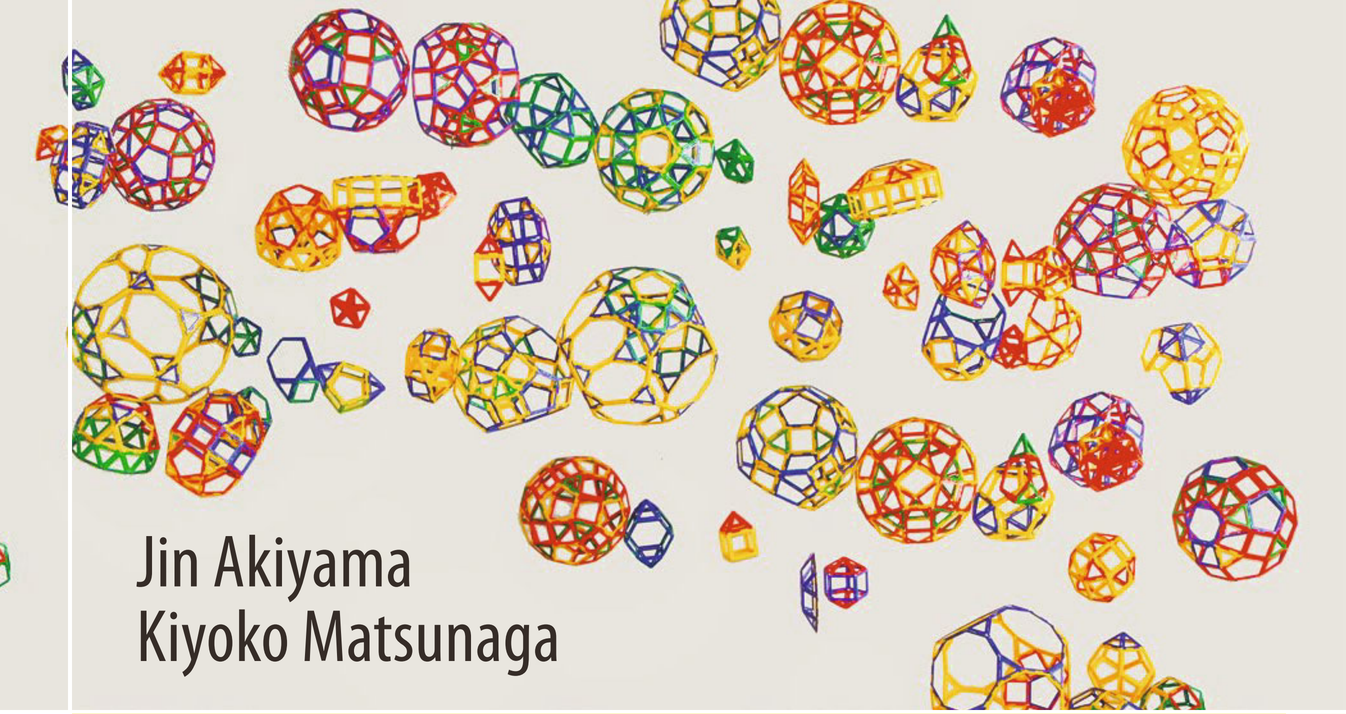

Jin Akiyama
Kiyoko Matsunaga

Treks into Intuitive Geometry

The World of Polygons and Polyhedra

Springer

Treks into Intuitive Geometry

Jin Akiyama • Kiyoko Matsunaga

Treks into Intuitive Geometry

The World of Polygons and Polyhedra

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Preface

Mathematical truth remains uninfluenced by speculation or perspective, and its truth and reality transcend time and space. Mathematical theories do not allow for vagueness or compromise. Therefore, once a theory has been constructed on the basis of its proof, it cannot be altered by anyone. Furthermore, everyone can admire the beautiful fruit borne of mathematical theories, no matter what their race, age, religious beliefs, historical views or perception of life may be.

The authors of this book aim to demonstrate the elegant mathematical proofs of various theorems to as many people as possible, especially to university students. Additionally, this book was written so that readers can enjoy discovering new theories, regardless of their simplicity.

One of the motivating factors behind this book was the fact that we have been involved in mathematical programs on television and radio for over two decades and have found that many viewers and listeners were tired of the dull math taught in school. We strove to engage the audience's senses to demonstrate how wonderful math can be. In the programs we tried as much as possible to avoid any top-down teaching methods. We wanted viewers to discover the processes by which the founders of the theories came to their various conclusions – their trials, errors, and tribulations.

We have always kept in mind the words of Ernest Rutherford, "If you can't explain a result in simple, nontechnical terms, then you don't really understand it." In other words, "you don't really understand it" means not that your result is wrong, but that you do not fully understand its origin, meaning, or implications.

Fortunately, as a result of our approach, many viewers have told us that for the first time in their lives they understood the joy of mathematics. However, there are still many theories that are difficult to grasp no matter how much they are broken down. Generally speaking, in order to understand complex theories it is imperative to have a high level of knowledge and understanding of abstract concepts. At the same time we cannot enjoy or be grateful for such theories unless we fully recognize the significance of the theories themselves.

There are large, beautiful, unusual flowers on the peaks of high mountains, but there are also common flowers such as violets, bellflowers and dandelions that are just as beautiful on the small hills near our houses. Similarly, in this book we highlight and focus on wisdom taken from daily life – such as examples from various works of art, traditional crafts, patterns that appear in nature, music, and mathematical mechanisms in techniques that craftspeople use. This book is written in a style that unearths the mathematical theories buried in our everyday lives. Our goal is for our readers to enjoy the process of applying mathematical rules to beautiful art and design by highlighting examples of wonders and mysteries from our daily lives. To fulfill these aims, this book deals with polygons and polyhedra that can be found around us. There are detailed explanations concerning their nets, cross sections, surface areas, and volumes – as well as their filling properties, their transformations, and their decomposabilities.

In this book, Kyuta – a student – is led by a geometry researcher –Gen – through a forest of geometry. Through a series of discussions they solve mathematical problems step by step. They trek through this vivid forest to dig up mysterious treasure boxes.

Intuitive geometry is not a field well explored within mathematics. The term "Intuitive Geometry" does not appear in the AMS subject classification. It was coined by Hungarian mathematician László Fejes Tóth to refer to the kind of geometry which, in Hilbert's words, can be explained to and can appeal to the "man on the street."

This book allows people to enjoy intuitive geometry casually and instinctively. It does not require more than high school level knowledge, but does call for a sense of wonder, intuition, and mathematical maturity.

