill-defined category. These are built by considering a variant of the theory over an open cone over Z and taking a quotient category. In this quotient category every morphisms has for every $\varepsilon > 0$ an ε-controlled representative. Pedersen–Weibel [38] used this to construct homology of a space E with coefficients in the K-theory spectrum as the K-theory of an additive category. Similar constructions can be used to