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LQG for the Bewildered

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Abstract

We present a pedagogical introduction to the notions underlying the connection formulation of General Relativity—Loop Quantum Gravity (LQG)—with an emphasis on the physical aspects of the framework. We begin by reviewing General Relativity and Quantum Field Theory, to emphasise the similarities between them which establish a foundation upon which to build a theory of quantum gravity. We then explain, in a concise and clear manner, the steps leading from the Einstein–Hilbert action for gravity to the construction of the quantum states of geometry, known as *spin-networks*, which provide the basis for the kinematical Hilbert space of quantum general relativity. Along the way we introduce the various associated concepts of *tetrads*, *spin-connection* and *holonomies* which are a pre-requisite for understanding the LQG formalism. Having provided a minimal introduction to the LQG framework, we discuss its applications to the problems of black hole entropy and of quantum cosmology. A list of the most common criticisms of LQG is presented, which are then tackled one by one in order to convince the reader of the physical viability of the theory.

An extensive set of appendices provide accessible introductions to several key notions such as the Peter–Weyl theorem, duality of differential forms and Regge calculus, among others. The presentation is aimed at graduate students and researchers who have some familiarity with the tools of quantum mechanics and field theory and/or General Relativity, but are intimidated by the seeming technical prowess required to browse through the existing LQG literature. Our hope is to make the formalism appear a little less bewildering to the uninitiated and to help lower the barrier for entry into the field.

Chapter 1
Introduction

The goal of Loop Quantum Gravity (LQG) is to take two extremely well-developed and successful theories, General Relativity and Quantum Field Theory, at "face value" and attempt to combine them into a single theory with a minimum of assumptions and deviations from established physics. Our goal, as authors of this paper, is to provide a succinct but clear description of LQG—the main body of concepts in the current formulation of LQG relying primarily on the self-dual variables approach, some of the historical basis underlying these concepts, and a few simple yet interesting results—aimed at the reader who has more curiosity than familiarity with the underlying concepts, and hence desires a broad, pedagogical overview before attempting to read more technical discussions. This paper is inspired by the view that one never truly understands a subject until one tries to explain it to others. Accordingly we have attempted to create a discussion which we would have wanted to read when first encountering LQG. Everyone's learning style is different, and accordingly we make note of several other reviews of this subject [1–11], which the reader may refer to in order to gain a broader understanding, and to sample the various points of view held by researchers in the field.

We will begin with a brief review of the history of the field of quantum gravity in the remainder of this section. Following this we review some topics in general relativity in Chap. 2 and Quantum Field Theory in Chap. 3, which hopefully fall into the "Goldilocks zone", providing all the necessary concepts for LQG, and nothing more. We may occasionally introduce concepts in greater detail than the reader considers necessary, but we feel that when introducing concepts to a (hopefully) wide audience who find them unfamiliar, insufficient detail is more harmful than excessive detail. We will discuss the Lagrangian and Hamiltonian approaches to classical GR in more depth and set the stage for its quantization in Chap. 4 then sketch a conceptual outline of the broad program of quantization of the gravitational field in Chap. 5, before moving on to our main discussion of the loop quantum gravity approach in Chap. 6. The pros and cons of the self-dual variables approach are also discussed in some depth in Sect. 5.3. In Chap. 7 we cover applications of the ideas and methods of LQG to the counting of microstates of black holes (Sect. 7.1) and to the problem of quantum cosmology (Sect. 7.2). Last, but not least, some common criticisms of

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LQG and our rebuttals thereof are presented in Chap. 8 along with a discussion of its present status and future prospects.

It is assumed that the reader has a minimal familiarity with the tools and concepts of differential geometry, quantum field theory and general relativity, though we aim to remind the reader of any relevant technical details as necessary.

Before we begin, it would be helpful to give the reader a historical perspective of the developments in theoretical physics which have led us to the present stage.

We are all familiar with classical geometry consisting of points, lines and surfaces. The framework of Euclidean geometry provided the mathematical foundation for Newton's work on inertia and the laws of motion. In the 19th century Gauss, Riemann and Lobachevsky, among others, developed notions of *curved* geometries in which one or more of Euclid's postulates were loosened. The resulting structures allowed Einstein and Hilbert to formulate the theory of general relativity which describes the motion of matter through spacetime as a consequence of the curvature of the background geometry. This curvature in turn is induced by the matter content as encoded in Einstein's equations (2.10). Just as the parallel postulate was the unstated assumption of Newtonian mechanics, whose rejection led to Riemmanian geometry, the unstated assumption underlying the framework of general relativity is that of the smoothness and continuity of spacetime on all scales.