Now, let us begin our trek to hunt for mathematical treasures in this forest of geometry, where beautiful flowers bloom and small woodland creatures await!

Acknowledgement

The study of intuitive geometry and discrete geometry in Japan has a history dating back to the 1980s.

In those days, David Avis, Vašek Chvátal, Michel Deza, Paul Erdős, Peter Frankl, Ron Graham and Jorge Urrutia played the role of missionaries by giving lectures on this academic field at their respective universities.

Thanks to their efforts, the first JCDCG (Japan Conference on Discrete and Computational Geometry) was held in Tokyo in 1997, and since then it has been held annually; at the time of this writing up to the 19th conference this year (2015).

Guest speakers at these conferences included Takao Asano, Tetsuo Asano, David Avis, Imre Barany, Sergey Bereg, William Y. C. Chen, Vašek Chvátal, William Cook, Erik Demaine, Martin Demaine, Nikolai Dolbilin, Rudolf Fleischer, Greg Frederickson, Ferran Hurtado, Hiroshi Imai, Hiro Ito, Mikio Kano, Naoki Katoh, Ken-ichi Kawarabayashi, David Kirkpatrick, Stefan Langerman, Xueliang Li, Hiroshi Maehara, Jiri Matoušek, Alberto Márquez, Frank Nielsen, Joseph O'Rourke, János Pach, Rom Pinchasi, Pedro Ramos, Eduardo Rivera-Campo, Akira Saito, Jonathan Shewchuk, Gyula Solymosi, William Steiger, Kokichi Sugihara, Endre Szemerédi, Xuehou Tan, Takeshi Tokuyama, Godfried Toussaint, Géza Tóth, Ryuhei Uehara, Jorge Urrutia, and Chuanming Zong as well as many other respected speakers. We owe many topics dealt with in this book to the series of conferences.

Additionally, research meetings specialized in intuitive geometry were started by Jin-ichi Itoh in 2009 at Kumamoto University. At the time of this writing, they are holding their seventh meeting this year (2015). With international support from various people, this field has also been able to grow in Japan and many universities have now courses on this subject.

This book was actually written based on the lecture notes of the Intuitive Geometry and Discrete Geometry course taught at Tokyo University of Science.

The course was programmed for junior and senior university students majoring in math. In the class, many of the models and works that appeared in this book were used as visual aids, in order to enhance the understanding of abstract concepts. Mathematical exercises enabled the students to utilize the theories in this book to create mathematical works. It was necessary for students to go over the theories they learned repeatedly in order to complete the artwork that required mathematical accuracy.

Although it had a somewhat different atmosphere than that of a normal math class, many of the students were absorbed in their work. I would like to express my thanks to my dedicated students who completed artworks of mathematical beauty.

Last but not least, we would like to extend thanks for all the support we received in completing this book. We would like to express our deep and sincere gratitude to Vašek Chvátal, Alex Cole, Erik Demaine, Martin Demaine, Agnes Garciano, Mark Goldsmith, Mikio Kano, Keiko Kotani, Stefan Langerman, Hiroshi Maehara, Pauline Ann Mangulabnan, Gisaku Nakamura, Chie Nara, Amy Ota, David Rappaport, Mari-Jo Ruiz, Ikuro Sato, Jorge Urrutia, Margaret Schroeder and Teruhisa Sugimoto who corrected the manuscript and gave valuable comments.

We would also like to gratefully acknowledge Kaoru Yumi, who provided many illustrations and pictures of art; Yasuyuki Yamaguchi and Hynwoo Seong, who created the complicated drawings and diagrams; and Toru Takemura, Etsuko Watanabe, and Hidetoshi Okazaki, who contributed an enormous amount of illustrations and typing.

Contents

Chapter 1 Art From Tiling Patterns **1**

1. Geometric Patterns 1
2. Tilings 9
3. Conway Criterion 17
4. Math Behind Masterpieces 24

Appendix 1.1.1 30
Appendix 1.2.1 33
References 34

Chapter 2 The Tile-Maker Theorem and Its Applications to Art and Designs **35**

1. How to Draw Repeated Patterns Like Escher's 35
2. Tile-Makers 38
3. Characteristics of a Net of an Isotetrahedron 41
4. Polyhedral Stampers 45
5. Dihedral Stampers and Dihedral Tile-Maker Theorem 49
6. From Shark Heads to Tile-Makers 53
7. Exhibition of Collage Art 54

Appendix 2.3.1 60
Appendix 2.4.1 62
Appendix 2.5.1 63
References 64

Chapter 3 Patchwork **67**

1. Patchwork — Equidecomposable Pairs of Polygons 67
2. How to Turn a Polygon into a Rectangle 74
3. Theorem of Equidecomposability 78

References 80

Chapter 4 Reversible Pairs of Figures **81**

1. Envelope Magic 81
2. The Envelope Magic Trick 84
3. Trapezoidal Envelope Magic 88
4. Dudeney's Haberdasher's Puzzle 92
5. Reversible Transformation of Figures 97
6. Tessellability and Shapes of Trunks 107
7. Miscellaneous Properties of Uni-Trunk Holders 112

8. Conway Polygons ··· 118
9. Art Based on Reversible Pairs of Figures ··· 125
Appendix 4.4.1 ··· 127
Appendix 4.5.1 ··· 128
Appendix 4.7.1 ··· 130
Appendix 4.8.1 ··· 131
Appendix 4.8.2 ··· 137
References ··· 141

Chapter 5 Platonic Solids **143**
1. The Platonic Solids ··· 143
2. Euler's Formula for Polyhedra ··· 145
3. Plato's Image for Platonic Solids ··· 147
4. Duality Between a Pair of Polyhedra ··· 150
5. The Cycle of Platonic Solids ··· 151
Appendix 5.2.1 ··· 153
Appendix 5.3.1 ··· 156
References ··· 158

Chapter 6 Cross-Sections of Polyhedra **159**
1. CT Scanners and 3-D Copiers ··· 159
2. Cross-Sections of a Cube ··· 161
3. Universal Tilers ··· 165
4. Mitsubishi and the Star of David ··· 166
References ··· 174

Chapter 7 Symmetry of Platonic Solids **175**
1. Diagonal Weights ··· 175
2. A Soccer Ball and a Fullerene ··· 180
3. The Platonic Solids through Reflecting Mirrors ··· 183
Appendix 7.1.1 ··· 189
Appendix 7.1.2 ··· 189
Appendix 7.1.3 ··· 190
References ··· 191

Chapter 8 Double Duty Solids **193**
1. TetraPak® ··· 193
2. Double Duty Solids ··· 197
3. Finding More Double Duty Solids ··· 200
4. Eleven Types of e-Nets of a Cube ··· 205
5. References ··· 212

Chapter 9 Nets of Small Solids with Minimum Perimeter Lengths **213**
1. Minimum Steiner Tree Problem ··· 213
2. Melzak's Algorithm ··· 219
3. Minimum Perimeter Nets for a Regular Octahedron $\mathfrak{O}$ ··· 221
4. Minimum Perimeter Net for a Cube $\mathcal{C}$ ··· 223
5. Soap Experiments for Minimum Steiner Trees ··· 225
6. Soap Experiments for Nets with Minimum Perimeter Lengths for Small Solids ··· 227

Appendix 9.1.1 231
Appendix 9.2.1 232
References 233

Chapter 10 Tessellation Polyhedra **235**
1. Regular Faced Polyhedra 235
2. Tessellation Polyhedra 238
3. Art 245
Appendix 10.2.1 254
References 255

Chapter 11 Universal Measuring Boxes **257**
1. How to Measure Water Using a 6 dL Box 257
2. Changing the Shape of the Box 262
3. Orthogonal Measuring Boxes 265
4. An Application of the Ternary System 272
5. Non-Orthogonal Universal Measuring Boxes 277
Appendix 11.4.1 282
References 283

Chapter 12 Wrapping a Box **285**
1. Wrapping Technique of the Experts 285
2. Point Symmetric Skew Wrapping 286
3. Patchwork Surgery 290
4. Investigating Efficient Wrappings 295
5. Six Remarkable Wrappings Obtained by Patchwork Surgery 300
6. The Most Saving But Mummy-Like Wrapping 303
References 308

Chapter 13 Bees, Pomegranates and Parallelohedra **309**
1. How to Pack Cans Efficiently 309
2. Honeycombs and Pomegranate Seeds 312
3. Applications of Sphere Packing 315
4. Rhombic Dodecahedron 319
5. Truncated Octahedron 320
6. Lattices and Voronoi Domains 323
7. Parallelohedra 329
References 333

Chapter 14 Reversible Polyhedra **335**
1. A Pig to a Ham, a Fox to a Snake, and Panda Magic 335
2. Transformation of a Pig to Ham 339
3. How to Make a Fox-Snake Solid 341
4. Make a Fox-Snake Solid with Paper 347
5. Making Panda Magic! 349
6. How to Make Reversible Polyhedra 350
7. Who Will Come Out of the U.F.O.? 352
8. P_1-lattices and (P, Q)-Chimera Superimpositions 354
9. P_1-Reversible Pairs Among Canonical Parallelohedra 359

10. Reversibility for Parallelohedra and the Double-Reversal-Plates Method ····· 367
References ····· 372

Chapter 15 Elements of Polygons and Polyhedra 373
1. The Power of Triangulation ····· 373
2. Element Sets and Element Numbers ····· 378
3. Hilbert's Third Problem and Dehn Theorem ····· 382
4. An Element Set and the Element Number for Platonic Solids ····· 388
5. Another Element Set for the Platonic Solids ····· 390
Appendix 15.4.1 ····· 395
References ····· 398

Chapter 16 The Pentadron 399
1. Atoms for Parallelohedra ····· 399
2. Another Way to Find a Pair of Pentadra ····· 406
3. The Chart for Composing Five Parallelohedra with Pentadra ····· 407
4. Pentadral Complices ····· 410
5. A Bridge Between Platonic Solids and Parallelohedra ····· 412
6. Art and Design ····· 413
Appendix 16.1.1 ····· 414
References ····· 416

Figure Permissions ····· 417
© J Art 2015 ····· 418
Copyright ····· 419
Endnote ····· 420
Index ····· 421
Symbols ····· 424

About this Book

The characteristics of this book are the following:

(a) The theorems and formulas that are presented encourage the reader to discover for him or herself (heuristic approach).

(b) Most of the theorems are presented with a story of how the authors were inspired and came up with the idea for the theorem.

(c) This book introduces not only key results and tools in each topic, but also many original results obtained by the authors. This is done through casual conversation between Gen and Kyuta. Gen (a mathematician), visits Kyuta and teaches him to appreciate the excitement of creating theorems together with art using mathematics. Most of the topics are original and have not been introduced in other books.

(d) The target readers are undergraduate students; however, this book is self-contained and only requires knowledge at the high school level.

(e) Many of the latest unique and beautiful results in geometry (in particular on polygons and polyhedra) and the dynamism of mathematical research history may also captivate adults and even researchers.

(f) Readers can create their own works of art by applying the theorems presented in this book, as the procedures are explained explicitly.

(g) The book is illustrated with color photographs of works of art and design that have been created using the theorems and procedures in the book.

About the Authors

Jin Akiyama is a mathematician at heart. Currently, he is a fellow of the European Academy of Science, the founding editor of the journal *Graphs and Combinatorics*, and the director of the Research Institute of Math Education at Tokyo University of Science. He is particularly interested in graph theory and discrete and solid geometry and has published many papers in these fields. Aside from his research, he is best known for popularizing mathematics, first in Japan and then in other parts of the globe: his lecture series was broadcast on NHK (Japan's only public broadcaster) television and radio from 1991 to 2013. He was a founding member of the Organizing Committee of the UNESCO-sponsored traveling exhibition "Experiencing Mathematics", which debuted in Denmark in 2004. In 2013, he built a hands-on mathematics museum called Akiyama's Math Experience Plaza in Tokyo. He has authored and co-authored more than 100 books, including *Factors and Factorizations of Graphs* (jointly with Mikio Kano, Lect. Notes Math., Springer, 2011) and *A Day's Adventure in Math Wonderland* (jointly with Mari-Jo Ruiz, World Scientific, 2008), which has been translated into nine languages.

Kiyoko Matsunaga is a science writer for mathematics. She has contributed not only through Japanese-language books, but also in NHK TV programs on mathematics, including *Math, I Like It* for elementary students, *Math Time Travel* for junior and senior high school students, and *The Joy of Math* and *Math Wonderland* for the broader public.

Chapter 1
Art From Tiling Patterns

1. Geometric Patterns

Gen Hi, Kyuta. You look bored.

Kyu Oh, Gen. What kind of mathematical topics will you tell me about today?