Loop quantum gravity and related approaches invite us to consider the possibility that our notion of spacetime as a smooth continuum must give way to an atomistic description of geometry in which the classical spacetime we observe around us emerges from the interactions of countless (truly indivisible) *atoms* of spacetime. This idea is grounded in mathematically rigorous results, but is also a natural continuation of the trend that began when 19th century attempts to reconcile classical thermodynamics with the physics of radiation encountered fatal difficulties—such as James Jeans' "ultraviolet catastrophe". These difficulties were resolved only when work by Planck, Einstein and others in the early 20th century provided an atomistic description of electromagnetic radiation in terms of particles or "quanta" of light known as *photons*. This development spawned quantum mechanics, and in turn quantum field theory, while around the same time the special and general theories of relativity were being developed.

In the latter part of the 20th century physicists attempted, without much success, to unify the two great frameworks of quantum mechanics and general relativity. For the most part it was assumed that gravity was a phenomenon whose ultimate description was to be found in the form of a quantum field theory as had been so dramatically and successfully accomplished for the electromagnetic, weak and strong forces in the framework known as the Standard Model. These three forces could be understood as arising due to interactions between elementary particles mediated by gauge bosons whose symmetries were encoded in the groups $U(1)$, $SU(2)$ and $SU(3)$ for the electromagnetic, weak and strong forces, respectively. The universal presumption was that the final missing piece of this "grand unified" picture, gravity, would eventually be found in the form of the QFT of some suitable gauge group. This was the motivation for the various grand unified theories (GUTs) developed by Glashow, Pati-Salam, Weinberg and others where the hope was that it would be possible to embed the gravitational interaction along with the Standard Model in some larger group (such $SO(5)$, $SO(10)$ or E_8 depending on the particular scheme). Such schemes could be said to be in conflict with Occam's dictum of simplicity and Einstein and Dirac's notions of beauty and elegance. *More importantly all these models assumed implicitly that spacetime remains continuous at all scales*. As we shall see this assumption lies at the heart of the difficulties encountered in unifying gravity with quantum mechanics.

A significant obstacle to the development of a theory of quantum gravity is the fact that GR is not renormalizable. The gravitational coupling constant G (or equivalently $1/M_{\text{Planck}}^2$ in dimensionless units where $G = c = \hbar = 1$) is not dimensionless, unlike the fine-structure constant α in QED. This means that successive terms in any perturbative series have increasing powers of momenta in the numerator. Rejecting the notion that systems could absorb or transmit energy in arbitrarily small amounts led to the photonic picture of electromagnetic radiation and the discovery of quantum mechanics. Likewise, rejecting the notion that spacetime is arbitrarily smooth at all scales—and replacing it with the idea that geometry at the Planck scale must have a discrete character—leads us to a possible resolution of the ultraviolet infinities encountered in quantum field theory and to a theory of "quantum gravity".

Bekenstein's observation [12–14] of the relationship between the entropy of a black hole and the area of its horizon combined with Hawking's work on black hole thermodynamics led to the realization that there were profound connections between thermodynamics, information theory and black hole physics. These can be succinctly summarized by the famous *area law* relating the entropy of a *macroscopic* black hole S_{BH} to its surface area A:

$$S_{BH} = \gamma A \tag{1.1}$$

where γ is a universal constant and $A \gg A_{\text{P}}$, with $A_{\text{P}} \propto l_{\text{P}}^2$ being the Planck area. While a more detailed discussion will wait until Sect. 7.1, we note here that if geometrical observables such as area are quantized Eq. (1.1) can be seen as arising from the number of ways that one can join together $\mathcal{N}$ quanta of area to form a horizon. In LQG the quantization of geometry arises naturally—though not all theorists are convinced that geometry should be quantized or that LQG is the right way to do so.

With this historical overview in mind, it is now worth summarizing the basic notions of general relativity and QFT before we attempt to see how these two disciplines may be unified in a single framework.

1.1 Conventions

Before we proceed, a quick description of our conventions for indices will hopefully be useful to the reader;

- Greek letters $\mu, \nu, \rho, \lambda, \ldots \in \{0, 1, 2, 3\}$ from the middle of the alphabet are four-dimensional spacetime indices. Other Greek letters, $\alpha, \beta, \ldots$ will be used for general cases in N dimensions.
- Lowercase letters from the start of the Latin alphabet, $a, b, c, \ldots \in \{1, 2, 3\}$ are three-dimensional spatial indices. These will often be used when dealing exclusively with the spatial part of a four-dimensional quantity that would otherwise have Greek indices.
- Lowercase Latin letters $j, k, l, \ldots \in 1, 2, 3, \ldots, N$ from the middle of the alphabet are indices for a space of N dimensions. Equations involving these indices are the general cases, which can be applied to Minkowski space, $\mathbb{R}^3$, etc. They will also sometimes be used as $\mathfrak{su}(2)$ Lie algebra indices.
- Uppercase letters $I, J, K, \ldots$ are specifically "internal" indices used for Lie algebra elements, which take values in the appropriate range, such as $\{1, 2, 3\}$ for the Pauli matrices, or $\{0, 1, 2, 3\}$ for the $\mathfrak{sl}(2, \mathbb{C})$ Lorentz Lie algebra.