Gen points at a spot on the globe and says…

Gen Have you ever been here?

Kyu Is it in Spain!?

Gen Well, wait and see. Have a look at this slide.

Gen shows a picture of a stately palace (Fig. 1.1.1).

Kyu Wow! I've never seen such a spectacular view. What palace is that?

Gen It's the Alhambra Palace in Granada, built during the thirteenth and fourteenth centuries. In those days, the Islamic Empire was flourishing and had a great deal of influence over the whole region of Spain.

Gen recites the beginning of the old Japanese story, "The Tale of the Heike".

Gen "The knell of the bells at the Gion temple echoes the impermanence of all things…"

Kyu Oh, what happened?

J. Akiyama, K. Matsunaga, *Treks into Intuitive Geometry*, DOI 10.1007/978-4-431-55843-9_1

Fig. 1.1.1 The Alhambra Palace

Gen Do you know "The Tale of the Heike"? It is a historical account of when the Heike clan flourished. They held power in Japan around the twelfth century (the first time the Samurai warriors seized power in Japan), but lost the war against the Genji clan and disappeared by the end of the century.
The fate of the Islamic Moorish family who built the Alhambra Palace is just the same as the Heike, don't you think? That ancient Islamic palace stands alone in the south of Spain, and seems to embody their rise and fall.

Kyu Everything prospers and declines. I thought that was just Samurai philosophy.
Walls, ceilings, floors… all of the surfaces of the palace have different repeated geometrical patterns in tiles, plaster work and woodcarvings (Fig. 1.1.2).
How amazing! The beautiful patterns are so elegant.

Gen Islam prohibits images, so Islamic artists can't put people or animals in their art. That's why they've developed exclusively geometrical art.

Kyu I see.

Gen The emperor of the Islamic Moors ordered that every surface of the palace be decorated with geometric patterns in order to make the palace as close to paradise as possible, as described in the Koran. Look at the patterns carefully. Every pattern consists of the same identical shape, repeating over and over again while tiling the plane without gaps or overlaps. Do you see that?

Kyu Oh yes, I do.

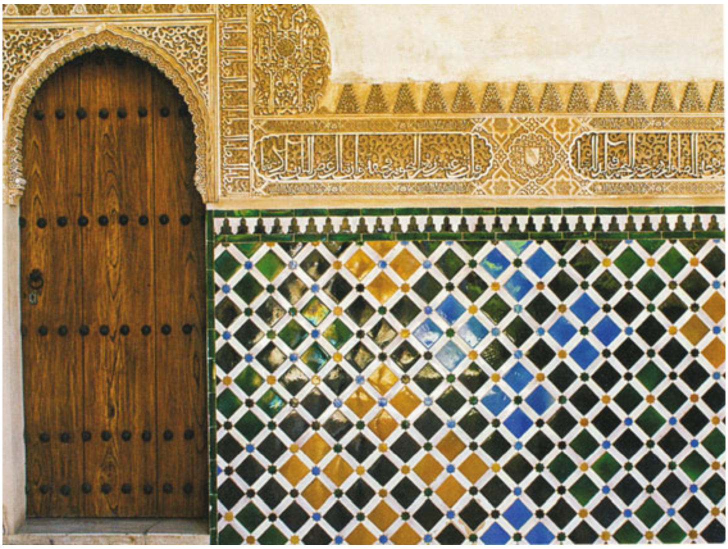


Fig. 1.1.2

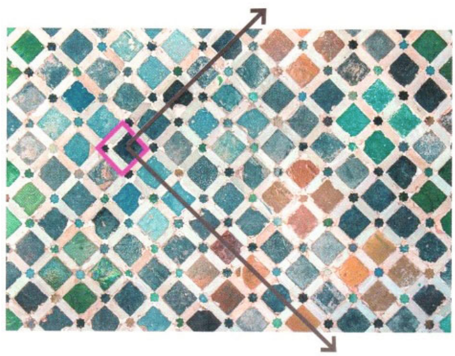


Fig. 1.1.3

Gen There are many patterns, but all of them have one feature in common. There are two different directions along which the pattern can be translated in such a way that the translated pattern coincides with the original pattern (Fig. 1.1.3). This feature is called "repeated (periodic) pattern symmetry."

Kyuta checks that the feature is true for several different patterns (Fig. 1.1.4).

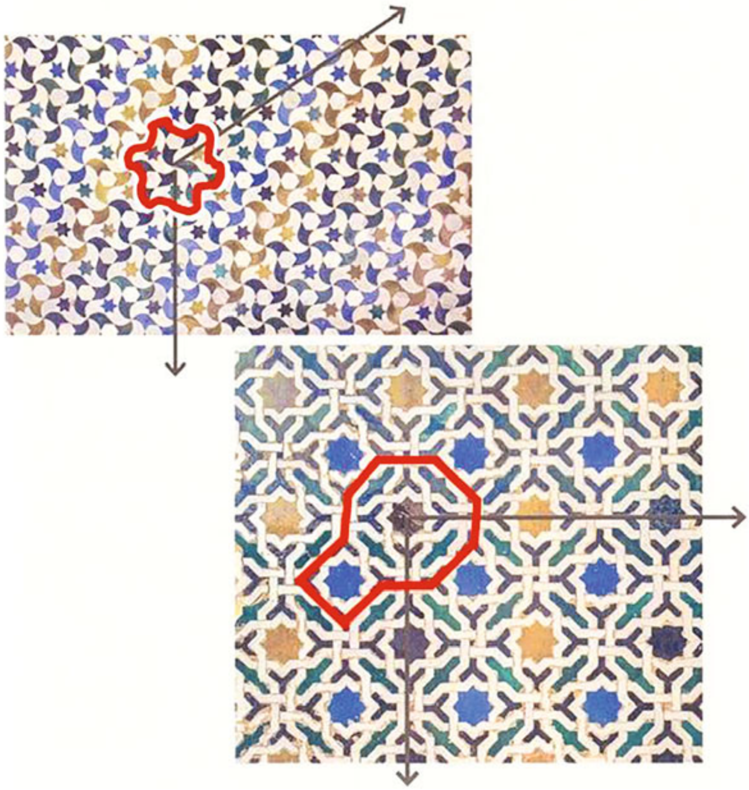


Fig. 1.1.4

Gen And there are exactly 17 different groups of repeated patterns in the patterns decorating the Alhambra.

Kyu Only 17 different groups in all the different patterns decorating the Alhambra?
Are there any other patterns that aren't in any of the 17 groups of the Alhambra and haven't been discovered yet?

Gen No. There are exactly 17 groups of repeated patterns that can possibly exist.
The Islamic Moorish artists in the thirteenth and fourteenth century had already found all 17 groups of repeated patterns through trial and error. You can classify repeated patterns into those 17 groups by asking these questions (Fig. 1.1.5) :

Criteria for Classification of the 17 Groups

1. *What is the smallest rotation around a certain point that makes the rotated pattern coincide with the original one?*
2. *Is the pattern a (line) reflection or not?*
3. *Is the pattern a glide reflection or not?*

A glide reflection is a combination of a translation and a (line) reflection.

Examples of the 17 groups

(1) 120° rotation

This kind is neither a reflection nor a glide reflection. When you rotate the pattern by 120° around any of the marked points, it coincides with the original pattern.

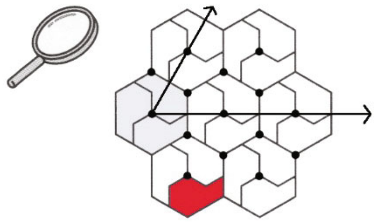


Fig. 1.1.5 (a) 120° rotation

(2) 60° rotation

This is neither a reflection nor a glide reflection. The smallest rotation is 60° around each of the marked points.

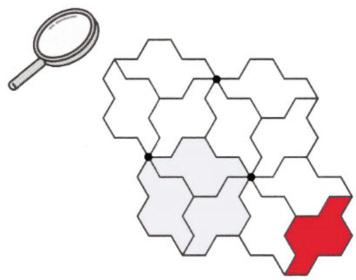


Fig. 1.1.5 (b) 60° rotation

(3) reflection

This is a reflection with respect to the line ℓ (not a glide reflection).
It has no rotation (i.e., the smallest rotation is 360°).

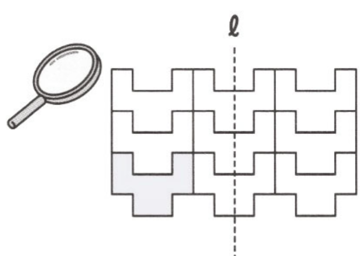


Fig. 1.1.5 (c) Reflection

(4) glide reflection

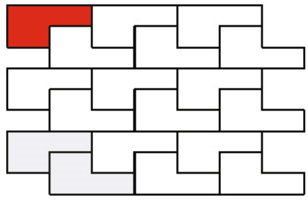


Fig. 1.1.5 (d) Glide reflection

Gen If you want to classify any pattern into one of the 17 groups, all you have to do is follow the flowchart in Appendix 1.1.1. I also give several examples of each tile of these 17 groups in Appendix 1.1.1. The International Crystallographic Union names the 17 groups "p1", "pg", "pm", "cm", "p2", "pgg", "pmg", "pmm", "cmm", "p4", "p4g", "p4m", "p3", "p3m1", "p31m", "p6" and "p6m" as shown in Appendix 1.1.1 [5, 7, 11, 14, 15, 20].

Kyu How is it proved that there are exactly 17 groups of repeated patterns?

Gen It is proved by group theory. Explaining it precisely would require too much space. If you want see the proof, I recommend the books, *Introduction to Geometry* by H. S. M. Coxeter, and *Groups and Symmetry* by M. A. Armstong, etc. I also recommend *Symmetry* by M. du Sautoy which gives vivid stories about mathematicians and artists who struggled to conquer repeated patterns.

Kyu OK. I'll read them someday.

Gen In short, every repeated pattern has one of two dimensional symmetry groups whose elements are products of translations, rotations and reflections. So, what you have to do is determine which features these symmetry groups possess and to numerate all of the symmetry groups. Whether there are more groups than these 17 or not wasn't known for certain until the Russian crystallographer E. S. Fedorov proved it in 1891 ([10, 14, 17]). But some other researchers, G. Polya, P. Nigli, A. M. Schönflies and W. Barlow also studied and proved it independently without knowing of each others' work ([7, 14, 17, 18]).
In the first place, group theory didn't exist until Evariste Galois laid the foundation of modern group theory around 1830, as Martin Gardner mentioned [17]. So, it took about 600 years for human beings to prove that these 17 groups were the only tiling patterns.

Kyu 600 years!

Gen The following theorem is a consequence of the theorem that J. H. Conway, H. Bulgiel and C. Goodman-Straus called the "Magic Theorem".

Theorem 1.1.1 *There are exactly 17 different groups of repeated patterns that can tile the plane.*

Gen Let me explain two terminologies, a fundamental region and a prototile. The pattern in Fig. 1.1.5 (a) is generated by translations of a fundamental region (a gray part) in two directions. This fundamental region consists of three congruent tiles (a red part). Such a tile is called a prototile. As shown in Fig. 1.1.6, some tilings have several prototiles.

(a) (b)

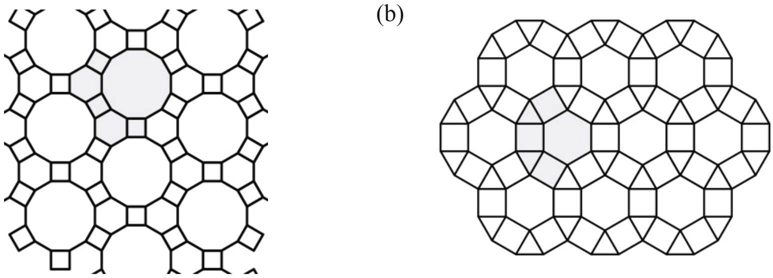


Fig. 1.1.6 A fundamental region and prototiles

Kyu In Fig. 1.1.6 (a), (b), a set of prototiles is of {a square, a regular hexagon, a regular dodecagon}, {an equilateral triangle, a square, a regular hexagon} respectively.

Gen Each gray part is a fundamental region of Fig. 1.1.6 (a), (b) respectively.

Kyu Prototiles for tilings are similar to be atoms for molecules or chemical compounds.

Gen Yes, I think so, too.

Kyu By the way, can there be such a thing as a non-periodic tiling?

Gen Yes, look at this tiling created by copies of a 2×1 rectangle (Fig. 1.1.7), for example. No matter how this tiling is translated, it never coincides with the original tiling pattern.