Wherever possible we will attempt to avoid using "special" letters (e.g. $\pi, i = \sqrt{-1}$, γ in the context of the Dirac matrices, σ in the context of the Pauli matrices) as indices, unless there is no chance of confusion.

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Chapter 2
Classical GR

General relativity (GR) is an extension of Einstein's special theory of relativity (SR), which was required in order to include observers in non-trivial gravitational backgrounds. SR applies in the absence of gravity, and in essence it describes the behavior of vector quantities in a four-dimensional spacetime, with the Minkowski metric[1]

$$\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1), \tag{2.1}$$

leading to a 4D line-element

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \,. \tag{2.2}$$

The speed of a light signal, measured by any inertial observer, is a constant, denoted c. If we denote the components of a vector in four-dimensional spacetime with Greek indices (e.g. v^μ) the Minkowski metric[2] divides vectors into three categories; *timelike* (those vectors for which $\eta_{\mu\nu} v^\mu v^\nu < 0$), *null* or *light-like* (those vectors for which $\eta_{\mu\nu} v^\mu v^\nu = 0$), and *spacelike* (those vectors for which $\eta_{\mu\nu} v^\mu v^\nu > 0$). Any point, with coordinates (ct, x, y, z), is referred to as an *event*, and the set of all null vectors having their origin at any event define the future light-cone and past light-cone of that event (Fig. 2.1). Events having time-like or null displacement from a given event E_0 (i.e. lying inside or on E_0's lightcones) are causally connected to E_0. Those in/on the past light-cone can influence E_0, those in/on the future lightcone can be influenced by E_0.

[1] Of course the choice $\text{diag}(+1, -1, -1, -1)$ is equally valid but we will have occasion later to restrict our attention to the spacial part of the metric, in which case a positive (spatial) line-element is cleaner to work with.

[2] Strictly speaking it is a pseudo-metric, as the distance it measures between two distinct points can be zero.

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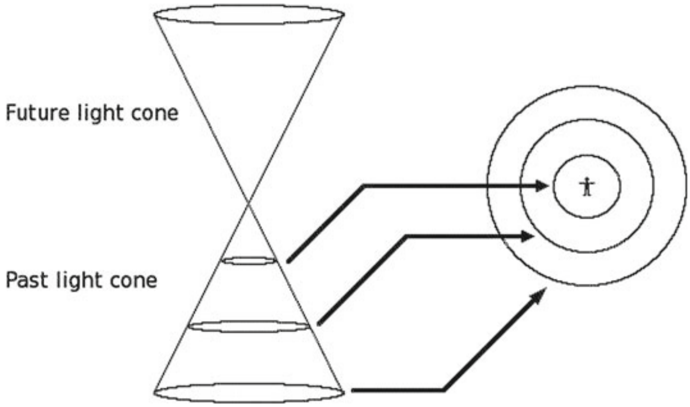

Fig. 2.1 The future-pointing and past-pointing null vectors at a point define the future and past light cones of that point. Slices (at constant time) through the past light cone of an observer are two-spheres centred on the observer, and hence map directly to that observer's celestial sphere

General relativity extends these concepts to non-Euclidean spacetime. The metric of this (possibly curved) spacetime is denoted $g_{\mu\nu}$. Around each event it is possible to consider a sufficiently small region such that the curvature of spacetime within this region is negligible, and hence the central concepts of Special Relativity apply locally. Rather than developing the idea that the curvature of spacetime gives rise to gravitational effects, we shall treat this as assumed knowledge, and discuss how the curvature of spacetime may be investigated. Since spacetime is not assumed to be flat (we'll define "flat" and "curved" rigorously below) and Euclidean, in general one cannot usefully extend the coordinate system from the neighborhood of one point in spacetime (one event) to the neighborhood of another arbitrary point. This can be seen from the fact that a Cartesian coordinate system which defines "up" to be the z-axis at one point on the surface of the Earth, would have to define "up" not to be parallel to the z-axis at most other points. In short, a freely-falling reference frame cannot be extended to each point in the vicinity of the surface of the Earth—or any other gravitating body. We are thus forced to work with local coordinate systems which vary from region to region. We shall refer to the basis vectors of these local coordinate systems by the symbols e_i. A set of four such basis vectors at any point is called a *tetrad* or *vierbein*. The metric is related to the dot product of basis vectors by $g_{ij} = e_i \cdot e_j$. As the basis vectors are not necessarily orthonormal, we also may define a set of dual basis vectors e^i, where $e^i \cdot e_j = \delta^i_j$.

2.1 Parallel Transport and Curvature

Given the basis vectors e_i of a local coordinate system, an arbitrary vector is written in terms of its components v^i as $\vec{V} = v^i e_i$. It is of course also possible to define vectors