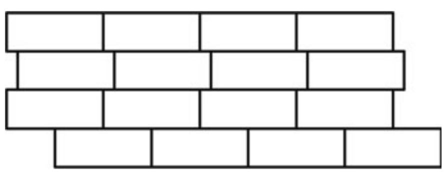


Fig. 1.1.7 A non-periodic tiling

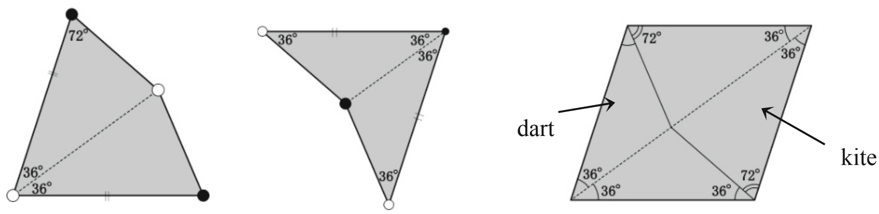
Kyu Wow, this tiling pattern is not periodic. I had no idea that such a simple case could produce a non-periodic tiling.

Gen Yes, but there are two kinds of tilings that are not periodic. One kind is called non-periodic and the other is called aperiodic. Grunbaum and Shephard wrote this in their book *Tilings and Patterns* (1987) [14]:

One of the most remarkable discoveries in the theory of tilings has taken place during the last few years —it concerns the existence of sets of prototiles which admit infinitely many tilings of the plane, yet no such tiling is periodic. Sets of prototiles with this property will be called aperiodic*.*
(Text partly omitted)
There are, of course, many sets of prototiles which admit non-periodic *tiling. Even a 2×1 rectangle has this property. However, the essential feature of an aperiodic set of prototiles is that every tiling admitted by them is necessarily non-periodic. We stress this fact because it seems that there has been some confusion in the past between the term "aperiodic" in the sense used here, and "non-periodic".*

Kyu I see. Researchers are more interested in aperiodic tilings.

Gen Right. Let me introduce you to some famous aperiodic tilings. In 1973 R. Penrose found aperiodic tilings that combine these two shapes (dart and kite) shown in Fig. 1.1.8 (a), (b) [5, 13, 15].



(a) kite (b) dart (c) A rhombus composed of a Kite and a Dart

Fig. 1.1.8 An aperiodic set of prototiles by Penrose

Gen In Penrose tilings, the vertices of the darts and kites are colored black and white as in (a) and (b). The two Penrose pieces, dart and kite, come from a rhombus (Fig. 1.1.8 (c)), and copies of a rhombus tile the plane periodically. In Penrose tilings, when you tile the plane with copies of the two Penrose pieces, each vertex of the tiling has either all black or all white tile vertices (Fig. 1.1.9).

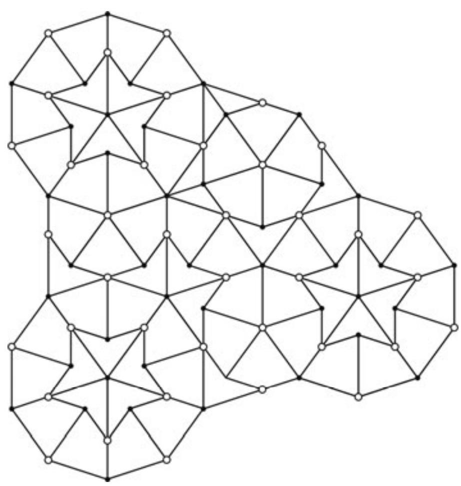
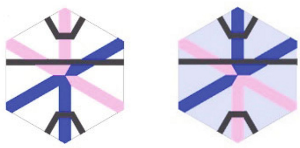


Fig. 1.1.9 Penrose tiling

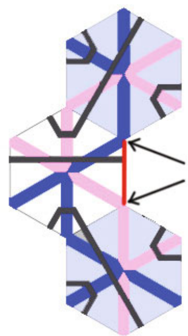
Gen There are many books on Penrose tilings, since they have a lot of intriguing properties and applications. Besides Penrose tilings, there is one particular result worthy of special mention. Recently Joshna E.S. Socolar and Joan Taylor, who lives in Tasmania, have created some beautiful aperiodic hexagonal colored tilings that consist of only a single type of piece under some matching conditions (Fig. 1.1.10) [23].

(a) (b)



Colored hexagons in (a) and (b) are related by reflection about a vertical line.

(c)

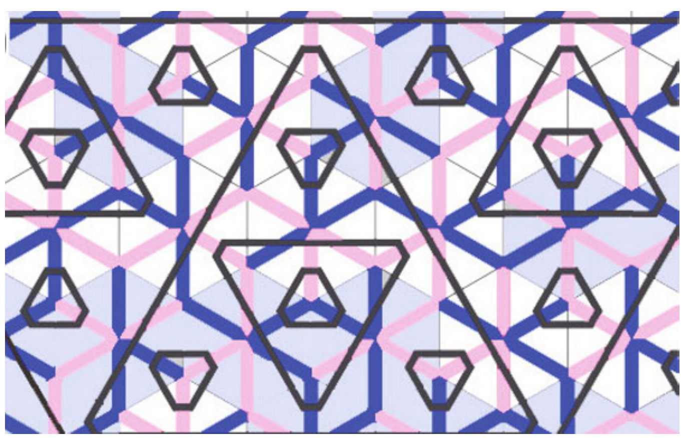


Matching conditions :

(1) Black stripes must be continuous across shared edges

(2) Pink or blue segments that meet a given edge (e.g. a red one in (c)) at opposite endpoints and are collinear with that edge must not be the same color.

(d)



(d) A portion of the infinite tiling

Fig. 1.1.10 An aperiodic tiling

2. Tilings

Gen We've really enjoyed looking at the many tiling patterns in the Alhambra, so let's study tiling by congruent convex polygons a little more and make some crafts in the next chapter.

Kyu All right!

Gen It is easy to see that both squares and rectangles can tile the plane without gaps or overlaps. Before I ask you the next question, let me explain the definition of “convex” and “concave”. A polygon P is said to be convex if every inner angle of P is less than 180°, and concave otherwise.
Now, can you tile the plane with each of these quadrangles (Fig. 1.2.1)?

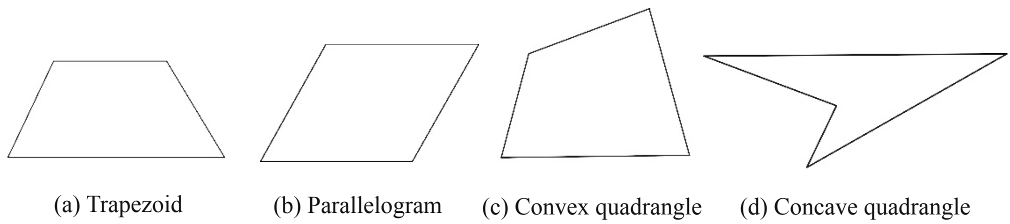


Fig. 1.2.1 Convex and Concave Quadrangles

Kyu It’s trivial that parallelograms can tile the plane.
Two congruent trapezoids combine to form a parallelogram, so any trapezoid can tile the plane (Fig. 1.2.2).

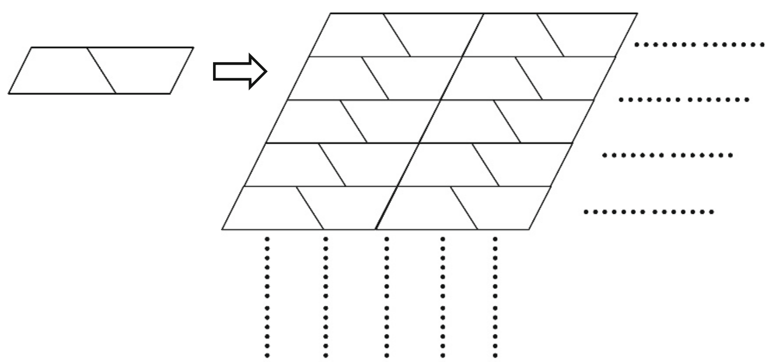


Fig. 1.2.2 A tiling by a trapezoid

Gen Good!

Kyu Next, the convex and concave quadrangles in Fig. 1.2.1 (c) and (d). Hmm….

While Kyuta thinks, Gen gives him some hints.

Gen Look at these tilings, Kyuta (Fig. 1.2.3).

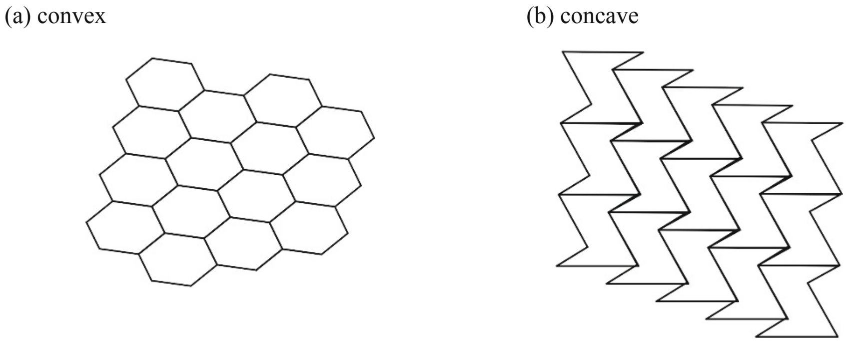


Fig. 1.2.3 Tilings by a hexagon

Kyu Both hexagons tile the plane!

Gen That's right. Do you notice that those hexagons are special hexagons? A hexagon P is called a parallelohexagon if P has three parallel pairs of edges such that members of the same pair have the same length (Fig. 1.2.4). Any parallelohexagon can tile the plane.

Parallelohexagons

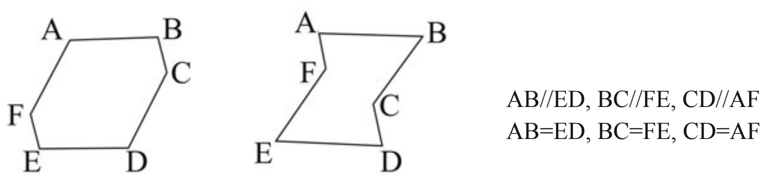


Fig. 1.2.4 Parallelohexagons

Kyu I see. Then what do these tilings with parallelohexagons have to do with tiling with those quadrangles?

Gen A very good question! What kind of shape appears if you combine the two congruent quadrangles in Fig. 1.2.1 (c) and (d) respectively along their common edge as in Fig. 1.2.5?

Kyu Wow! Each of them forms a parallelohexagon. Any kind of parallelohexagon can tile the plane. That means any kind of quadrangle can tile the plane!

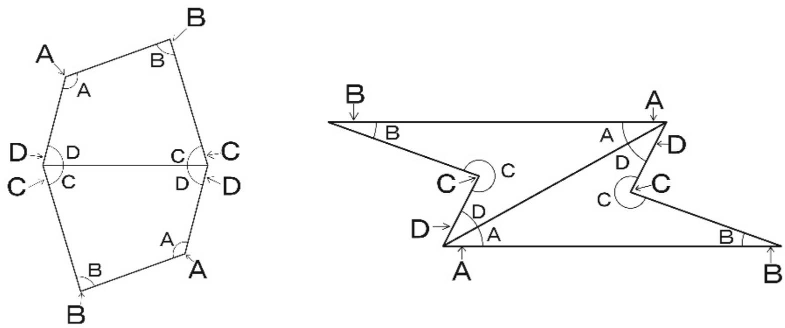
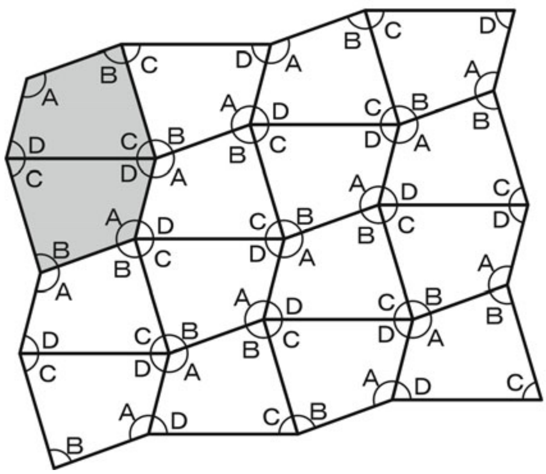


Fig. 1.2.5 Two congruent quadrangles form a parallelohexagon

Gen Good!

Kyuta tiled the plane using each of the quadrangle tiles (Fig. 1.2.6 (a), (b)).

(a)



(b)

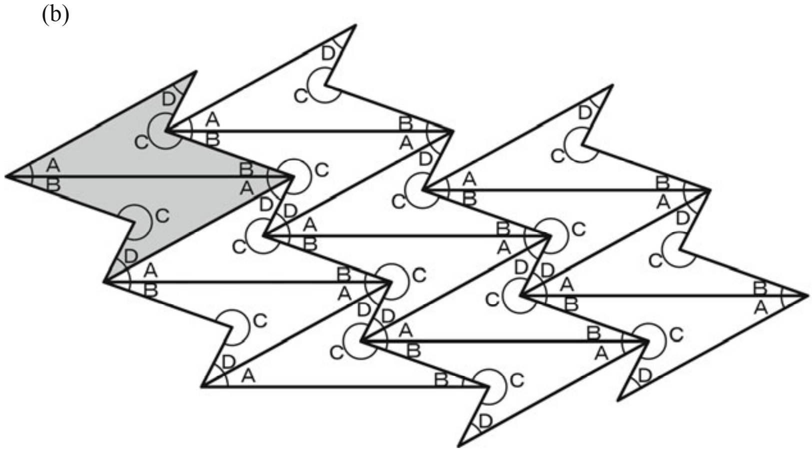


Fig. 1.2.6 Tilings by quadrangles

Gen Next, what about a triangle?

Kyu If two copies of a triangle are joined along a common edge, it makes a parallelogram. This means that any triangle can tile the plane (Fig. 1.2.7).

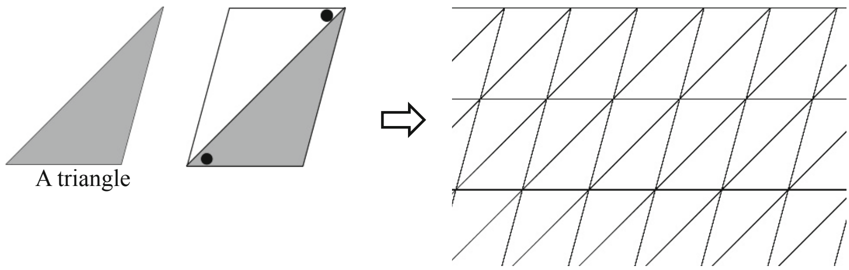


Fig. 1.2.7 A tiling by a triangle

Gen You're right. Let's summarize what we've observed so far.

Summary

(1) Any parallelogram (including a square, rectangle, or rhombus) can tile the plane.
(2) Any triangle can tile the plane, because a combination of two congruent copies forms a parallelogram.
(3) Any parallelohexagon can tile the plane.
(4) Any quadrangle can tile the plane, because a combination of two congruent copies forms a parallelogram or a parallelohexagon.

Gen It was known in ancient Greece that any kind of triangle or quadrangle (even a concave one) can tile the plane with congruent copies.

Kyu It dates back to the period of ancient Greece more than 2000 years ago!

Gen Next, what about a regular pentagon?

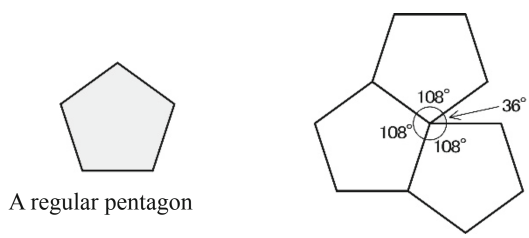


Fig. 1.2.8 A regular pentagon can't tile the plane

Kyu Impossible. No matter how I place regular pentagonal tiles, there will be gaps (Fig. 1.2.8).

Gen That's right. Next, what about a regular hexagon?

Kyu I can tile the plane with regular hexagons (Fig. 1.2.9).

Gen Does this tiling pattern remind you of something?

Kyu Well…

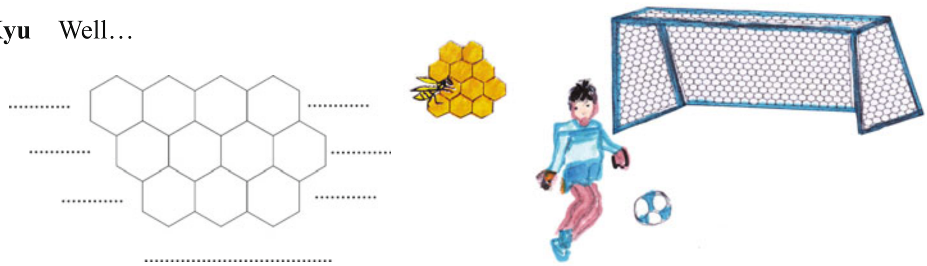


Fig. 1.2.9 A tiling by a regular hexagon

Gen A honeycomb has the same pattern. These days, they make the nets of football goals using the honeycomb pattern (i.e., regular hexagonal tiling patterns). I wish I employed bees as my private secretaries! They are very wise, skillful, and work very hard!

Kyu I admit that bees are more skillful than I am. I wonder how bees are able to construct regular hexagonal structures without rulers, protractors or compasses.

Gen Now, what about a regular heptagon, a regular octagon and so on (Fig. 1.2.10)? And what about convex pentagons, hexagons, heptagons, and more?

Kyu Well … I guess none of them can tile the plane.

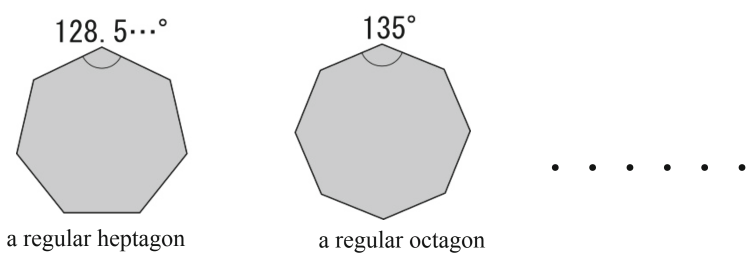


Fig. 1.2.10 Regular n-gons ($n \geq 7$)

Gen Indeed, K. Reinhardt proved in [19] that for $n \geq 7$, no convex n-gons can tile the plane.
Among convex pentagons, 14 types had been found that can tile the plane (see Appendix 1.2.1), until the surprising news came in that type 15 was discovered by C. Mann, J. McLoud and D. V. Derau at the end of July in 2015 (Fig. 1.2.11) [28]. 30 years have passed since type 14 of tessellative convex pentagon was found in 1985. If you tile the plane with convex pentagons under the condition that only edge-to-edge tiling is allowed (i.e., every edge of a tile touches exactly one edge of another tile), then it has been proved that the convex pentagonal tiles must be of these 15 types ([2, 25]). But no one has discovered whether any convex pentagons exist outside the 15 types that can tile the plane in a non-edge-to-edge manner ([5, 12, 14, 24, 25, 26]) as of 2015.